ORF522 – Linear and Nonlinear Optimization

12. Introduction to nonlinear optimization

Bartolomeo Stellato — Fall 2022

Homogeneous self-dual embedding

Optimality conditions

Primal

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax + s = b \\ & s \geq 0 \end{array}$$

Dual

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c = 0 \\ & y \geq 0 \end{array}$$

Optimality conditions

$$\begin{bmatrix} 0 \\ s \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & A^T \\ -A & 0 \\ c^T & b^T \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ b \\ 0 \end{bmatrix}$$

$$s, y \geq 0$$

Any $(x^\star, s^\star, y^\star)$ satisfying these conditions is **optimal**

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Any $(x^\star, s^\star, y^\star)$ satisfying these conditions is **optimal**

What happens if the problem is infeasible?

How do you detect infeasibility/unboundedness?

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Alternatives (Farkas lemma) Write feasibility problem and dualize…

- **primal feasible:** $Ax + s = b, \quad s \geq 0$
- **primal infeasible:** $A^T y = 0, \quad b^T y < 0, \quad y \geq 0$

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How do you detect infeasibility/unboundedness?

Primal		**Dual**	
minimize	$c^T x$	maximize	$-b^T y$
subject to	$Ax + s = b$	subject to	$A^T y + c = 0$
	$s \geq 0$		$y \geq 0$

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- **dual feasible:** $A^T y + c = 0, \quad y \geq 0$
- **dual infeasible:** $Ax \leq 0, \quad c^T x < 0$

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The homogeneous self-dual embedding

Derivation

Introduce two new variables $\kappa, \tau \geq 0$

Homogeneous self-dual embedding

$$\begin{bmatrix} 0 \\ s \\ \kappa \end{bmatrix} = \begin{bmatrix} 0 & A^T & c \\ -A & 0 & b \\ -c^T & -b^T & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ \tau \end{bmatrix}$$

$$s, y, \kappa, \tau \geq 0$$

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$$\longrightarrow$$

$$Qu = v$$
$$u, v \geq 0$$

$$Q = \begin{bmatrix} 0 & A^T & c \\ -A & 0 & b \\ -c^T & -b^T & 0 \end{bmatrix}$$

$$u = (x, y, \tau)$$
$$v = (0, s, \kappa)$$

The homogeneous self-dual embedding

Properties

$$Qu = v$$
$$u, v \geq 0$$

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Matrix

- Q is **skew-symmetric**: $Q^T = -Q \quad \Rightarrow \quad u^T Q u = 0$
- $u \perp v$ **proof** $Qu - v = 0 \quad \Rightarrow \quad u^T Q u - u^T v = 0 \quad \Rightarrow \quad u^T v = 0 \quad \blacksquare$

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Homogeneous

(u, v) satisfy $Qu = v,\ (v, u) \geq 0 \quad \Rightarrow \quad \alpha(u, v)$ with $\alpha \geq 0$ feasible

The homogeneous self-dual embedding

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Homogeneous

(u, v) satisfy $Qu = v,\ (v, u) \geq 0 \quad \Rightarrow \quad \alpha(u, v)$ with $\alpha \geq 0$ feasible

Always feasible

$\alpha = 0 \quad \Rightarrow \quad (0, 0)$ is feasible

The homogeneous self-dual embedding

Outcomes

Find x, s, y, κ, τ such that
$$\begin{bmatrix} 0 \\ s \\ \kappa \end{bmatrix} = \begin{bmatrix} 0 & A^T & c \\ -A & 0 & b \\ -c^T & -b^T & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ \tau \end{bmatrix}$$
$$s, y, \kappa, \tau \geq 0$$

Note. By strict complementarity, we can ensure $\kappa + \tau > 0$

The homogeneous self-dual embedding

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Case 1: feasibility

$\tau > 0, \kappa = 0$ define $(\hat{x}, \hat{s}, \hat{y}) = (x^\star/\tau, s^\star/\tau, y^\star/\tau)$

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$$\begin{aligned} 0 &= A^T\hat{y} + c \\ \hat{s} &= -A\hat{x} + b \end{aligned} \qquad \hat{s} \geq 0, \quad \hat{y} \geq 0, \quad \hat{s}^T\hat{y} = 0$$

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$\longrightarrow$ $(\hat{x}, \hat{s}, \hat{y})$ is a **solution** to the original problem

The homogeneous self-dual embedding

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Case 2: infeasibility

$\tau = 0, \kappa > 0 \longrightarrow c^T x + b^T y < 0$ (**impossible**). Must have infeasibility

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Case 2: infeasibility

$\tau = 0, \kappa > 0 \longrightarrow c^T x + b^T y < 0$ (**impossible**). Must have infeasibility

If $b^T y < 0$ then $\hat{y} = y/(-b^T y)$ is a **certificate of primal infeasibility**
$$A^T \hat{y} = 0, \quad b^T \hat{y} = -1 < 0, \quad \hat{y} \geq 0$$

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Outcomes

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Case 2: infeasibility

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If $c^T x < 0$ then $\hat{x} = x/(-c^T x)$ is a **certificate of dual infeasibility**
$$A\hat{x} \leq 0, \quad c^T \hat{x} = -1 < 0$$

Interior-point method for homogeneous self-dual embedding

Linear complementarity problem

$$\begin{aligned} Qu &= v \\ u^T v &= 0 \\ u, v &\geq 0 \end{aligned}$$

Equations

$$h(u, v) = \begin{bmatrix} Qu - v \\ UV\mathbf{1} \end{bmatrix} = 0$$

$$u, v \geq 0$$

Interior-point method for homogeneous self-dual embedding

Linear complementarity problem

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$$
u, v \geq 0
$$

Directions

$$
\begin{bmatrix} Q & -I \\ V & U \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix} = \begin{bmatrix} -r_e \\ -UV\mathbf{1} + \sigma\mu\mathbf{1} \end{bmatrix}
$$

$$
\begin{aligned}
r_e &= Qu - v \\
\mu &= (u^T v)/d
\end{aligned}
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Interior-point method for homogeneous self-dual embedding

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Line search to enforce $u, v > 0$

$$
(u, v) \leftarrow (u, v) + \alpha(\Delta u, \Delta v)
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Interior-point method for homogeneous self-dual embedding

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Line search to enforce $u, v > 0$

$$
(u, v) \leftarrow (u, v) + \alpha(\Delta u, \Delta v)
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Interior-point methods can solve **linear complementarity problems**

Today's lecture

[Chapter 2-4 and 6, CO] [Chapter A and B, FCA]

- Nonlinear optimization
- Examples
- Convex analysis review
- Convex optimization

What if the problem is no longer linear?

Nonlinear optimization

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_i(x) \leq 0, \quad i = 1, \ldots, m \end{array}$$

$x = (x_1, \ldots, x_n)$ Variables

$f : \mathbf{R}^n \to \mathbf{R}$ Nonlinear objective function

$g_i : \mathbf{R}^n \to \mathbf{R}$ Nonlinear constraints functions

Feasible set

$$C = \{x \mid g_i(x) \leq 0, \quad i = 1, \ldots, m\}$$

Small example

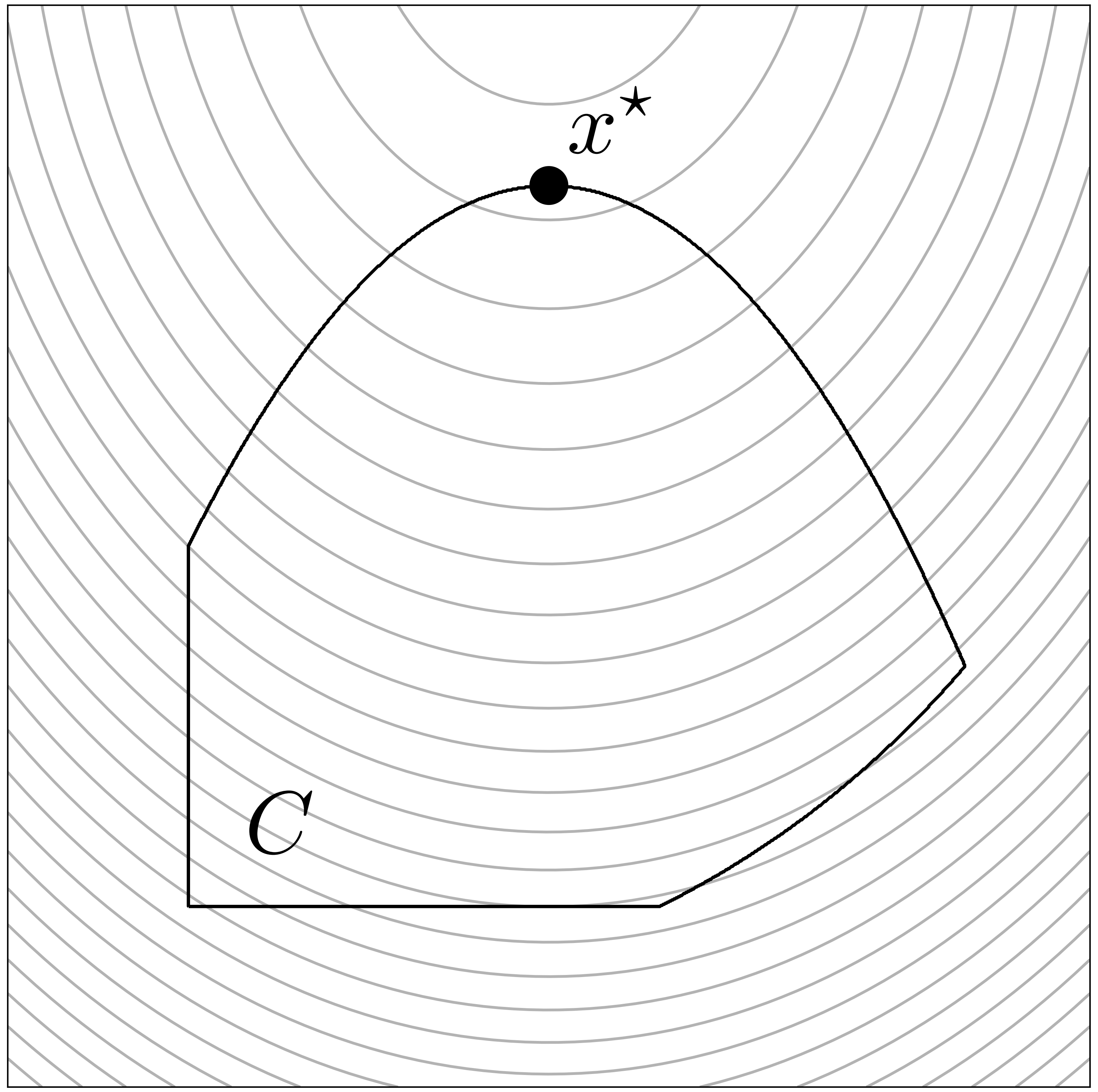
minimize $\quad 0.5x_1^2 + 0.25x_2^2$

subject to $\quad e^{x_1} - 2 - x_2 \leq 0$

$$(x_1 - 1)^2 + x_2 - 3 \leq 0$$

$$x_1 \geq 0$$

$$x_2 \geq 1$$



Contour plot has curves (no longer lines)

Feasible set is no longer a polyhedron

Integer optimization

It's still nonlinear optimization

minimize $\quad f(x)$

subject to $\quad x \in \mathbf{Z}$

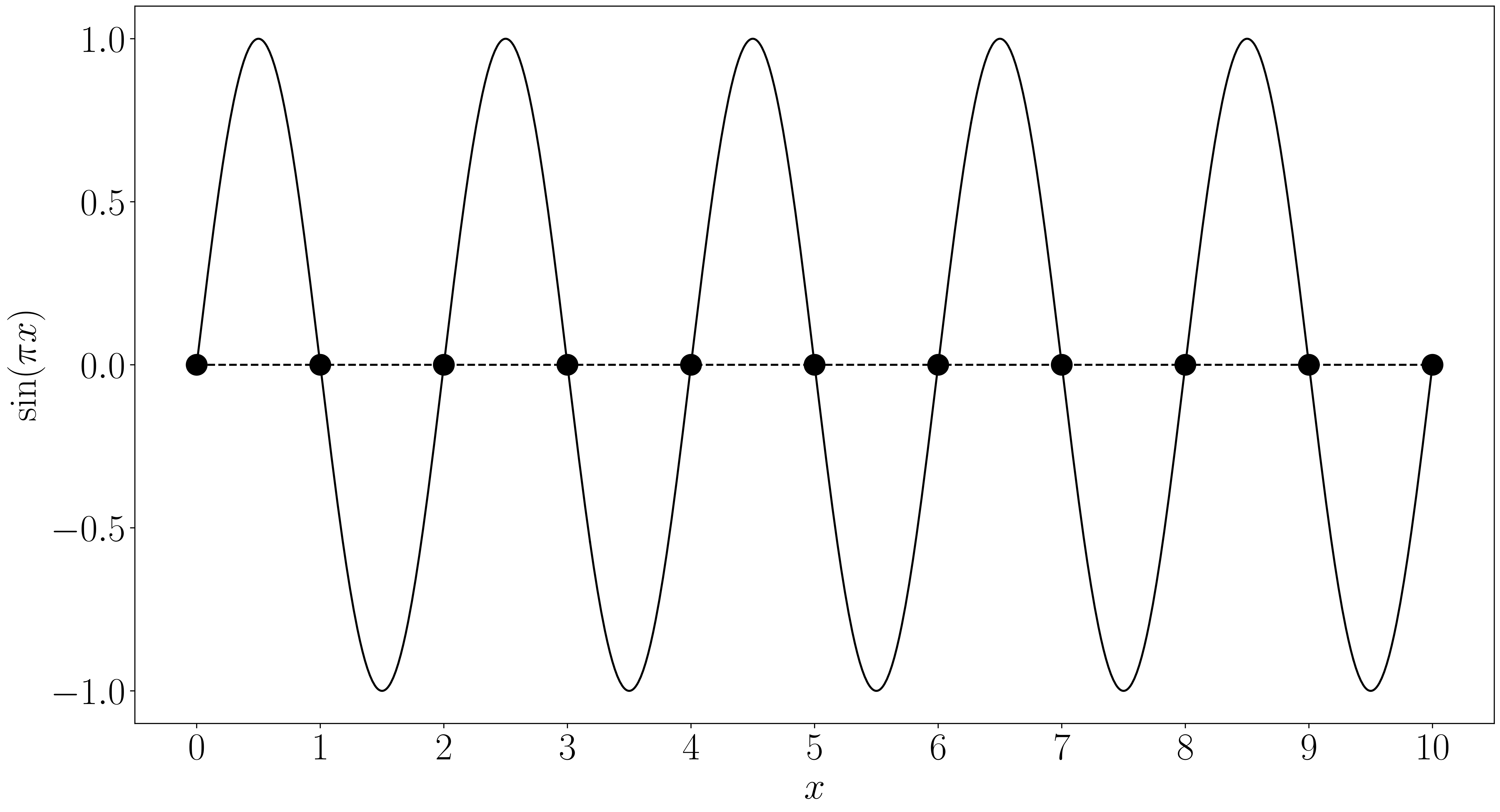
Integer optimization

It's still nonlinear optimization

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathbf{Z} \end{array}$$

$\Updownarrow$

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & \sin(\pi x) = 0 \end{array}$$



We cannot solve most nonlinear optimization problems

Examples of (solvable) nonlinear optimization

Regression

Fit affine function $f(z) = \alpha + \beta z$ to m points (z_i, y_i)

Approximation problem $Ax \approx b$ where
$$A = \begin{bmatrix} 1 & z_1 \\ \vdots & \vdots \\ 1 & z_m \end{bmatrix}, \quad x = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \quad b = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

Goal

minimize $\quad \|Ax - b\|$

Regression

Fit affine function $f(z) = \alpha + \beta z$ to m points (z_i, y_i)

Approximation problem $Ax \approx b$ where

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Goal

minimize $\quad \|Ax - b\|$

1-norm or ∞-norm $\quad \Longrightarrow \quad$ **linear optimization**

2-norm $\quad \Longrightarrow \quad$ **least-squares** $\qquad \|Ax - b\|_2^2 = \sum_i (f(z_i) - y_i)^2$

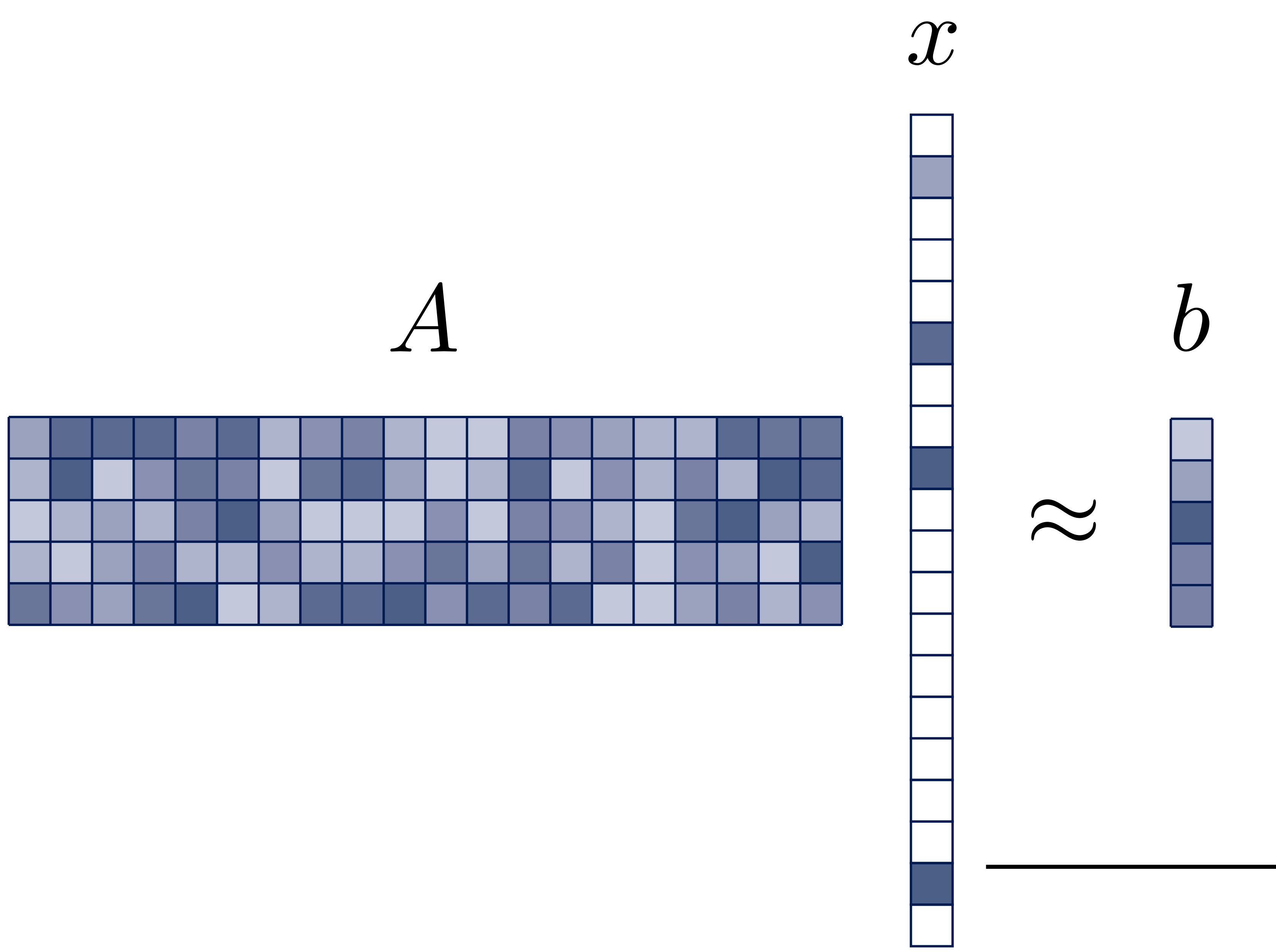
Sparse regression

Regressor selection

minimize $\quad \|Ax - b\|_2^2$

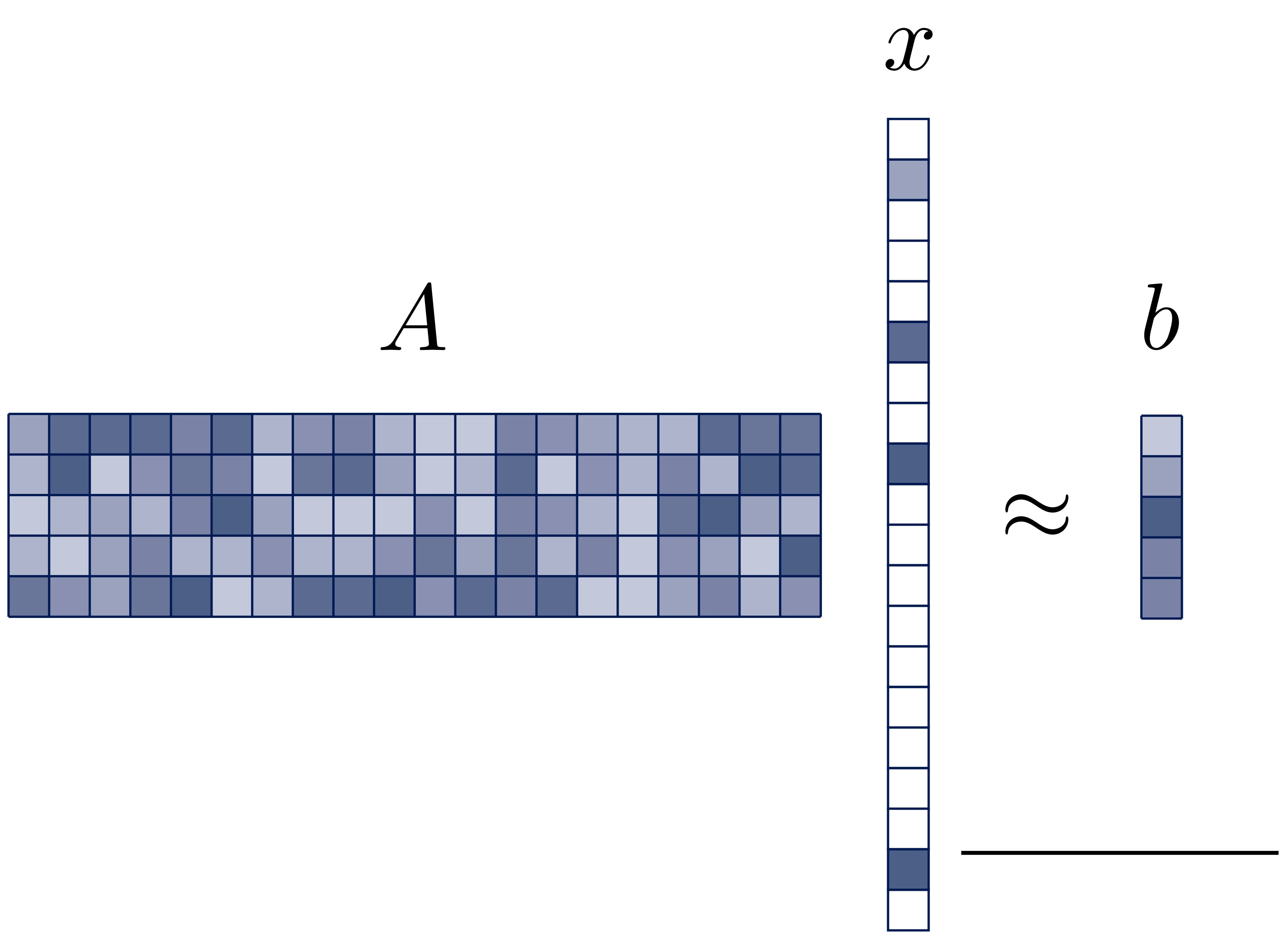
subject to $\quad \mathrm{card}(x) \leq k$

(very hard)



sparse x is
more robust
and interpretable

Sparse regression



Regressor selection

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_2^2 \\ \text{subject to} & \mathbf{card}(x) \leq k \end{array}$$

(very hard)

sparse x is **more robust and interpretable**

Add regularization to the objective

Regularized regression (ridge)

$$\text{minimize} \quad \|Ax - b\|_2^2 + \gamma \|x\|_2^2$$

Regularized regression (lasso)

$$\text{minimize} \quad \|Ax - b\|_2^2 + \gamma \|x\|_1$$

Lasso vs ridge regression

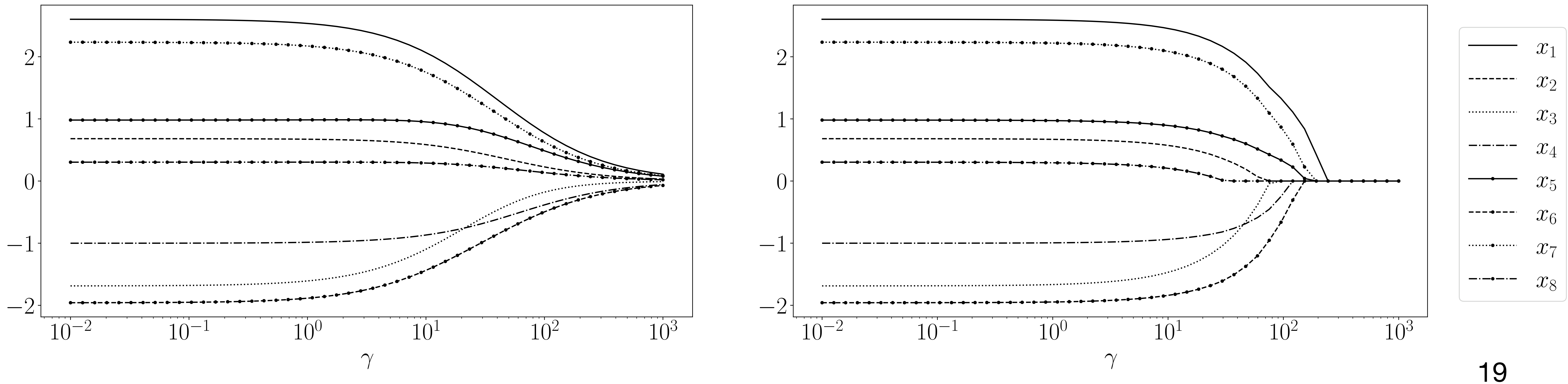
Regularized regression (ridge)

minimize $\quad \|Ax - b\|_2^2 + \gamma \|x\|_2^2$

Regularized regression (lasso)

minimize $\quad \|Ax - b\|_2^2 + \gamma \|x\|_1$

Regularization paths



Portfolio optimization

We have a total of n assets

x_i is fraction of money invested in asset i
p_i is the relative price change of asset i

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p random variable: mean μ, covariance Σ

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Portfolio optimization

$$\begin{array}{ll} \text{maximize} & \mu^T x - \gamma x^T \Sigma x \\ \text{subject to} & \mathbf{1}^T x = 1 \\ & x \geq 0 \end{array}$$

Portfolio optimization

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Returns
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Portfolio optimization

Expected return

$$\begin{array}{ll} \text{maximize} & \mu^T x - \gamma x^T \Sigma x \\ \text{subject to} & \mathbf{1}^T x = 1 \\ & x \geq 0 \end{array}$$

Portfolio optimization

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p random variable: mean μ, covariance Σ

Portfolio optimization

$$\begin{array}{ll} \text{maximize} & \underbrace{\mu^T x}_{\text{Expected return}} - \gamma \underbrace{x^T \Sigma x}_{\text{Risk}} \\ \text{subject to} & \mathbf{1}^T x = 1 \\ & x \geq 0 \end{array}$$

Portfolio optimization

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& x \geq 0
\end{array}
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Expected return

Risk

Risk-aversion parameter

Convex analysis review

Extended real-value functions

$f(x)$ on $\mathbf{dom} f$

Extended-value extension

$$\tilde{f}(x) = \begin{cases} f(x) & x \in \mathbf{dom} f \\ \infty & x \notin \mathbf{dom} f \end{cases}$$

Always possible to evaluate functions

$$\mathbf{dom} \tilde{f} = \{x \mid \tilde{f}(x) < \infty\}$$

Indicator functions

Indicator function

$$\mathcal{I}_C(x) = \begin{cases} 0 & x \in C \\ \infty & x \notin C \end{cases}$$

Constrained form

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in C \end{array}$$

$\longleftrightarrow$

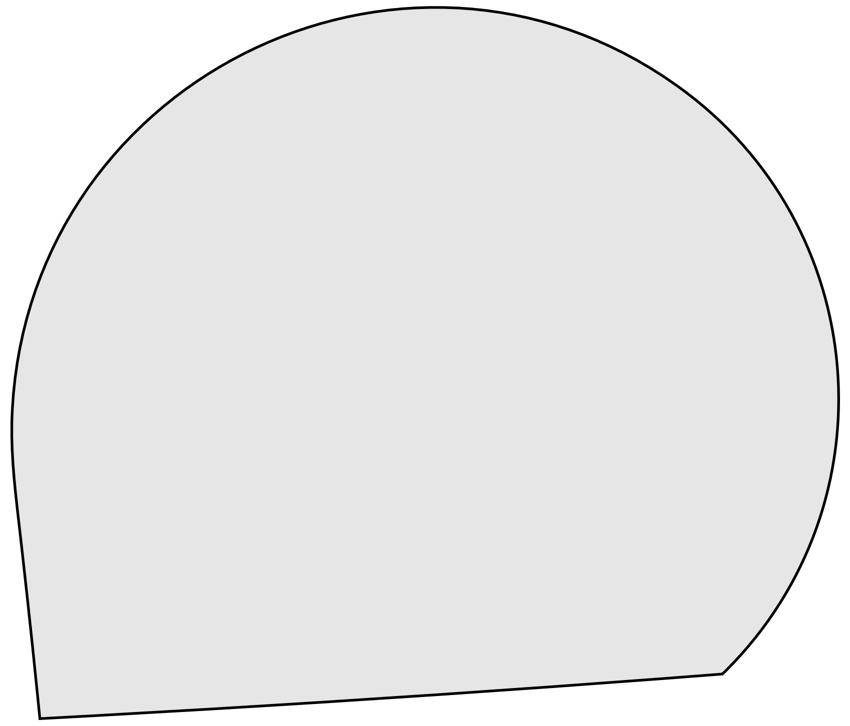
Unconstrained form

$$\text{minimize} \quad f(x) + \mathcal{I}_C(x)$$

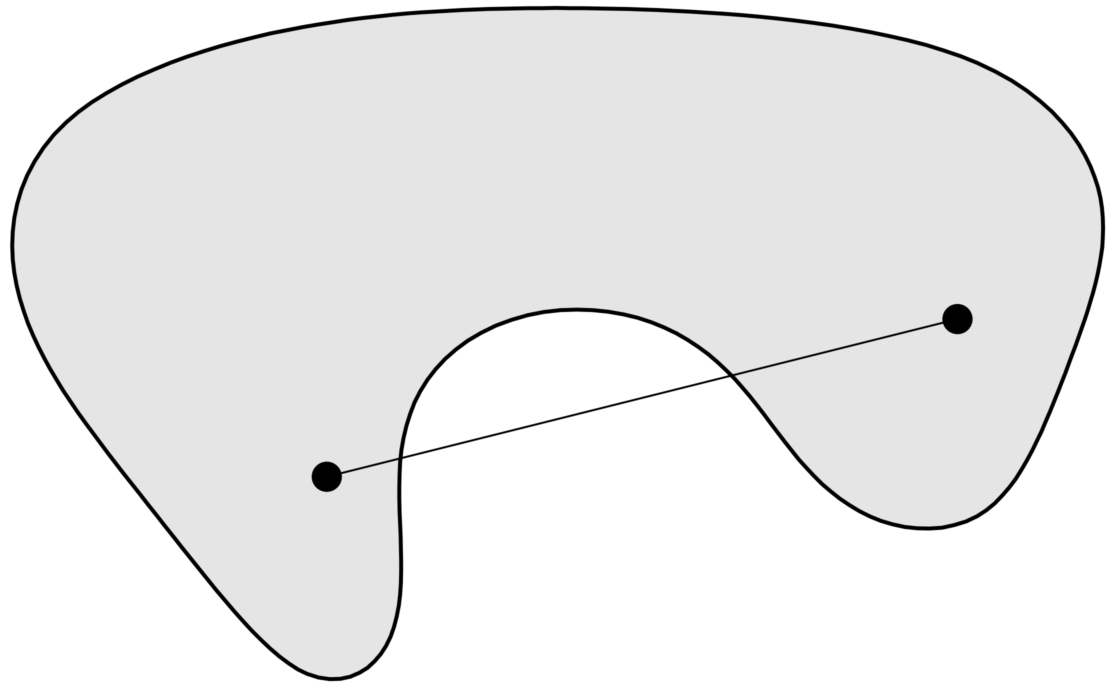
Convex set

Definition

For any $x, y \in C$ and any $\alpha \in [0,1]$ $\qquad \alpha x + (1-\alpha) y \in C$



Convex



Not convex

Examples

- $\mathbf{R}^n$
- Hyperplanes
- Hyperspheres
- Polyhedra

Examples

- Cardinality constraint $\mathrm{card}(x) \leq k$
- $\mathbf{Z}^n$
- Any disjoint set

Convex combinations

Convex combination

$\alpha_1 x_1 + \cdots + \alpha_k x_k$ for any $x_1, \ldots, x_k$ and $\alpha_1, \ldots, \alpha_k$ such that $\alpha_i \geq 0,\ \sum_{i=1}^{k} \alpha_i = 1$

Convex combinations

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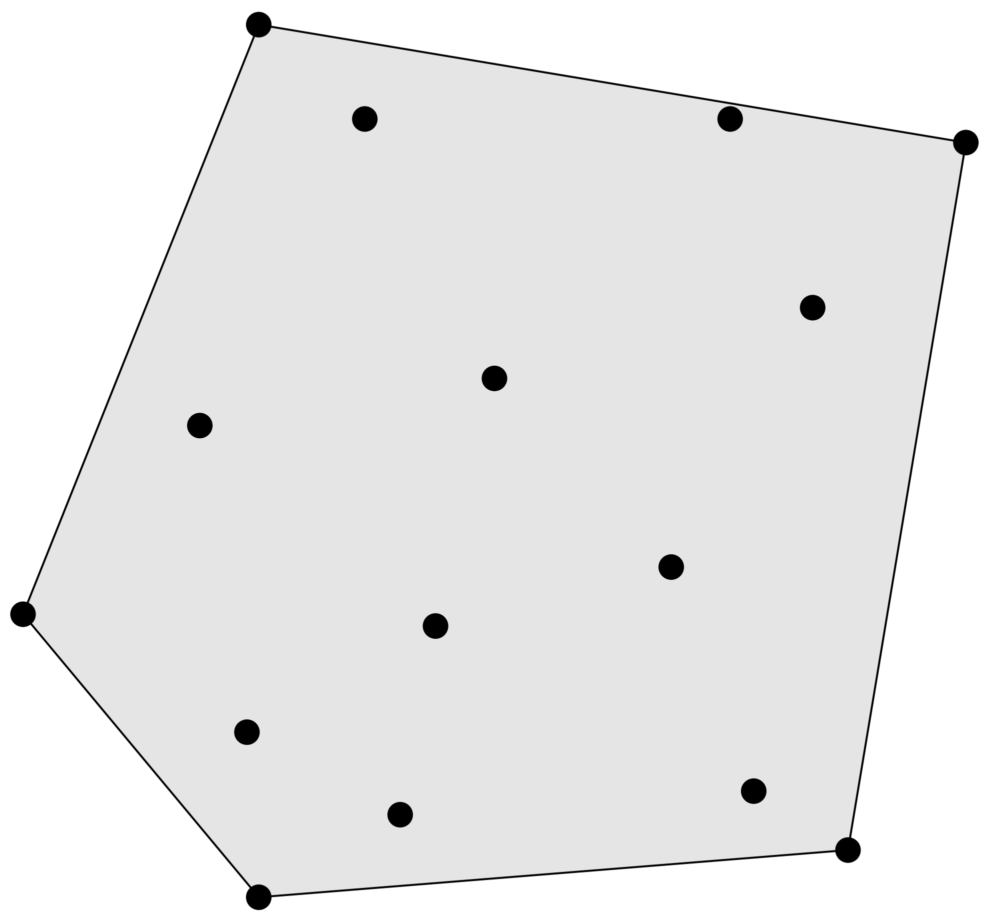
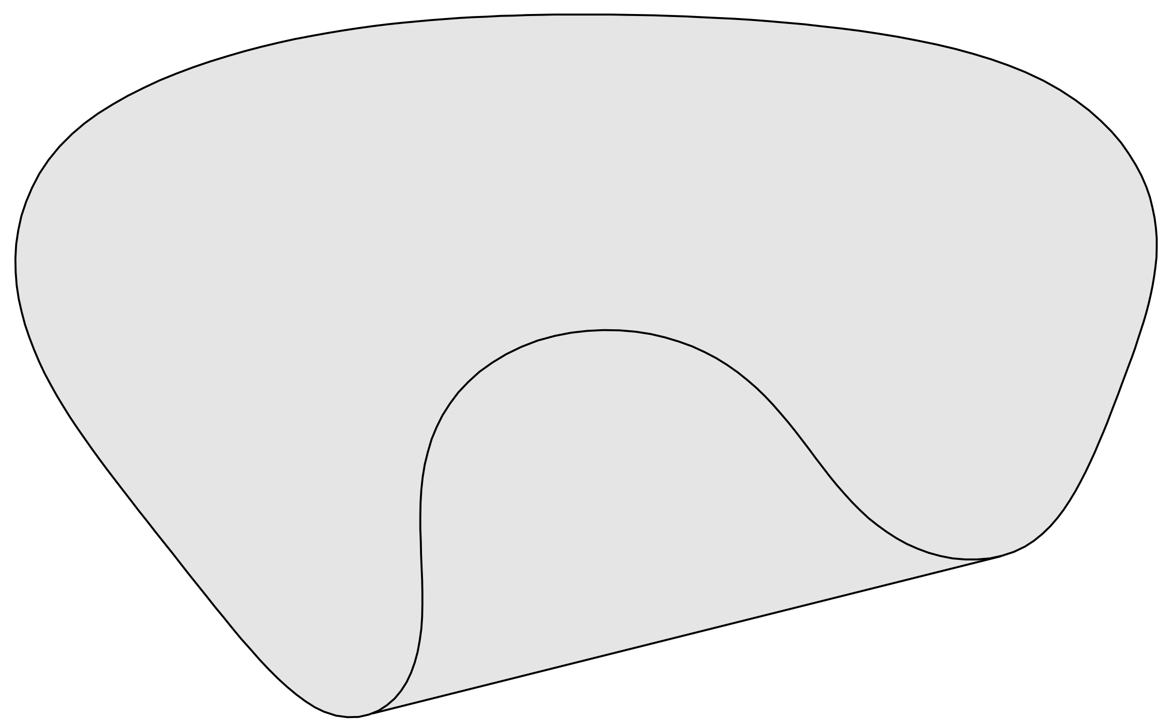
Convex hull

$$\mathbf{conv}\, C = \left\{ \sum_{i=1}^{k} \alpha_i x_i \mid x_i \in C, \quad \alpha_i \geq 0, \quad i = 1, \ldots, k, \quad \mathbf{1}^T \alpha = 1 \right\}$$





Cones

Cone

$x \in C \quad \implies \quad tx \in C \quad$ for all $\quad t \geq 0$

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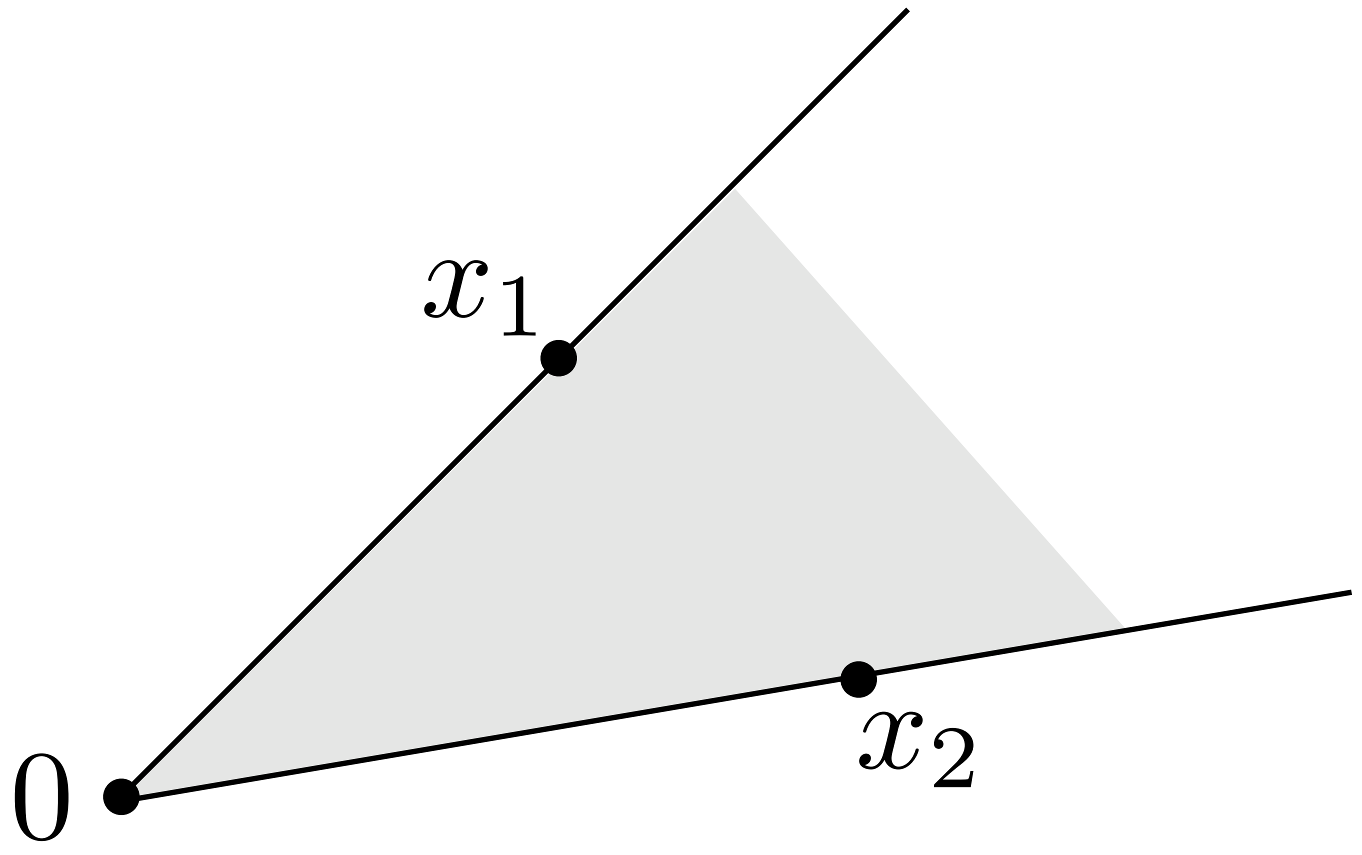
Convex cone

$$x_1, x_2 \in C \quad \implies \quad t_1 x_1 + t_2 x_2 \in C \text{ for all } t_1, t_2 \geq 0$$



Conic combinations

Conic combination

$\alpha_1 x_1 + \cdots + \alpha_k x_k$ for any $x_1, \ldots, x_k$ and $\alpha_1, \ldots, \alpha_k$ such that $\alpha_i \geq 0$

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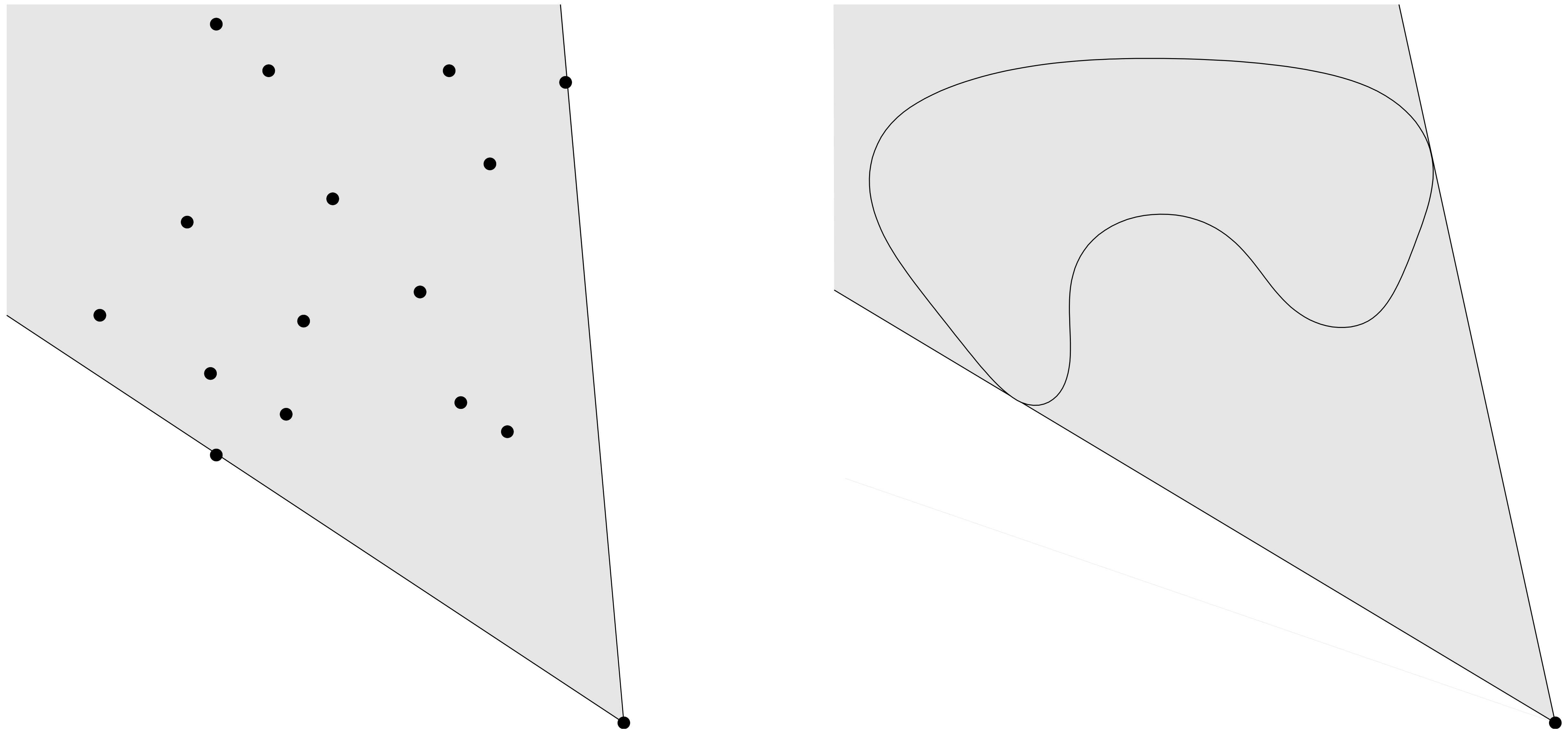
Conic hull

$$\left\{ \sum_{i=1}^{k} \alpha_i x_i \mid x_i \in C, \quad \alpha_i \geq 0, \quad i = 1, \ldots, k \right\}$$



Cones

Examples

Nonnegative orthant

$$\mathbf{R}^n_+ = \{x \in \mathbf{R}^n \mid x \geq 0\}$$

Norm-cone

$$\{(x, t) \mid \|x\| \leq t\}$$ (if 2-norm, second-order cone)

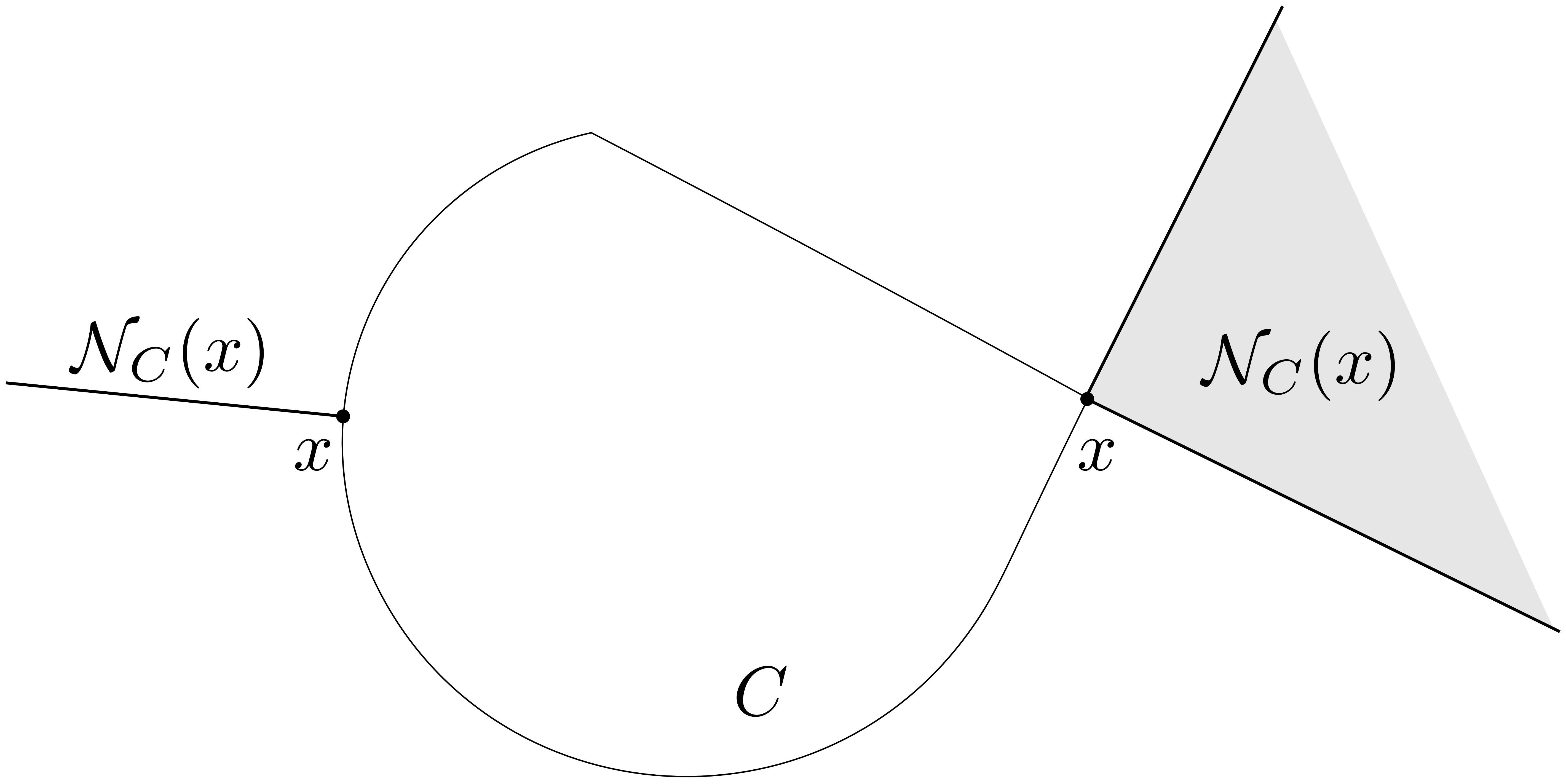
Positive semidefinite cone

$$\mathbf{S}^n_+ = \{X \in \mathbf{S}^n \mid z^T X z \geq 0, \quad \text{for all} \quad z \in \mathbf{R}^n\}$$

Normal cone

For any set C and point $x \in C$, we define

$$\mathcal{N}_C(x) = \left\{ g \mid g^T(y - x) \leq 0, \quad \text{for all } y \in C \right\}$$



$\mathcal{N}_C(x)$ is **always convex**

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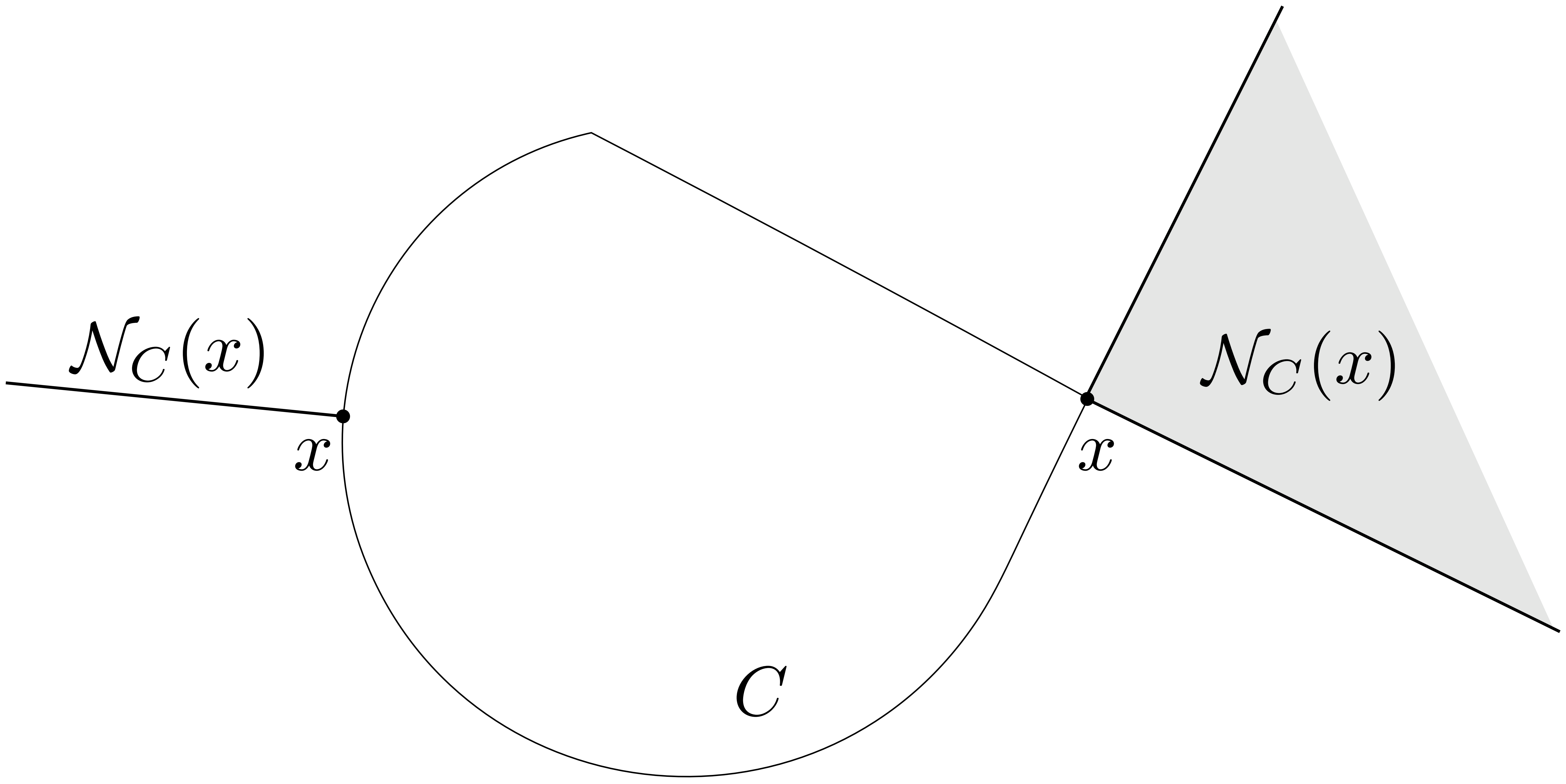

$\mathcal{N}_C(x)$ is **always convex**

What if $x \in \mathbf{int} S$?

Gradient

Derivative

If $f(x) : \mathbf{R}^n \to \mathbf{R}^m$ continuously differentiable, we define

$$Df(x)_{ij} = \frac{\partial f_i(x)}{\partial x_j}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n$$

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Gradient

If $f : \mathbf{R}^n \rightarrow \mathbf{R}$, we define

$$\nabla f(x) = Df(x)^T$$

Example

$$f(x) = (1/2)x^T P x + q^T x$$

$$\nabla f(x) = Px + q$$

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First-order approximation

$$f(y) \approx f(x) + \nabla f(x)^T (y - x)$$

(affine function of y)

Hessian

Hessian matrix (second derivative)

If $f(x) : \mathbf{R}^n \to \mathbf{R}$ second-order differentiable, we define

$$\nabla^2 f(x)_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}, \quad i, j = 1, \dots, n$$

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Second-order approximation

$$f(y) \approx f(x) + \nabla f(x)^T (y - x) + (1/2)(y - x)^T \nabla^2 f(x)(y - x)$$

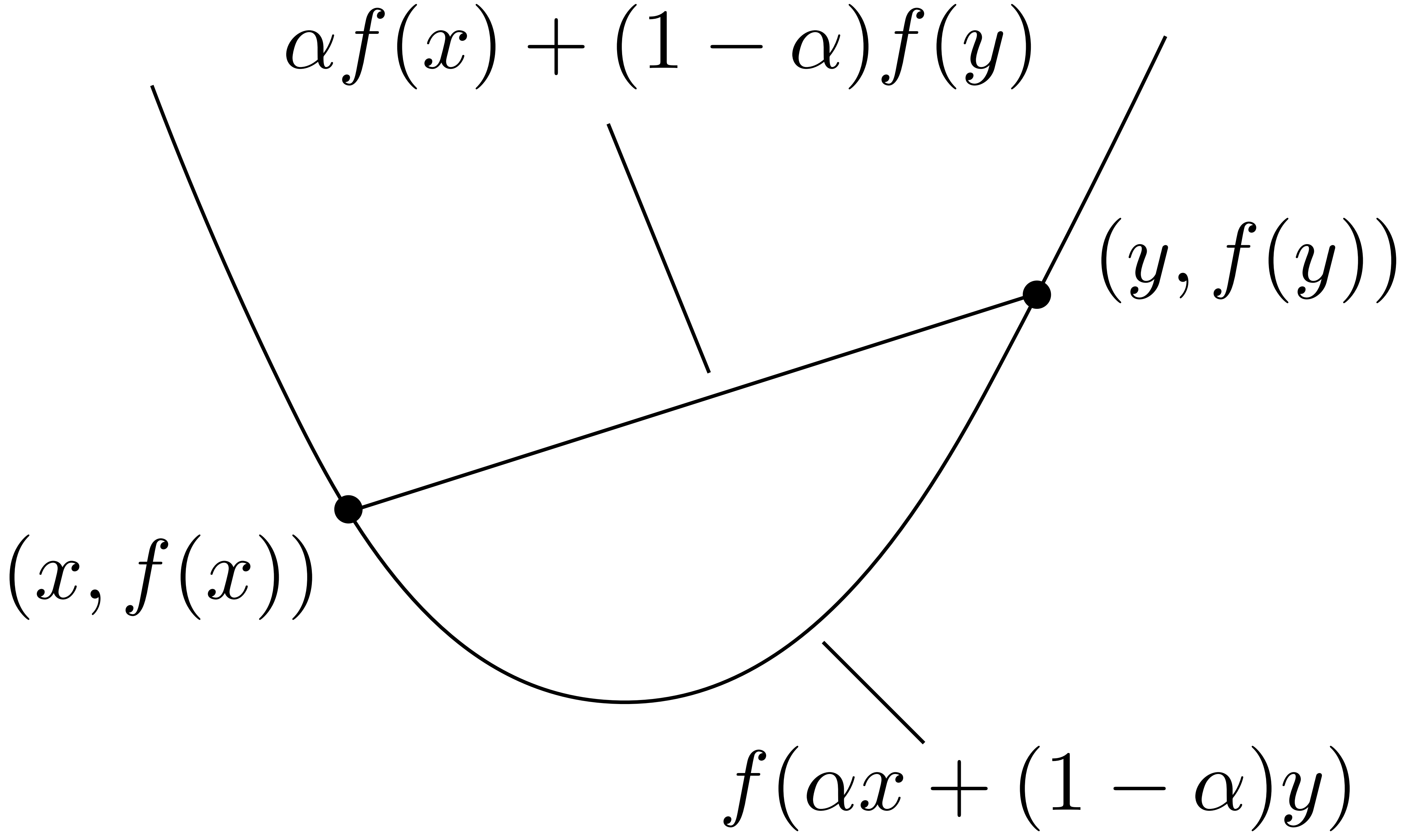
(quadratic function of y)

Convex optimization

Convex functions

Convex function

For every $x, y \in \mathbf{R}^n,\ \alpha \in [0,1]$ $$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$$



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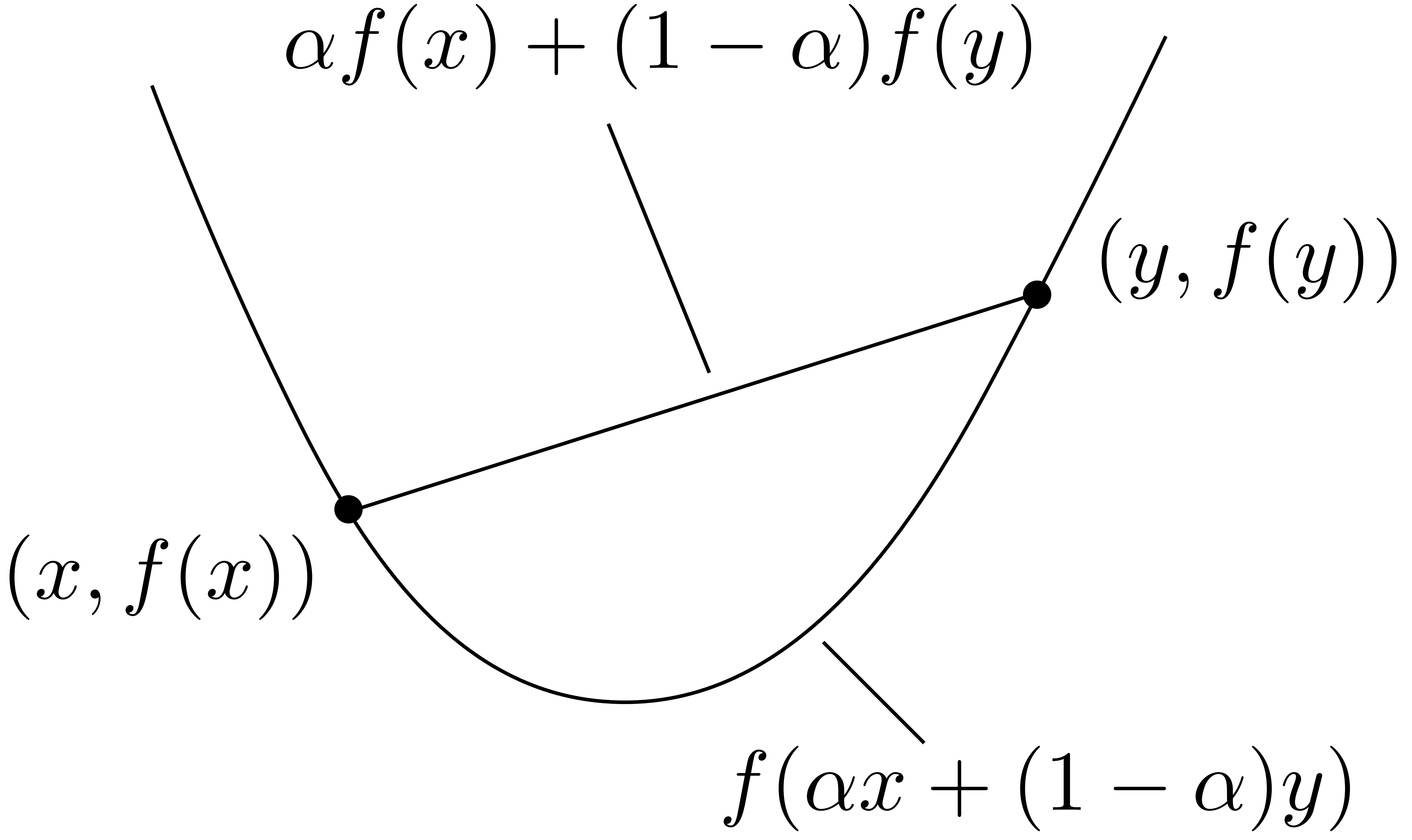


Concave function

f is concave if and only if $-f$ is convex

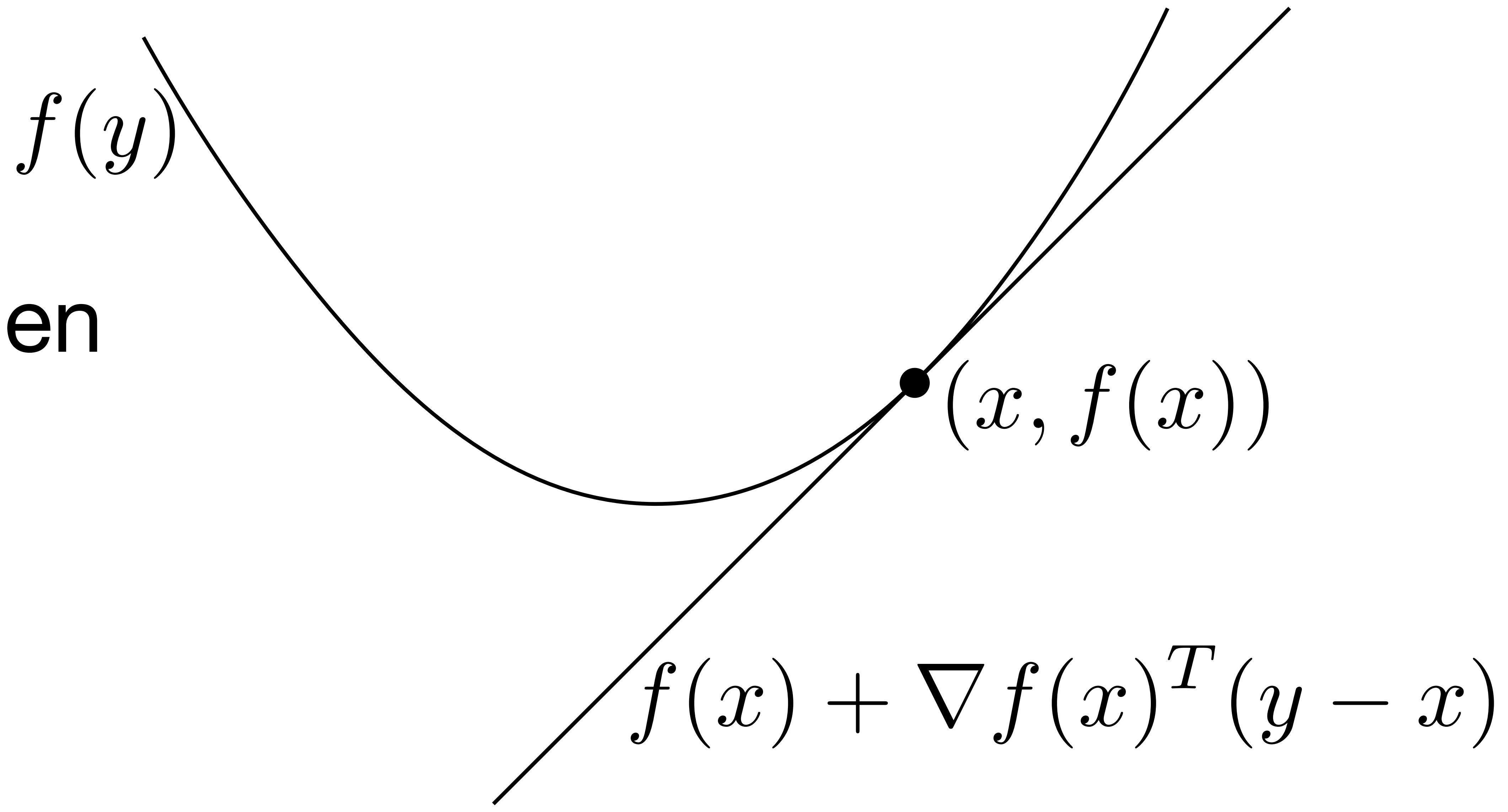
Convex conditions

First-order

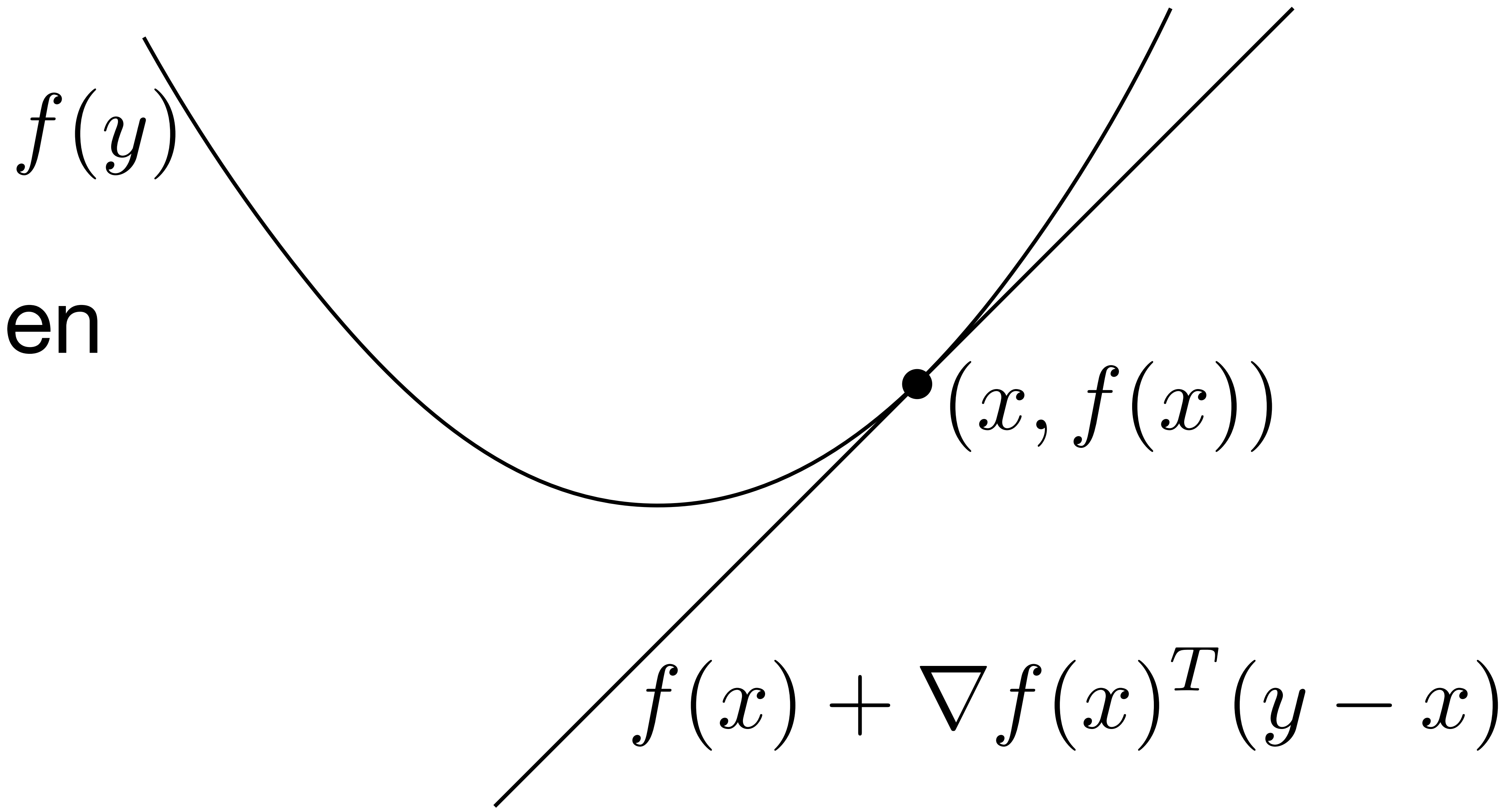
Let f be a continuous differentiable function, then it is convex if and only if $\mathbf{dom} f$ is convex and

$$f(y) \geq f(x) + \nabla f(x)^T (y - x)$$

for all $x, y \in \mathbf{dom} f$



Convex conditions



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for all $x, y \in \mathbf{dom} f$

Second-order

If f is twice differentiable, then f is convex if and only if $\mathbf{dom} f$ is convex and

$$\nabla^2 f(x) \succeq 0$$

for all $x \in \mathbf{dom} f$

Verifying convexity

Basic definition (inequality)

First and second order conditions (gradient, hessian)

Verifying convexity

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First and second order conditions (gradient, hessian) $\longrightarrow$ **Hard!**

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Convex calculus (directly construct convex functions)

- Library of basic functions that are convex/concave
- Calculus rules or transformations that preserve convexity

Verifying convexity

Basic definition (inequality)

First and second order conditions (gradient, hessian) $\longrightarrow$ **Hard!**

Convex calculus (directly construct convex functions)

- Library of basic functions that are convex/concave
- Calculus rules or transformations that preserve convexity

$\longrightarrow$ **Easy!**

Disciplined Convex Programming

Convexity by construction

General composition rule

$h(f_1(x), f_2(x), \ldots, f_k(x))$ is convex when h is convex and for each i

- h is nondecreasing in argument i and f_i is convex, or
- h is nonincreasing in argument i and f_i is concave, or
- f_i is affine

Disciplined Convex Programming

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Disciplined Convex Programming

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Only sufficient condition

Check your functions at `https://dcp.stanford.edu/`

More details and examples in ORF523

Convex optimization problems

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_i(x) \leq 0, \quad i = 1, \dots, m \end{array}$$

$f : \mathbf{R}^n \to \mathbf{R}$ Convex objective function

$g_i : \mathbf{R}^n \to \mathbf{R}$ Convex constraints functions

Convex feasible set

$$C = \{x \mid g_i(x) \leq 0, \quad i = 1, \dots, m\}$$

Modelling software for convex optimization

Modelling tools simplify the formulation of convex optimization problems

- **Construct problems** using library of basic functions
- **Verify convexity** by general composition rule
- Express the problem in input format required by a specific solver

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Examples

- CVX, YALMIP (Matlab)
- CVXPY (Python)
- Convex.jl (Julia)

Solving convex optimization problems

CVXPY

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_2 \\ \text{subject to} & 0 \leq x \leq 1 \end{array}$$

```
x = cp.Variable(n)
objective = cp.Minimize(cp.norm(A*x - b))
constraints = [0 <= x, x <= 1]
problem = cp.Problem(objective, constraints)

# The optimal objective value is returned by `problem.solve()`.
result = problem.solve()

# The optimal value for x is stored in `x.value`.
print(x.value)
```

Local vs global minima (optimizers)

minimize $\quad f(x)$

subject to $\quad x \in C$

Local optimizer x

$f(y) \geq f(x), \quad \forall y$

such that $\|x - y\|_2 \leq R$

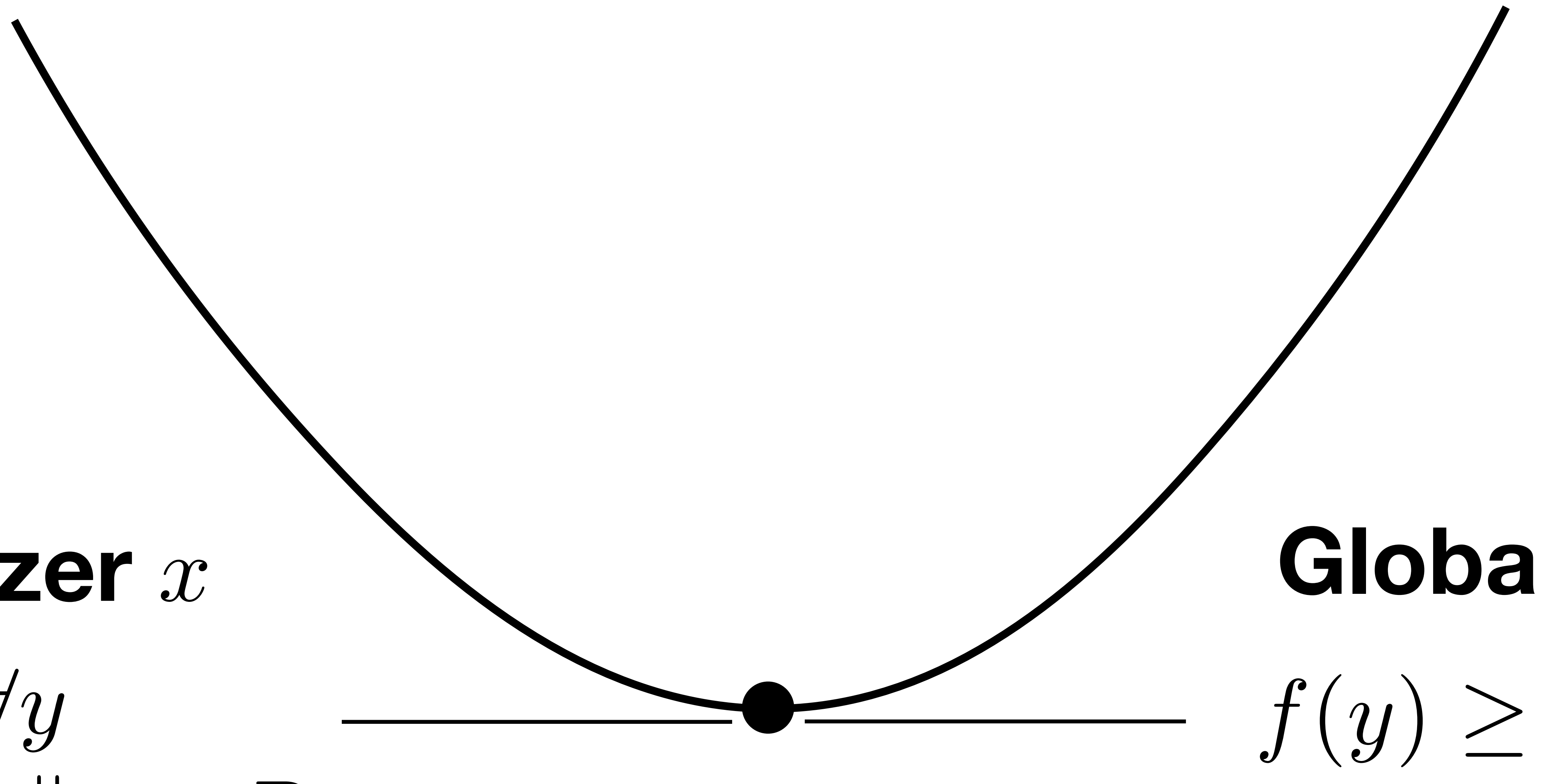
Global optimizer x

$f(y) \geq f(x), \quad \forall y \in C$

Optimality and convexity

Theorem

For a convex optimization problem, any **local minimum** is a **global minimum**



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Optimality and convexity

Proof (contradiction)

Suppose that f is convex and x is a local (not global) minimum for f, i.e.,

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Therefore, there exists a feasible z such that $\|z - x\| > R$ and $f(z) < f(x)$.

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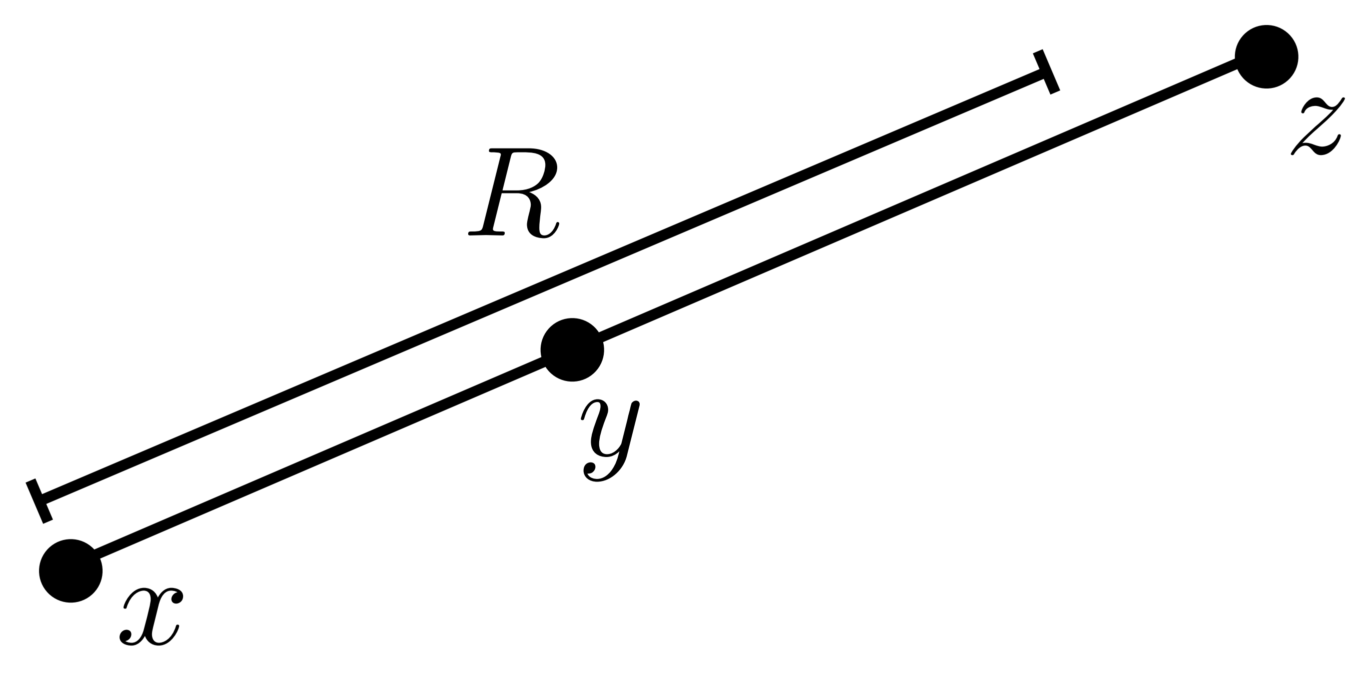
Consider $y = (1-\alpha)x + \alpha z$ with $\alpha = \dfrac{R}{2\|z - x\|_2}$.

Then, $\|y - x\|_2 = \alpha\|z - x\|_2 = R/2 < R$, and
by convexity of the feasible set, y is feasible.



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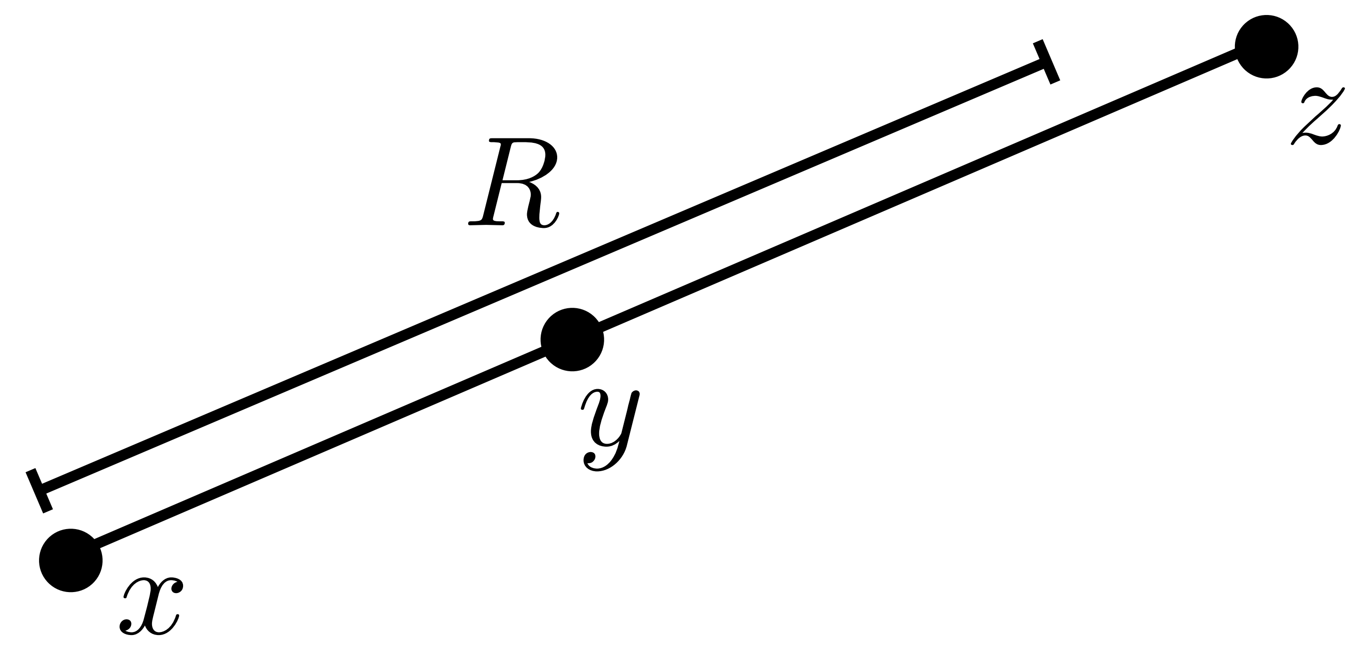

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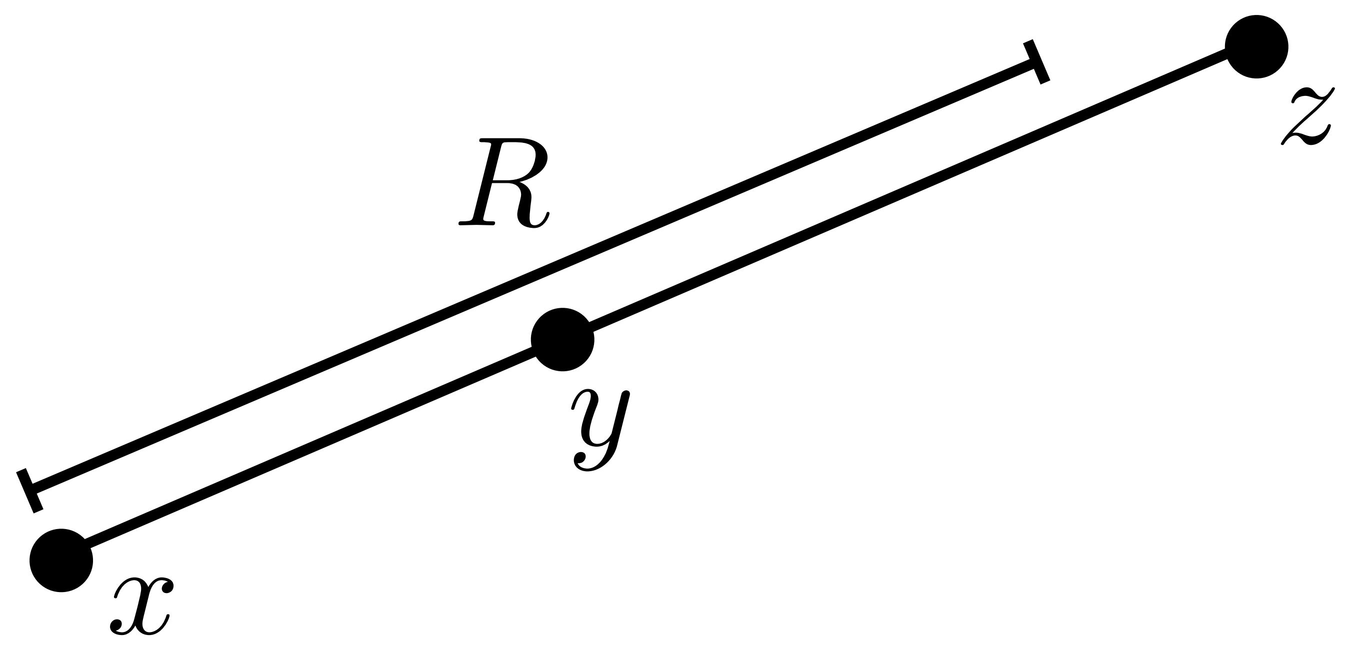

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Therefore, x is globally optimal. $\blacksquare$

"...in fact, the great watershed in optimization isn't between linearity and nonlinearity, but convexity and nonconvexity."

R. Tyrrell Rockafellar, in SIAM Review, 1993

Nonlinear optimization

Topics of this part of the course

Conditions to **characterize** minima

Algorithms to **find** (local) minima

(if applied to **convex problems,** they find **global minima**)

Introduction to nonlinear optimization

Today, we learned to:

- **Define** nonlinear optimization problems
- **Understand** convex analysis fundamentals (sets, cones, functions, and gradients)
- **Verify** convexity and **construct** convex optimization problems
- **Define** convex optimization problems in CVXPY
- **Understand** the importance of *convexity vs nonconvexity* in optimization

Next lecture

- Optimality conditions in nonlinear optimization