High-Speed Integer Optimal Control using ADP

Bartolomeo Stellato and Paul Goulart
Medium-voltage drives market is worth 4bln$ … and is growing 10% every year!
Grid Rectifier Inverter Motor Load
Power Distribution

Inverter  ≈  Motor
Power Distribution

Inverter  Motor

=  \approx

\begin{tikzpicture}
  \node (inverter) at (0,0) {\includegraphics{inverter.png}};
  \node (motor) at (3,0) {\includegraphics{motor.png}};
  \draw[->] (inverter) -- (motor);
\end{tikzpicture}

\begin{tikzpicture}
  \node (inverter) at (-3,-3) {\includegraphics{inverter.png}};
  \node (motor) at (3,-3) {\includegraphics{motor.png}};
  \draw[->] (inverter) -- (motor);
  \node (current) at (3,0) {$i_s$};
\end{tikzpicture}
Traditional Control Schemes Are Suboptimal

Traditional Control

\[ T^* \quad u_{cont} \quad \text{Modulator} \quad u \quad \text{Inverter and Motor} \quad X \]
Traditional Control Schemes Are Suboptimal

Direct MPC

$T^*$

Controller

$u_{sw}$

Inverter and Motor

$x$
... But Direct MPC is Difficult

![Graph showing THD and currents over time](image)

- **THD**
- **Inputs**

Bartolomeo Stellato
... But Direct MPC is Difficult

Tradeoff

\( THD \) vs \( f_{sw} \)
... But Direct MPC is Difficult

Very fast dynamics
... But Direct MPC is Difficult

Very fast dynamics

Mixed Integer Optimization Problems in 25 µs!
... But Direct MPC is Difficult

Control Objectives

\[ THD \]

\[ f_{sw} \]

Tradeoff

\[ THD \text{ vs } f_{sw} \]

Timing

Mixed Integer Optimization Problems in 25 \( \mu s \)!
Problem Formulation
Total Harmonic Distortion

\[ THD \sim \lim_{M \to \infty} \sqrt{\frac{1}{M} \sum_{k=0}^{M-1} \| i(k) - i^*(k) \|_2^2} \]
Total Harmonic Distortion

\[ THD \sim \lim_{M \to \infty} \sqrt{\frac{1}{M} \sum_{k=0}^{M-1} \| i(k) - i^*(k) \|_2^2} \]

**THD Component in Cost Function**

\[ \sum_{k=0}^{\infty} \gamma^k \| i(k) - i^*(k) \|_2^2 \]
Total Harmonic Distortion

\[ THD \sim \lim_{M \to \infty} \sqrt{\frac{1}{M} \sum_{k=0}^{M-1} \|i(k) - i^*(k)\|_2^2} \]

\[ THD \text{ Component in Cost Function} \]

\[ \sum_{k=0}^{\infty} \gamma^k \|i(k) - i^*(k)\|_2^2 \]

Internal Motor States \( x_m \)
Total Harmonic Distortion

\[
THD \sim \lim_{M \to \infty} \sqrt{\frac{1}{M} \sum_{k=0}^{M-1} \|i(k) - i^*(k)\|^2_2}
\]

**THD Component in Cost Function**

\[
\sum_{k=0}^{\infty} \gamma^k \|i(k) - i^*(k)\|^2_2
\]

- Internal Motor States \(x_m\)
- Oscillating References \(x_{osc}\)
$$f_{sw} = \lim_{T \to \infty} \frac{1}{T} \sum_{k=0}^{T} \| u(k) - u(k - 1) \|_1$$
Switching Frequency

\[ f_{sw} = \lim_{T \to \infty} \frac{1}{T} \sum_{k=0}^{T} \| u(k) - u(k - 1) \|_1 \]

FIR Filter
Switching Frequency

\[ f_{sw} = \lim_{T \to \infty} \frac{1}{T} \sum_{k=0}^{T} \| u(k) - u(k-1) \|_1 \]

FIR Filter

Approximate with IIR Filter
Switching Frequency

\[ f_{sw} = \lim_{T \to \infty} \frac{1}{T} \sum_{k=0}^{T} \| u(k) - u(k - 1) \|_1 \]

Approximate with IIR Filter

FIR Filter

\[ f_{sw} \text{ Component in Cost Function} \]

\[ \delta \sum_{k=0}^{\infty} \gamma^k \| \hat{f}_{sw}(k) - f_{sw}^* \|_2^2 \]
Switching Frequency

\[ f_{sw} = \lim_{T \to \infty} \frac{1}{T} \sum_{k=0}^{T} \| u(k) - u(k - 1) \|_1 \]

Approximate with IIR Filter

FIR Filter

\[ f_{sw} \text{ Component in Cost Function} \]

\[ \delta \sum_{k=0}^{\infty} \gamma^k \| \hat{f}_{sw}(k) - f_{sw}^* \|_2^2 \]

Frequency Estimate
Switching Frequency

\[ f_{SW} = \lim_{T \to \infty} \frac{1}{T} \sum_{k=0}^{T} \| u(k) - u(k-1) \|_1 \]

FIR Filter

\[ f_{SW} \] Component in Cost Function

\[ \delta \sum_{k=0}^{\infty} \gamma^k \| \hat{f}_{SW}(k) - f_{\text{SW}}^* \|_2^2 \]

Frequency Estimate

Desired Frequency

Approximate with IIR Filter
Switching Frequency

\[ f_{sw} = \lim_{T \to \infty} \frac{1}{T} \sum_{k=0}^{T} \| u(k) - u(k-1) \|_1 \]  

FIR Filter

Approximate with IIR Filter

\[ f_{sw} \text{ Component in Cost Function} \]

\[ \delta \sum_{k=0}^{\infty} \gamma^k \| f_{sw}(k) - f_{sw}^* \|_2^2 \]

Frequency Estimate

Desired Frequency

Filter States \( x_{iir} \)
States: $x = (\mathbf{x}_{osc}, \mathbf{x}_{im}, \mathbf{x}_{iir})$
Complete Block Diagram

Controller

$T^*$

$x_{osc}$

$x_m$

$x_{iir}$

MPC

Inverter and Motor

$u$

$x_m$

States: $x = (x_{osc}, x_m, x_{iir})$
Infinite Horizon

minimize \[ \sum_{k=0}^{\infty} \gamma^k l(x(k)) \]
subject to \[ x(k + 1) = Ax(k) + Bu(k) \]
\[ x(0) = x_0 \]
\[ x(k) \in \mathcal{X}, u(k) \in \{-1, 0, 1\}^3 \]
Optimal Control Problem

Infinite Horizon

minimize \( \sum_{k=0}^{\infty} \gamma^k l(x(k)) \)

subject to \( x(k+1) = Ax(k) + Bu(k) \)
\( x(0) = x_0 \)
\( x(k) \in \mathcal{X}, u(k) \in \{-1, 0, 1\}^3 \)

where \( l(x(k)) = \|i(k) - i^*(k)\|_2^2 + \delta\|\hat{f}_{sw}(k) - f^*_{sw}\|_2^2 \)

\( THD \) \( f_{sw} \)
Infinite Horizon

minimize \( \sum_{k=0}^{\infty} \gamma^k l(x(k)) \)

subject to \( x(k + 1) = Ax(k) + Bu(k) \)
\( x(0) = x_0 \)
\( x(k) \in \mathcal{X}, u(k) \in \{-1, 0, 1\}^3 \)

where \( l(x(k)) = \|i(k) - i^*(k)\|_2^2 + \delta \|\hat{f}_{sw}(k) - f^*_{sw}\|_2^2 \)

THD

\( f_{sw} \)
Optimal Control Problem

Infinite Horizon

minimize \[ \sum_{k=0}^{\infty} \gamma^k l(x(k)) \]

subject to \[
\begin{align*}
x(k + 1) &= A x(k) + B u(k) \\
x(0) &= x_0 \\
x(k) &\in \mathcal{X}, u(k) \in \{-1, 0, 1\}^3
\end{align*}
\]

where \[
l(x(k)) = \underbrace{\| i(k) - i^*(k) \|_2^2}_{THD} + \underbrace{\delta \| \hat{f}_{sw}(k) - f^*_{sw} \|_2^2}_{f_{sw}}
\]
Optimal Control Problem

Infinite Horizon

minimize \( \sum_{k=0}^{\infty} \gamma^k l(x(k)) \)

subject to \( x(k + 1) = Ax(k) + Bu(k) \)
\( x(0) = x_0 \)
\( x(k) \in \mathcal{X}, u(k) \in \{-1, 0, 1\}^3 \)

where \( l(x(k)) = \|i(k) - i^*(k)\|_2^2 + \delta \|\hat{f}_{sw}(k) - f_{sw}^*\|_2^2 \)

THD
\( f_{sw} \)

Impossible to Solve Online
Problem Solution
Optimal Control Problem

Infinite Horizon

minimize \[ \sum_{k=0}^{\infty} \gamma^k l(x(k)) \]
subject to \[ x(k + 1) = Ax(k) + Bu(k) \]
 \[ x(0) = x_0 \]
 \[ x(k) \in \mathcal{X}, u(k) \in \{-1, 0, 1\}^3 \]
Optimal Control Problem

Short Finite Horizon

minimize \[ \sum_{k=0}^{N-1} \gamma^k l(x(k)) + \gamma^N V(x(k)) \]
subject to \[ x(k+1) = Ax(k) + Bu(k) \]
\[ x(0) = x_0 \]
\[ x(k) \in \mathcal{X}, u(k) \in \{-1, 0, 1\}^3 \]
Optimal Control Problem

Short Finite Horizon

\[
\begin{align*}
\text{minimize} \quad & \sum_{k=0}^{N-1} \gamma^k l(x(k)) + \gamma^N V(x(k)) \\
\text{subject to} \quad & x(k + 1) = Ax(k) + Bu(k) \\
& x(0) = x_0 \\
& x(k) \in X, u(k) \in \{-1, 0, 1\}^3
\end{align*}
\]

Very Short Horizon N
Optimal Control Problem

Short Finite Horizon

\[
\begin{align*}
\text{minimize} & \quad \sum_{k=0}^{N-1} \gamma^k l(x(k)) + \gamma^N V(x(k)) \\
\text{subject to} & \quad x(k+1) = Ax(k) + Bu(k) \\
& \quad x(0) = x_0 \\
& \quad x(k) \in \mathcal{X}, u(k) \in \{-1, 0, 1\}^3
\end{align*}
\]

Very Short Horizon N

Approximate Offline $V(x(k))$
Bellman Equality

$$V^*(x) = \min_u \{ l(x) + \gamma V^* (Ax + Bu) \}$$
Bellman Equality

\[ V^*(x) = \min_u \{ l(x) + \gamma V^*(Ax + Bu) \} \]

Bellman Operator \( \mathcal{T} \)
Bellman Equality

\[ V^*(x) = \min_u \{ l(x) + \gamma V^*(Ax + Bu) \} \]

Bellman Operator \( \mathcal{T} \)

Contractive

\[ \lim_{M \to \infty} \mathcal{T}^M V = V^* \]
**Bellman Equality**

\[ V^*(x) = \min_u \{ l(x) + \gamma V^* (Ax + Bu) \} \]

\[ \mathcal{T} V^* \]

**Bellman Operator** \( \mathcal{T} \)

**Contractive**

\[ \lim_{M \to \infty} \mathcal{T}^M V = V^* \]

**Monotone**

\[ V_1 \leq V_2 \implies \mathcal{T} V_1 \leq \mathcal{T} V_2 \]
Approximate Dynamic Programming

Bellman Inequality

\[ V(x) \leq \min_u \{ l(x) + \gamma V(Ax + Bu) \} \]

Bellman Operator \( \mathcal{T} \)

Contractive

\[ \lim_{M \to \infty} \mathcal{T}^M V = V^* \]

Monotone

\[ V_1 \leq V_2 \implies \mathcal{T} V_1 \leq \mathcal{T} V_2 \]
Bellman Inequality

\[ V(x) \leq \min_u \{ l(x) + \gamma V(Ax + Bu) \} \]

Bellman Operator \( \mathcal{T} \)

Contractive

\[ \lim_{M \to \infty} \mathcal{T}^M V = V^* \]

Monotone

\[ V_1 \leq V_2 \implies \mathcal{T}V_1 \leq \mathcal{T}V_2 \]

Sufficient condition for underestimating \( V^* \)

\[ V \leq \mathcal{T}V \implies V \leq V^* \]
Finding the Best Underestimator

Infinite Dimensional LP

maximize \( \int_{\mathcal{X}} V(x) c(dx) \)

subject to \( V(x) \leq TV(x) \quad \forall x \)
Infinite Dimensional LP

maximize \[ \int_{X} V(x)c(dx) \]
subject to \[ V(x) \leq T^M V(x) \quad \forall x \]

Iterated Bellman Inequality
Infinite Dimensional LP

maximize \( \int_{x} V(x)c(dx) \)
subject to \( V(x) \leq T^M V(x) \quad \forall x \)

Iterated Bellman Inequality

Restrict to Quadratic Functions
\[ V(x) = x^T P x + 2q^T x + r \]
Infinite Dimensional LP

\[
\begin{align*}
\text{maximize} \quad & \int_{\mathcal{X}} V(x) c(dx) \\
\text{subject to} \quad & V(x) \leq T^M V(x) \quad \forall x
\end{align*}
\]

Iterated Bellman Inequality

Restrict to Quadratic Functions

\[V(x) = x^T P x + 2q^T x + r\]

Tractable SDP (Offline)
FPGA Implementation

PROTOIP Toolbox
(Suardi et al.)

- Brute force enumeration
- Pipelined Evaluation of Switch Sequences
- Parallelized Matrix Computations
Timing Benchmarks

Computation Times under 25 µs!
Hardware in the Loop Tests
Steady-State Performance at $f_{SW} = 300$ Hz

<table>
<thead>
<tr>
<th>Horizon Length N</th>
<th>THD [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.44</td>
</tr>
<tr>
<td>2</td>
<td>5.43</td>
</tr>
<tr>
<td>3</td>
<td>5.39</td>
</tr>
<tr>
<td>10</td>
<td>5.29</td>
</tr>
</tbody>
</table>

- **ADP**
- **State of the art**
Steady-State Performance at $f_{SW} = 300$ Hz

Better Performance with Short Horizons

THD [%]

- ADP
- State of the art

Horizon Length N

1  2  3  10

5.44  5.43  5.39  5.29

5.24  5.13  5.1  4.8
Transients

Torque

Inputs

Time [ms]

0 5 10 15 20 25 30

0 0.25 0.5 0.75 1

0 1

0 5 10 15 20 25 30

0 1 0

0 5 10 15 20 25 30

0 1 0

0 5 10 15 20 25 30

0 1 0
Transients

Torque

Inputs

0.35 ms
Transients

Torque

Time [ms]

0 5 10 15 20 25 30

0 0.25 0.5 0.75 1

Inputs

Time [ms]

0 5 10 15 20 25 30

-1 0 1

0.35 ms 3.5 ms

Bartolomeo Stellato
Transients

Torque

Inputs

Very Fast Transient Times!

0.35 ms  3.5 ms
• Meaningful formulation ($THD$ vs $f_{sw}$)
• Meaningful formulation \((THD \text{ vs } f_{sw})\)

• Complexity reduction with good performance (ADP)
Conclusions - New Approach to Direct MPC

- Meaningful formulation ($THD$ vs $f_{sw}$)
- Complexity reduction with good performance (ADP)
- Real-Time FPGA implementation ($< 25\mu s$)
• Meaningful formulation ($THD$ vs $f_{sw}$)

• Complexity reduction with good performance (ADP)

• Real-Time FPGA implementation ($< 25\mu s$)

“High-Speed Finite Control Set Model Predictive Control for Power Electronics”
B. Stellato, T. Geyer, P. J. Goulart

IEEE Transactions on Power Electronics (In Press)
\[ T = C(\psi_r \times i_s) \]
**Integer Program**

minimize \( \frac{1}{2} U^T Q U + f(x_0)^T U \)

subject to \( A_{ineq} U \leq b_{ineq}(x_0) \)

\( U \in \{-1, 0, 1\}^{3N} \)
Quadratic Cost Function

\[
\int_{X} V_0(z) c(dz) = \text{Tr}(P_0 \Sigma_c) + 2q_0^T \mu_c + r_0
\]

SDP Formulation

maximize \[\text{Tr} (P_0 \Sigma_c) + 2q_0^T \mu_c + r_0\]

subject to \[\tilde{M}_i(m) \succeq 0, \quad \forall m \in M, \quad i = 1, \ldots, M\]
\[V_0 = V_M\]
\[P_i \in S^{n_x}, \quad q_i \in \mathbb{R}^{n_x}, \quad r_i \in \mathbb{R}, \quad i = 0, \ldots, M\]
Backup - Switch Positions