ORF522 – Linear and Nonlinear Optimization

10. Interior-point methods for linear optimization

Bartolomeo Stellato — Fall 2022

Midterm

- **Date/Time:** Thursday October 13, 11:00am — 12:20pm (80 min)
- **Where:** in class
- **Material allowed:** single sheet of paper, double-sided, hand-written or typed.
- **Topics:** linear optimization

Ed Forum

- Since introducing a new constraint to a primal problem is equivalent to introducing a new variable to the dual, is there a difference between choosing to solve one problem rather than another?
- How efficient is differentiable optimization? In neural network, people use ReLU because it is simple in evaluating both the forward pass values and backward differentiations. However, it seems like we need to solve a LP in every node in forward pass and to solve a linear system in backward pass.
- In lecture 9, we have seen that if we perturb the constraints a bit by some vector u, the optimal value behaves as a piecewise linear function in u. In case c^Tx and Ax are replaced by any suitable convex functions in x, is such a sensitivity analysis still possible?

MULTIPARAMETRIC PROGRAMMING

Recap

Adding new variables

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Solution $x^\star, y^\star$

Adding new variables

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & c^T x + c_{n+1} x_{n+1} \\ \text{subject to} & Ax + A_{n+1} x_{n+1} = b \\ & x, x_{n+1} \geq 0 \end{array}$$

Solution $x^\star, y^\star$

Adding new variables

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & c^T x + c_{n+1} x_{n+1} \\ \text{subject to} & Ax + A_{n+1} x_{n+1} = b \\ & x, x_{n+1} \geq 0 \end{array}$$

Solution $x^\star, y^\star$

Solution $(x^\star, 0), y^\star$ **optimal** for the new problem?

Adding new variables

Optimality conditions

$$\begin{array}{ll} \text{minimize} & c^T x + c_{n+1} x_{n+1} \\ \text{subject to} & Ax + A_{n+1} x_{n+1} = b \\ & x, x_{n+1} \geq 0 \end{array} \longrightarrow \text{Solution } (x^\star, 0) \text{ is still } \textbf{primal feasible}$$

Adding new variables

Optimality conditions

$$
\begin{array}{ll}
\text{minimize} & c^T x + c_{n+1} x_{n+1} \\
\text{subject to} & Ax + A_{n+1} x_{n+1} = b \\
& x, x_{n+1} \geq 0
\end{array}
\longrightarrow
$$

Solution $(x^\star, 0)$ is still **primal feasible**

Is $y^\star$ still **dual feasible**?

$$A_{n+1}^T y^\star + c_{n+1} \geq 0$$

Adding new variables

Optimality conditions

$$\begin{array}{ll} \text{minimize} & c^T x + c_{n+1} x_{n+1} \\ \text{subject to} & Ax + A_{n+1} x_{n+1} = b \\ & x, x_{n+1} \geq 0 \end{array} \longrightarrow$$

Solution $(x^\star, 0)$ is still **primal feasible**

Is $y^\star$ still **dual feasible**?

$$A_{n+1}^T y^\star + c_{n+1} \geq 0$$

Yes

$(x^\star, 0)$ still **optimal** for new problem

Otherwise

Primal simplex

Adding new constraints

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax = b \\
& x \geq 0
\end{array}
$$

Solution $x^\star, y^\star$

Adding new constraints

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & a_{m+1}^T x = b_{m+1} \\ & x \geq 0 \end{array}$$

Solution $x^\star, y^\star$

Adding new constraints

minimize $c^T x$
subject to $Ax = b$
$x \geq 0$

Solution $x^\star, y^\star$

$\longrightarrow$

minimize $c^T x$
subject to $Ax = b$
$a_{m+1}^T x = b_{m+1}$
$x \geq 0$

Dual

maximize $-b^T y - b_{m+1} y_{m+1}$
subject to $A^T y + a_{m+1} y_{m+1} + c \geq 0$

Adding new constraints

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & a_{m+1}^T x = b_{m+1} \\ & x \geq 0 \end{array}$$

Solution $x^\star, y^\star$

Dual

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + a_{m+1} y_{m+1} + c \geq 0 \end{array}$$

Solution $x^\star, (y^\star, 0)$ **optimal** for the new problem?

Adding new constraints

Optimality conditions

maximize $\quad -b^T y$

subject to $\quad A^T y + a_{m+1} y_{m+1} + c \geq 0 \quad \longrightarrow \quad$ Solution $(y^\star, 0)$ is still **dual feasible**

Adding new constraints
Optimality conditions

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Is $x^\star$ still **primal feasible**?

$$\begin{aligned} &Ax = b \\ &a_{m+1}^T x = b_{m+1} \\ &x \geq 0 \end{aligned}$$

Adding new constraints

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Is $x^\star$ still **primal feasible**?

$$
\begin{aligned}
&Ax = b \\
&a_{m+1}^T x = b_{m+1} \\
&x \geq 0
\end{aligned}
$$

Yes

$x^\star$ still **optimal** for new problem

Otherwise

Dual simplex

Today's lecture

[Chapter 14, NO][Chapters 17/18, LP]

- History
- Newton's method
- Central path
- Primal-dual path following method
- Logarithmic barrier functions
- Convergence

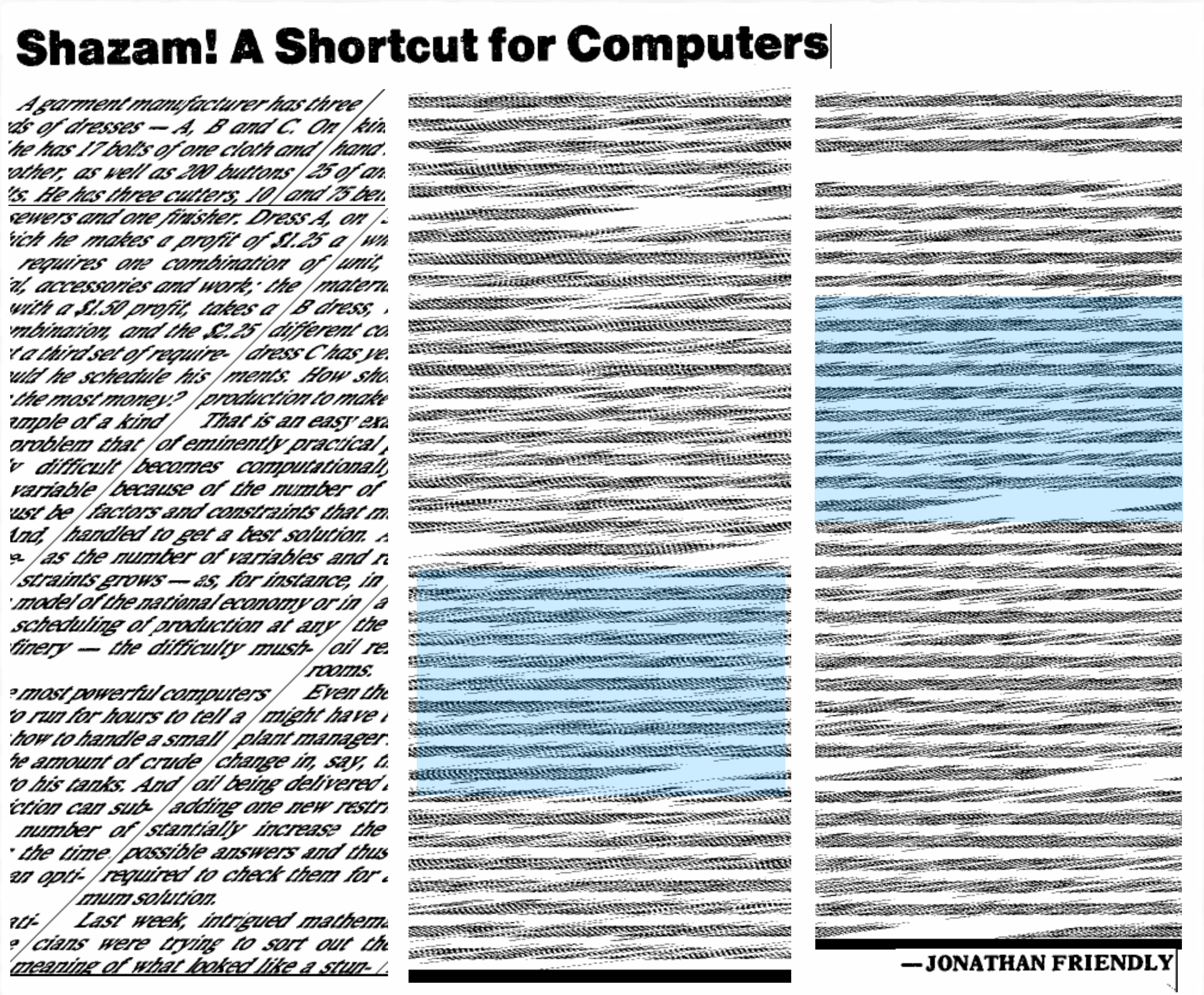
History

Ellipsoid method

Khachian (1979)

Answer to major question

Is worst-case LP complexity polynomial? Yes!



Shazam! A Shortcut for Computers

—JONATHAN FRIENDLY

The New York Times
Published: November 11, 1979

Ellipsoid method

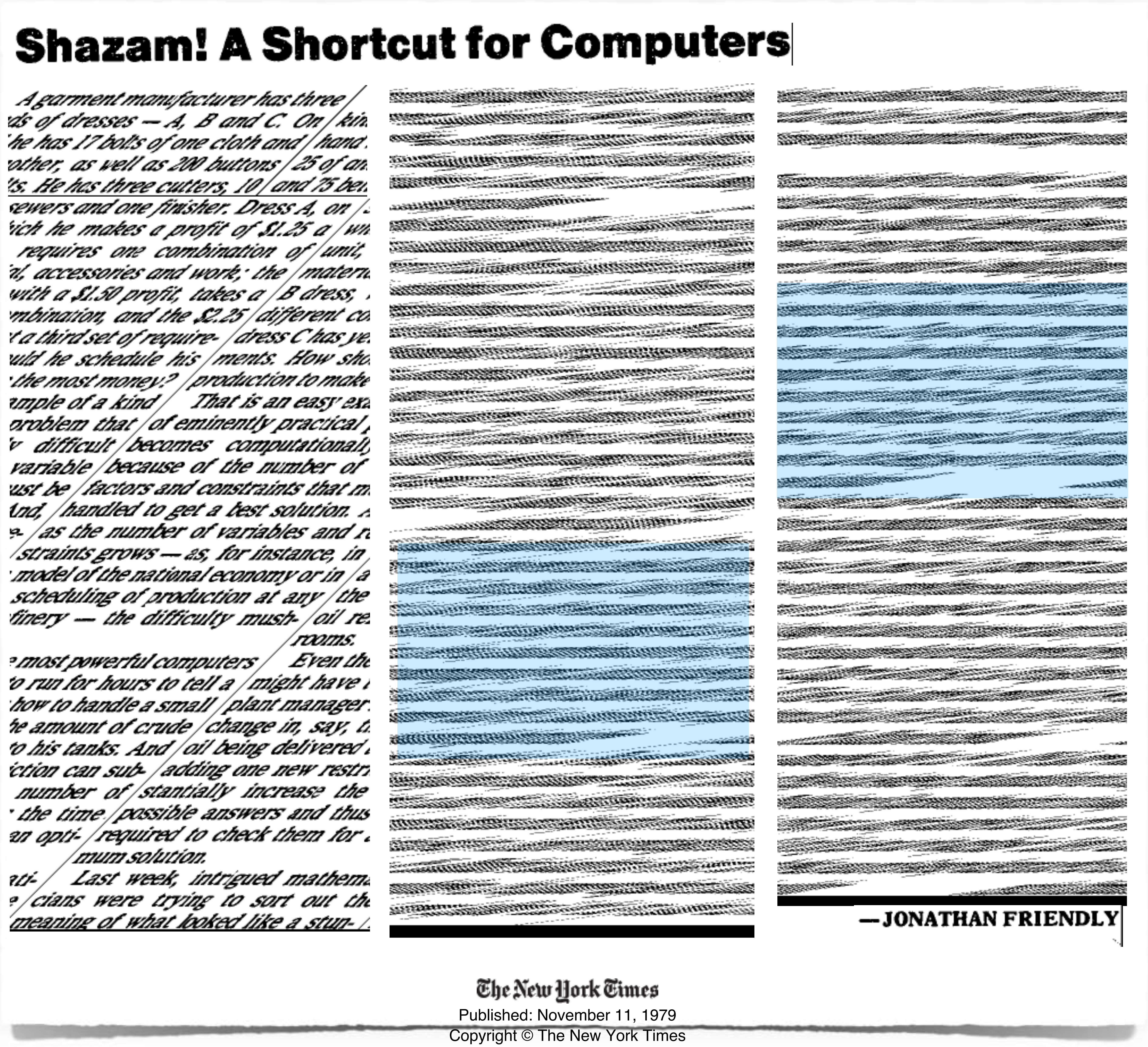
Khachian (1979)

Answer to major question

Is worst-case LP complexity polynomial? Yes!

Drawbacks

Very inefficient. Much slower than simplex!



Ellipsoid method

Khachian (1979)

Answer to major question

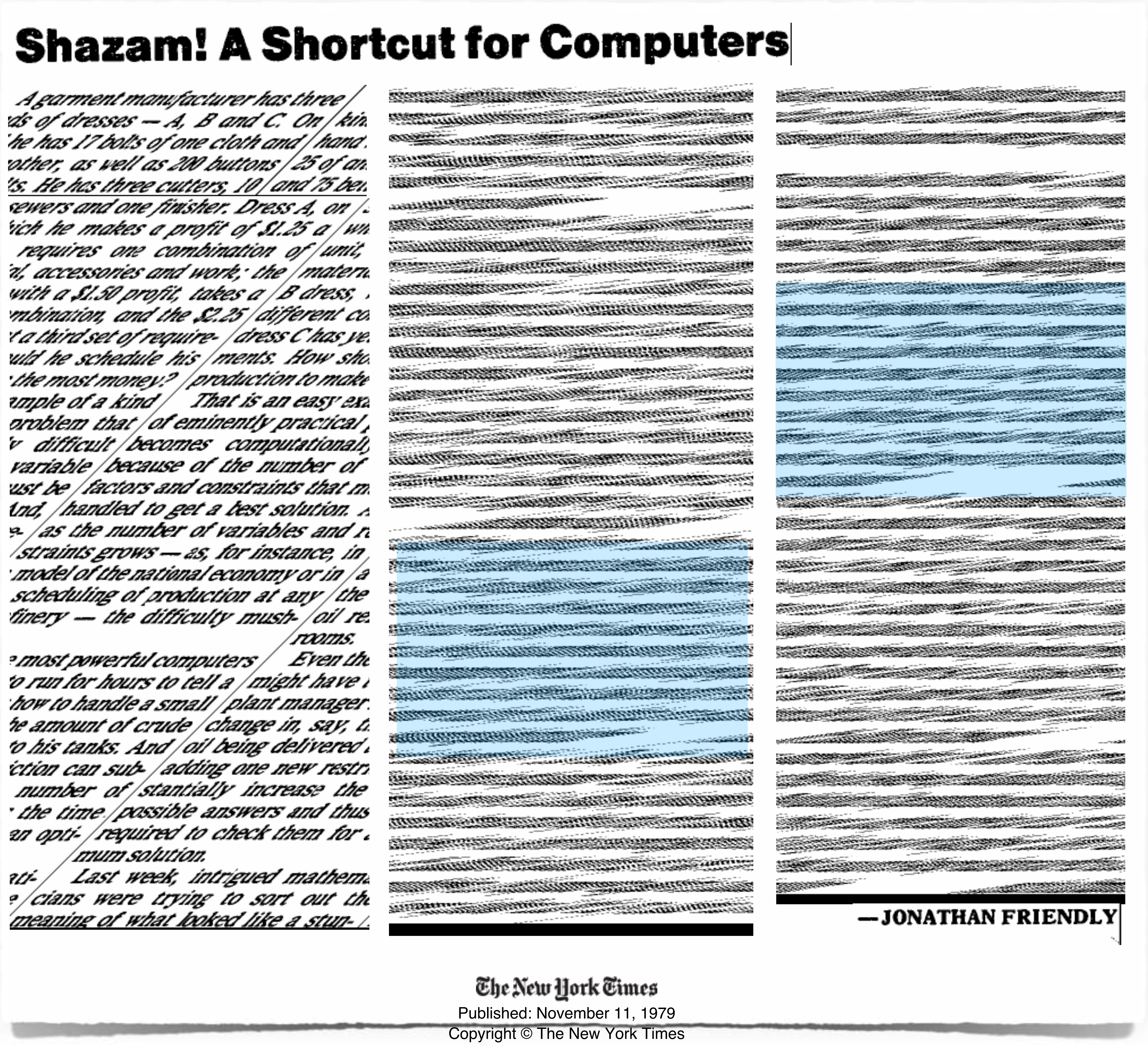
Is worst-case LP complexity polynomial? Yes!

Drawbacks

Very inefficient. Much slower than simplex!

Benefits

Motivated new research directions



Shazam! A Shortcut for Computers

A garment manufacturer has three
ds of dresses — A, B and C. On / kin
he has 17 bolts of one cloth and / hand
other, as well as 200 buttons / 25 of an
ts. He has three cutters, 10 / and 75 be
sewers and one finisher. Dress A, on
ich he makes a profit of $1.25 a / wh
requires one combination of / unit,
d, accessories and work; the / materi
with a $1.50 profit, takes a / B dress,
mbination, and the $2.25 / different co
t a third set of require- / dress C has ye
uld he schedule his / ments. How sho
the most money? / production to make
mple of a kind / That is an easy exa
problem that / of eminently practical
v difficult / becomes computationall
variable / because of the number of
ust be / factors and constraints that m
And, / handled to get a best solution.
e- / as the number of variables and r
straints grows — as, for instance, in
model of the national economy or in / a
scheduling of production at any / the
finery — the difficulty mush- / oil re
rooms.
most powerful computers / Even the
o run for hours to tell a / might have
how to handle a small / plant manager
he amount of crude / change in, say, t
o his tanks. And / oil being delivered
ction can sub- / adding one new restr
number of / stantially increase the
the time / possible answers and thus
an opti- / required to check them for
mum solution.
ati- / Last week, intrigued mathema
e / cians were trying to sort out the
meaning of what looked like a stun-

—JONATHAN FRIENDLY

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Interior-point methods

1950s-1960s: nonlinear convex optimization

- Sequential unconstrained optimization (Fiacco & McCormick), Logarithmic barrier method (Frish), affine scaling method (Dikin), etc.
- No worst-case complexity theory but often good practical performance

Interior-point methods

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The New York Times

NEW YORK, MONDAY, NOVEMBER 19, 1984

Breakthrough in Problem Solving

By JAMES GLEICK

A 28-year-old mathematician at A.T.&T. Bell Laboratories has made a startling theoretical breakthrough in the solving of systems of equations that often grow too vast and complex for the most powerful computers.

The discovery, which is to be formally published next month, is already circulating rapidly through the mathematical world. It has also set off a deluge of inquiries from brokerage houses, oil companies and airlines, industries with millions of dollars at stake in problems known as linear programming.

These problems are fiendishly complicated systems, often with thousands of variables. They arise in a variety of commercial and government applications, ranging from allocating time on a communications satellite to routing millions of telephone calls over long distances. Linear programming is particularly useful whenever a limited, expensive resource must be spread most efficiently among competing users. And investment companies use the approach in creating portfolios with the best mix of stocks and bonds.

Faster Solutions Seen

The Bell Labs mathematician, Dr. Narendra Karmarkar, has devised a radically new procedure that may speed the routine handling of such problems by businesses and Government agencies and also make it possible to tackle problems that are now far out of reach.

"This is a path-breaking result," said Dr. Ronald L. Graham, director of mathematical sciences for Bell Labs in Murray Hill, N.J. "Science has its moments of great progress, and this may well be one of them."

Because problems in linear programming can have billions or more possible answers, even high-speed computers cannot check every one. So computers must use a special procedure, an algorithm, to examine as few answers as possible before finding the best one — typically the one that minimizes cost or maximizes efficiency.

A procedure devised in 1947, the simplex method, is now used for such prob-

Continued on Page A19, Column 1

1980s-1990s: interior point methods

- Karmarkar's algorithm (1984)
- Competitive with simplex, often faster for larger problems

Newton’s method

Newton’s method for nonlinear equations

Goal: solve

$$h(x) = 0$$

Derivative

$$Dh = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial h_m}{\partial x_1} & \cdots & \frac{\partial h_m}{\partial x_n} \end{bmatrix}$$

Newton's method for nonlinear equations

Derivative

$$Dh = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial h_m}{\partial x_1} & \cdots & \frac{\partial h_m}{\partial x_n} \end{bmatrix}$$

Goal: solve

$$h(x) = 0$$

First-order approximation

$$h(x) \approx h(\bar{x}) + Dh(\bar{x})(x - \bar{x})$$

Iteratively set to zero

$$h(x^k) + Dh(x^k)(\underbrace{x^{k+1} - x^k}_{\Delta x}) = 0$$

Newton’s method for nonlinear equations

Goal: solve

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Derivative

$$Dh = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial h_m}{\partial x_1} & \cdots & \frac{\partial h_m}{\partial x_n} \end{bmatrix}$$

First-order approximation

$$h(x) \approx h(\bar{x}) + Dh(\bar{x})(x - \bar{x})$$

Iteratively set to zero

$$h(x^k) + Dh(x^k)\underbrace{(x^{k+1} - x^k)}_{\Delta x} = 0$$

Iterations

- Solve $Dh(x^k)\Delta x = -h(x^k)$
- $x^{k+1} \leftarrow x^k + \Delta x$

Newton method

Convergence

Iterations

- Solve $Dh(x^k)\Delta x = -h(x^k)$
- $x^{k+1} \leftarrow x^k + \Delta x$

Remarks

- Iterations can be **expensive** (linear system solution)
- **Fast (quadratic) convergence** close to the solution $x^\star$

Optimality conditions

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax \leq b
\end{array}
$$

Optimality conditions

			Primal		**Dual**	
minimize	$c^T x$	$\longrightarrow$	minimize	$c^T x$	maximize	$-b^T y$
subject to	$Ax \leq b$		subject to	$Ax + s = b$	subject to	$A^T y + c = 0$
				$s \geq 0$		$y \geq 0$

Optimality conditions

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array} \longrightarrow$$

Primal

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax + s = b \\ & s \geq 0 \end{array}$$

Dual

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c = 0 \\ & y \geq 0 \end{array}$$

Optimality conditions

$$Ax + s - b = 0$$ PRIMAL FEAS

$$A^T y + c = 0$$ DUAL FEAS

$$s_i y_i = 0$$ COMPL SLACKNESS

$$s, y \geq 0$$

Main idea

$$SY\mathbf{1} = \begin{bmatrix} s_1 & & & \\ & s_2 & & \\ & & \ddots & \\ & & & s_m \end{bmatrix} \begin{bmatrix} y_1 & & & \\ & y_2 & & \\ & & \ddots & \\ & & & y_m \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} s_1 y_1 \\ s_2 y_2 \\ \vdots \\ s_m y_m \end{bmatrix}$$

Optimality conditions

$$h(x, s, y) = \begin{bmatrix} Ax + s - b \\ A^T y + c \\ SY\mathbf{1} \end{bmatrix} = 0 \qquad \begin{aligned} S &= \mathbf{diag}(s) \\ Y &= \mathbf{diag}(y) \end{aligned}$$

$$s, y \geq 0$$

- Apply variants of Newton's method to solve $h(x, s, y) = 0$
- Enforce $s, y > 0$ (strictly) at every iteration
- **Motivation** avoid getting stuck in "corners"

Newton's method for optimality conditions

Root-finding equation

$$h(x, s, y) = \begin{bmatrix} Ax + s - b \\ A^T y + c \\ SY\mathbf{1} \end{bmatrix} = 0$$

Linear system

$$\underbrace{\begin{bmatrix} 0 & A & I \\ A^T & 0 & 0 \\ S & 0 & Y \end{bmatrix}}_{Dh} \begin{bmatrix} \Delta y \\ \Delta x \\ \Delta s \end{bmatrix} = \underbrace{\begin{bmatrix} -r_p \\ -r_d \\ -SY\mathbf{1} \end{bmatrix}}_{-h}$$

Residuals

$$r_p = Ax + s - b$$

$$r_d = A^T y + c$$

Newton's method for optimality conditions

Root-finding equation

$$h(x, s, y) = \begin{bmatrix} Ax + s - b \\ A^T y + c \\ SY\mathbf{1} \end{bmatrix} = 0$$

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Residuals

$$r_p = Ax + s - b$$

$$r_d = A^T y + c$$

Line search to enforce $s, y > 0$

$$(x, s, y) \leftarrow (x, s, y) + \alpha(\Delta x, \Delta s, \Delta y)$$

Newton's method for optimality conditions

Root-finding equation

$$h(x, s, y) = \begin{bmatrix} Ax + s - b \\ A^T y + c \\ SY\mathbf{1} \end{bmatrix} = 0$$

Linear system

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Residuals

$$r_p = Ax + s - b$$

$$r_d = A^T y + c$$

Issue

Newton's step does not allow significant progress towards both $h(x, s, y) = 0$ **and** $s, y \geq 0$.

Line search to enforce $s, y > 0$

$$(x, s, y) \leftarrow (x, s, y) + \alpha(\Delta x, \Delta s, \Delta y)$$

Central path

Smoothed optimality conditions

Optimality conditions

$$\begin{aligned} Ax + s - b &= 0 \\ A^T y + c &= 0 \\ s_i y_i &= \boxed{\tau} \\ s, y &\geq 0 \end{aligned}$$

Same τ for every pair

Same optimality conditions for a "smoothed" version of our problem

Smoothed optimality conditions

Optimality conditions

$$Ax + s - b = 0$$

$$A^T y + c = 0$$

$$s_i y_i = \tau$$

$$s, y \geq 0$$

Same τ for every pair

Same optimality conditions for a "smoothed" version of our problem

Duality gap

$$s^T y = (b - Ax)^T y = b^T y - x^T A^T y = \underline{b^T y + c^T x}$$

$$\boxed{\frac{s^T y}{m} = \tau}$$

Newton's method for smoothed optimality conditions

Smoothed optimality conditions

$$h_\tau(x, s, y) = \begin{bmatrix} Ax + s - b \\ A^T y + c \\ SY\mathbf{1} - \tau\mathbf{1} \end{bmatrix} = 0$$

$$s, y \geq 0$$

Newton's method for smoothed optimality conditions

Smoothed optimality conditions

$$h_\tau(x, s, y) = \begin{bmatrix} Ax + s - b \\ A^T y + c \\ SY\mathbf{1} - \tau\mathbf{1} \end{bmatrix} = 0$$

$$s, y \geq 0$$

Linear system

$$\begin{bmatrix} 0 & A & I \\ A^T & 0 & 0 \\ S & 0 & Y \end{bmatrix} \begin{bmatrix} \Delta y \\ \Delta x \\ \Delta s \end{bmatrix} = \begin{bmatrix} -r_p \\ -r_d \\ -SY + \tau\mathbf{1} \end{bmatrix}$$

Line search to enforce $s, y > 0$

$$(x, s, y) \leftarrow (x, s, y) + \alpha(\Delta x, \Delta s, \Delta y)$$

The path parameter

Duality measure

$$\mu = \frac{s^T y}{m}$$ (average value of the pairs $s_i y_i$)

The path parameter

Linear system

$$\begin{bmatrix} 0 & A & I \\ A^T & 0 & 0 \\ S & 0 & Y \end{bmatrix} \begin{bmatrix} \Delta y \\ \Delta x \\ \Delta s \end{bmatrix} = \begin{bmatrix} -r_p \\ -r_d \\ -SY\mathbf{1} + \sigma\mu\mathbf{1} \end{bmatrix}$$

Duality measure

$$\mu = \frac{s^T y}{m}$$

(average value of the pairs $s_i y_i$)

The path parameter

Duality measure

$$\mu = \frac{s^T y}{m}$$ (average value of the pairs $s_i y_i$)

Centering parameter

$$\sigma \in [0, 1]$$

Linear system

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The path parameter

Linear system

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Duality measure

$$\mu = \frac{s^T y}{m}$$ (average value of the pairs $s_i y_i$)

Centering parameter

$\sigma \in [0, 1]$

$\sigma = 0 \quad \Rightarrow$ Newton step

$\sigma = 1 \quad \Rightarrow$ Centering step towards $(y^\star(\mu), x^\star(\mu), s^\star(\mu))$

The path parameter

Duality measure

$$\mu = \frac{s^T y}{m}$$ (average value of the pairs $s_i y_i$)

Linear system

$$\begin{bmatrix} 0 & A & I \\ A^T & 0 & 0 \\ S & 0 & Y \end{bmatrix} \begin{bmatrix} \Delta y \\ \Delta x \\ \Delta s \end{bmatrix} = \begin{bmatrix} -r_p \\ -r_d \\ -SY\mathbf{1} + \sigma\mu\mathbf{1} \end{bmatrix}$$

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Line search to enforce $s, y > 0$

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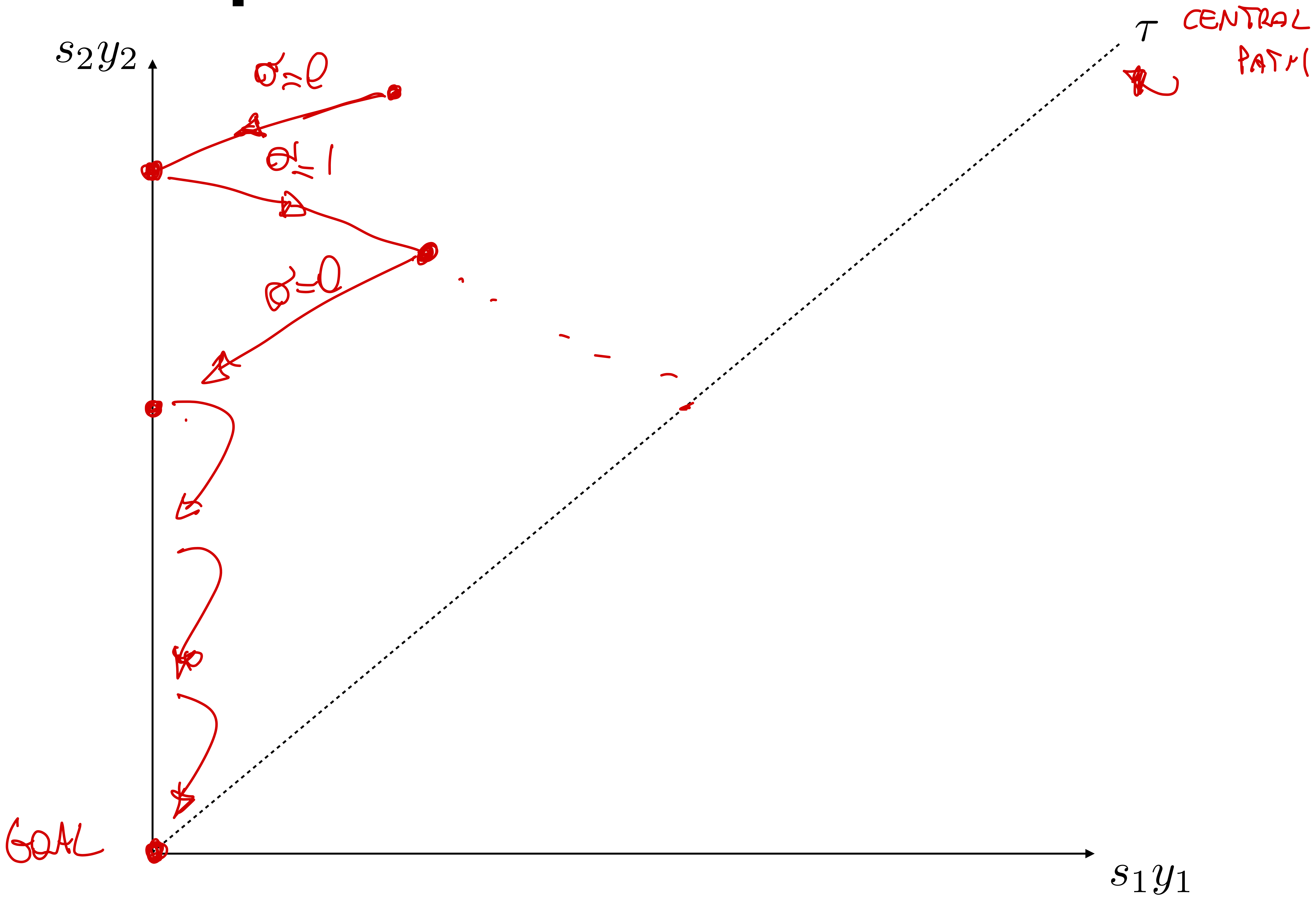
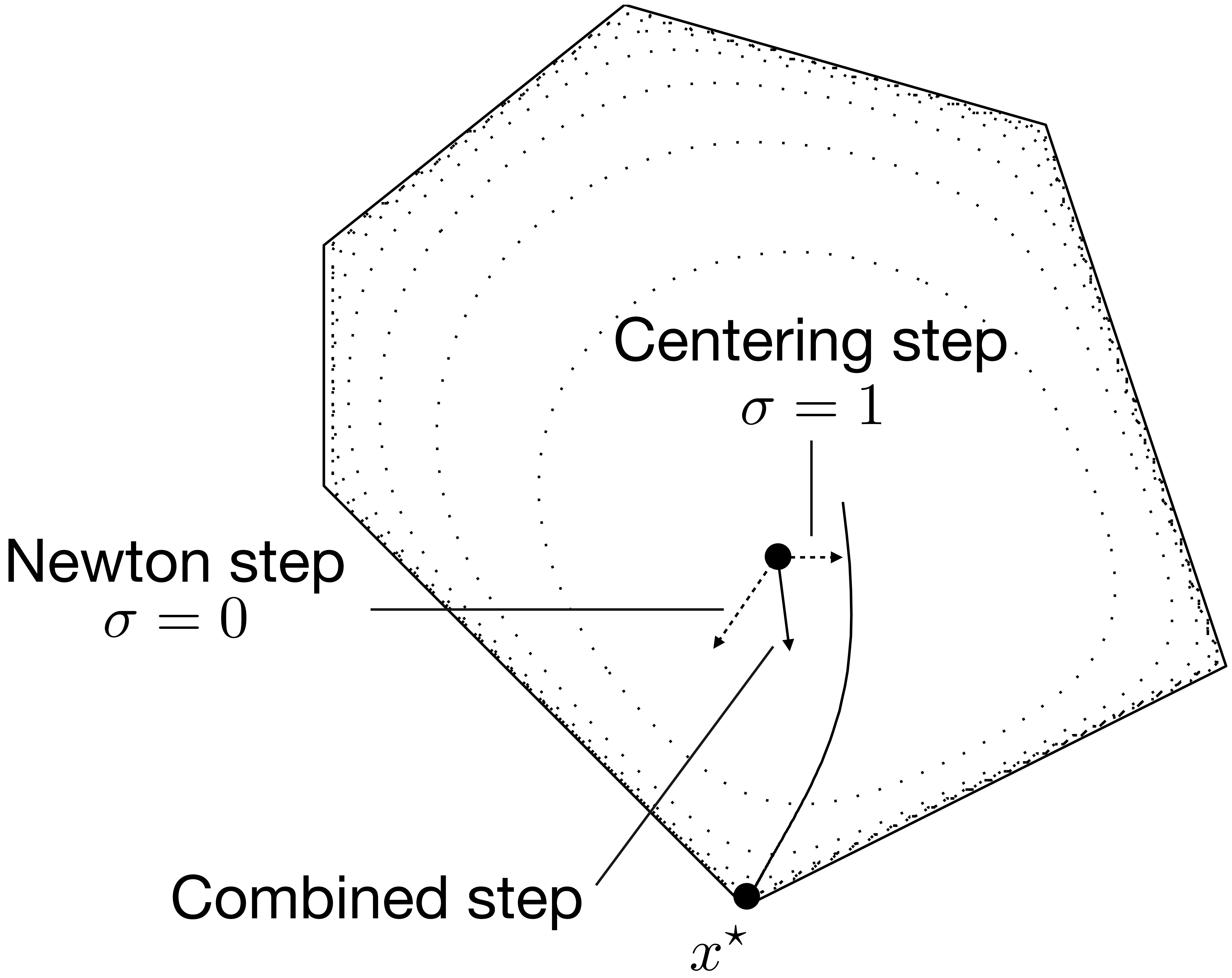
The central path

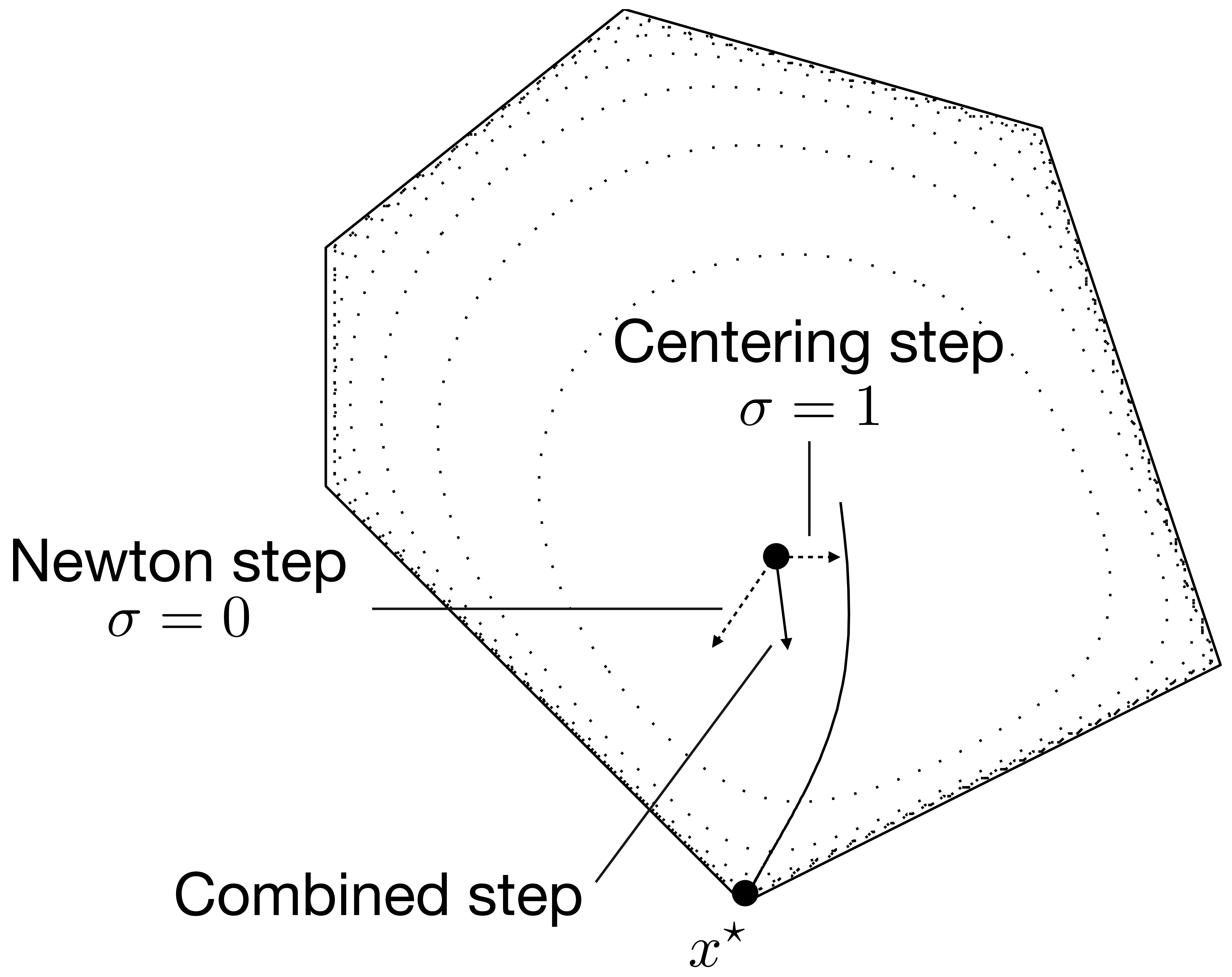


Primal-dual path-following method

Path-following algorithm idea

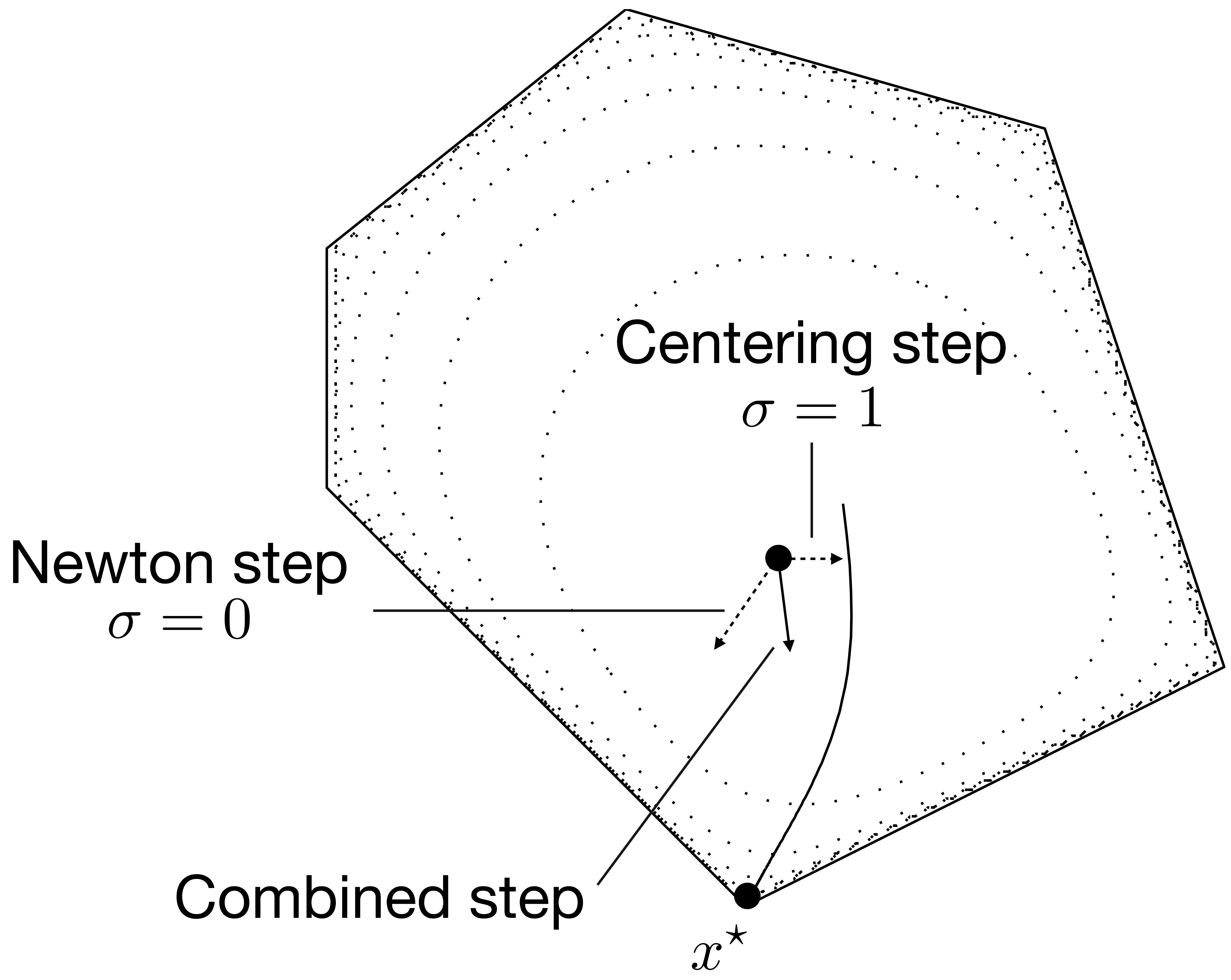


Path-following algorithm idea



Centering step
It brings towards the **central path** and is usually biased towards $s, y > 0$.
No progress on duality measure μ

Path-following algorithm idea



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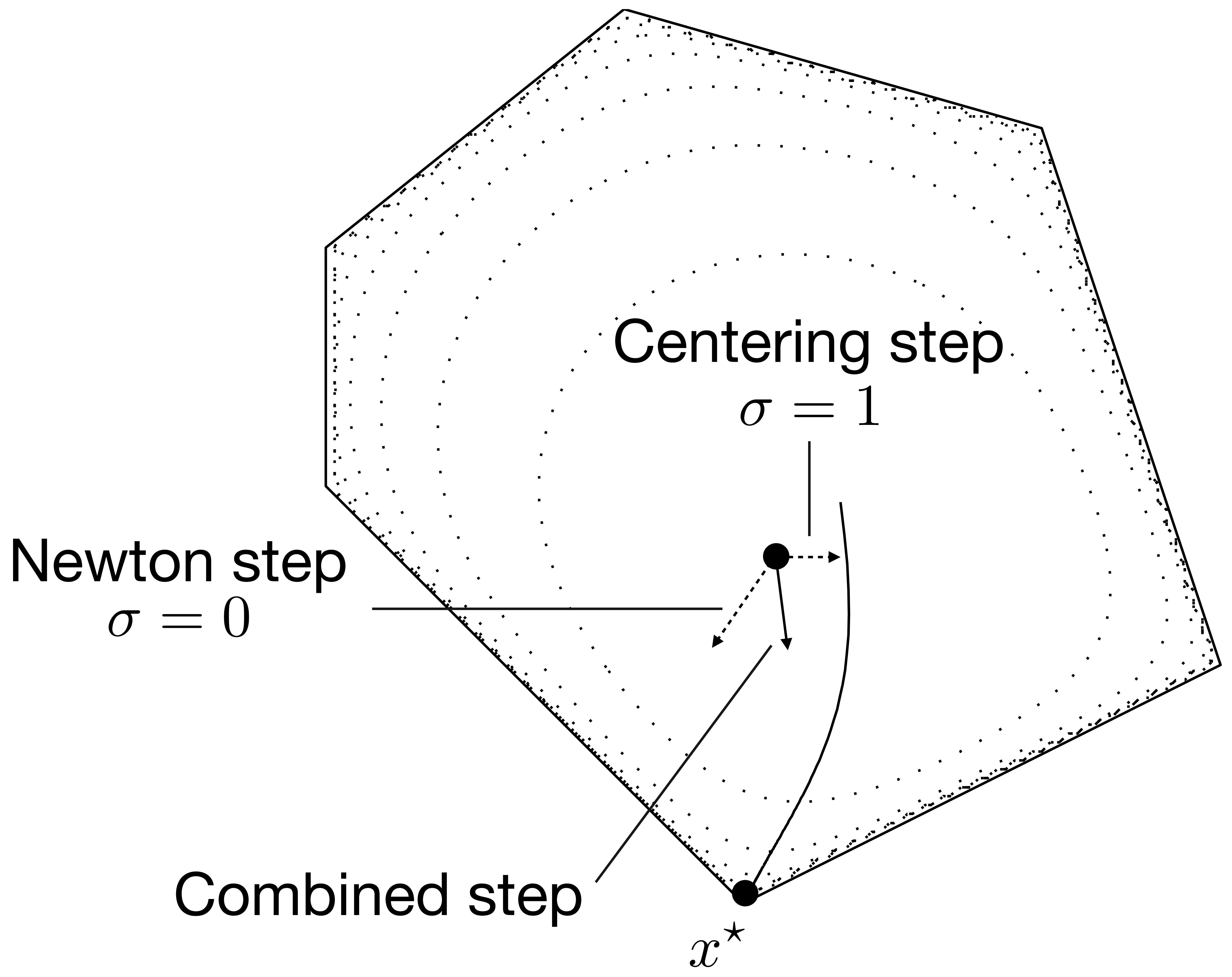
Newton step
It brings towards the **zero duality measure** μ. Quickly violates $s, y > 0$.

Path-following algorithm idea



Centering step
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No progress on duality measure μ

Newton step
It brings towards the **zero duality measure** μ. Quickly violates $s, y > 0$.

Combined step
Best of both worlds with longer steps

Primal-dual path-following algorithm

Initialization

1. Given (x_0, s_0, y_0) such that $s_0, y_0 > 0$

Iterations

1. Choose $\sigma \in [0, 1]$
2. Solve $\begin{bmatrix} 0 & A & I \\ A^T & 0 & 0 \\ S & 0 & Y \end{bmatrix} \begin{bmatrix} \Delta y \\ \Delta x \\ \Delta s \end{bmatrix} = \begin{bmatrix} -r_p \\ -r_d \\ -SY\mathbf{1} + \sigma\mu\mathbf{1} \end{bmatrix}$ where $\mu = s^T y/m$
3. Find maximum α such that $y + \alpha\Delta y > 0$ and $s + \alpha\Delta s > 0$
4. Update $(x, s, y) \leftarrow (x, s, y) + \alpha(\Delta x, \Delta s, \Delta y)$

Working towards optimality conditions

Optimality conditions satisfied **only at convergence**

Primal residual

$$r_p = Ax + s - b \to 0$$

Dual residual

$$r_d = A^T y + c \to 0$$

Complementary slackness

$$s^T y \to 0$$

Working towards optimality conditions

Optimality conditions satisfied **only at convergence**

$\boxed{10^{-6}}$

Primal residual

$$r_p = Ax + s - b \to 0$$

Dual residual

$$r_d = A^T y + c \to 0$$

Complementary slackness

$$s^T y \to 0$$

Stopping criteria

$$\|r_p\| \leq \epsilon_{\text{pri}}$$

$$\|r_d\| \leq \epsilon_{\text{dua}}$$

$$s^T y \leq \epsilon_{\text{gap}}$$

Logarithmic Barrier Functions

Smoothed optimality conditions

Optimality conditions

$$\begin{aligned} Ax + s - b &= 0 \\ A^T y + c &= 0 \\ s_i y_i &= \boxed{\tau} \\ s, y &\geq 0 \end{aligned}$$

Same τ for every pair

Same optimality conditions for a “smoothed” version of our problem

Smoothed optimality conditions

Optimality conditions

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Same τ for every pair

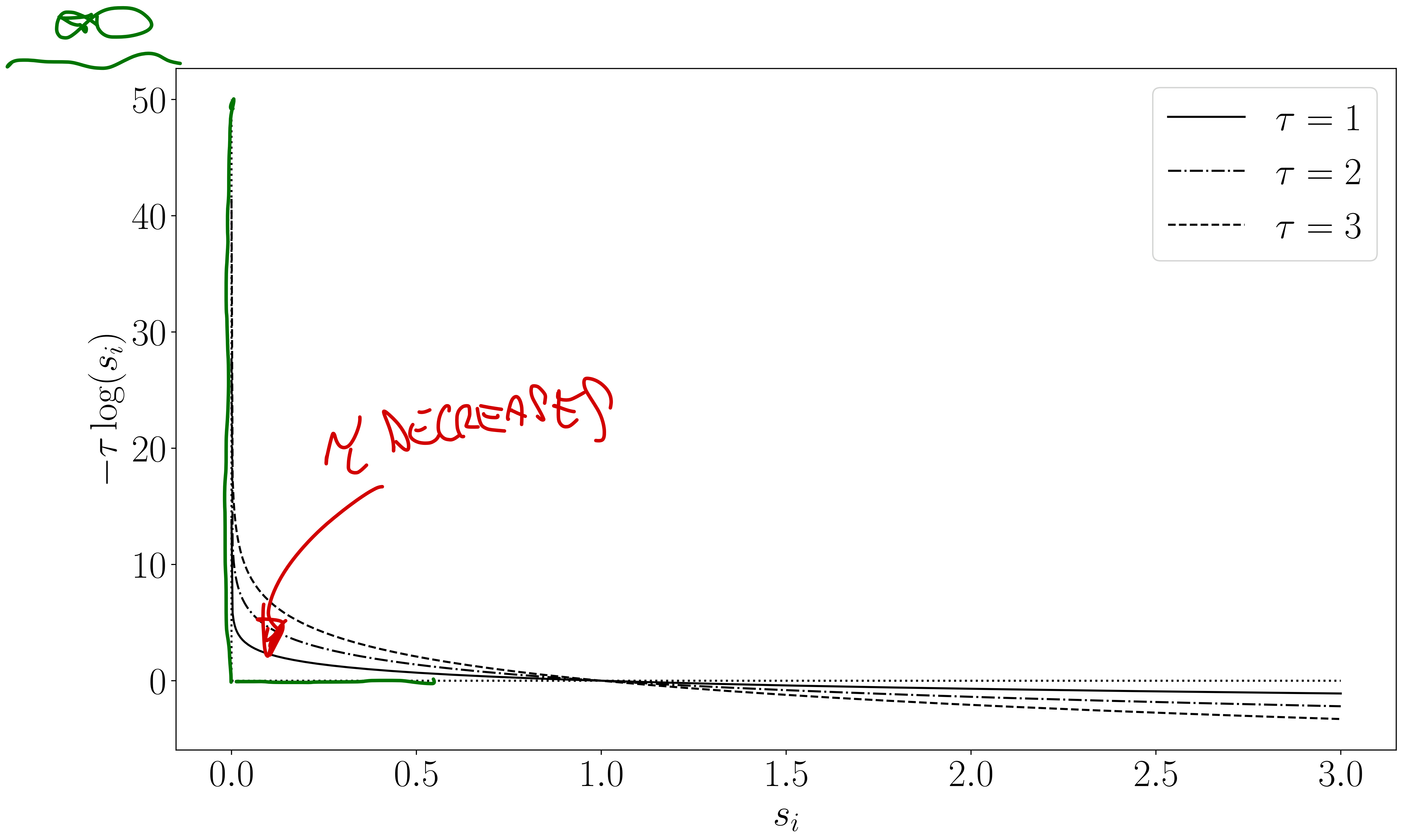
Same optimality conditions for a "smoothed" version of our problem

Do solutions actually exist?

What do they represent?

Logarithmic barrier

$$\phi(s) = -\tau \sum_{i=1}^{m} \log(s_i) \quad \text{on domain} \quad s_i > 0$$



As $\tau \to 0$ it approximates

$$\mathcal{I}_{s_i \geq 0} = \begin{cases} 0 & \text{if } s_i \geq 0 \\ \infty & \text{otherwise} \end{cases}$$

Smoothed problem

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax + s = b \\
& s \geq 0
\end{array}
$$

Smoothed problem

minimize	$c^T x$
subject to	$Ax + s = b$
	$s \geq 0$

$\longrightarrow$

minimize	$c^T x + \phi(x) = c^T x - \tau \sum_{i=1}^{m} \log(s_i)$
subject to	$Ax + s = b$

Smoothed problem

minimize $\quad c^T x$

subject to $\quad Ax + s = b$

$\quad s \geq 0$

$\longrightarrow$

minimize $\quad c^T x + \phi(s) = c^T x - \tau \sum_{i=1}^{m} \log(s_i)$

subject to $\quad Ax + s = b$

Dual cost

$$g(y) = \underset{x,s}{\text{minimize}}\ \mathcal{L}(x, s, y) = c^T x + \phi(s) + y^T (Ax + s - b)$$

Smoothed problem

minimize	$c^T x$
subject to	$Ax + s = b$
	$s \geq 0$

$\longrightarrow$

minimize	$c^T x + \phi(x) = c^T x - \tau \sum_{i=1}^{m} \log(s_i)$
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Dual cost

$$g(y) = \underset{x,s}{\text{minimize}}\ \mathcal{L}(x, s, y) = c^T x + \phi(s) + y^T (Ax + s - b)$$

$$\frac{\partial \mathcal{L}}{\partial x} = A^T y + c = 0$$

$$\frac{\partial \mathcal{L}}{\partial s_i} = -\tau \frac{1}{s_i} + y_i = 0 \quad \Longrightarrow s_i y_i = \tau$$

Central path

minimize $\quad c^T x - \tau \sum_{i=1}^m \log(s_i)$

subject to $\quad Ax + s = b$

Set of points $(x^\star(\tau), s^\star(\tau), y^\star(\tau))$
with $\tau > 0$ such that

$$
\begin{aligned}
Ax + s - b &= 0 \\
A^T y + c &= 0 \\
s_i y_i &= \tau \\
s, y &\geq 0
\end{aligned}
$$

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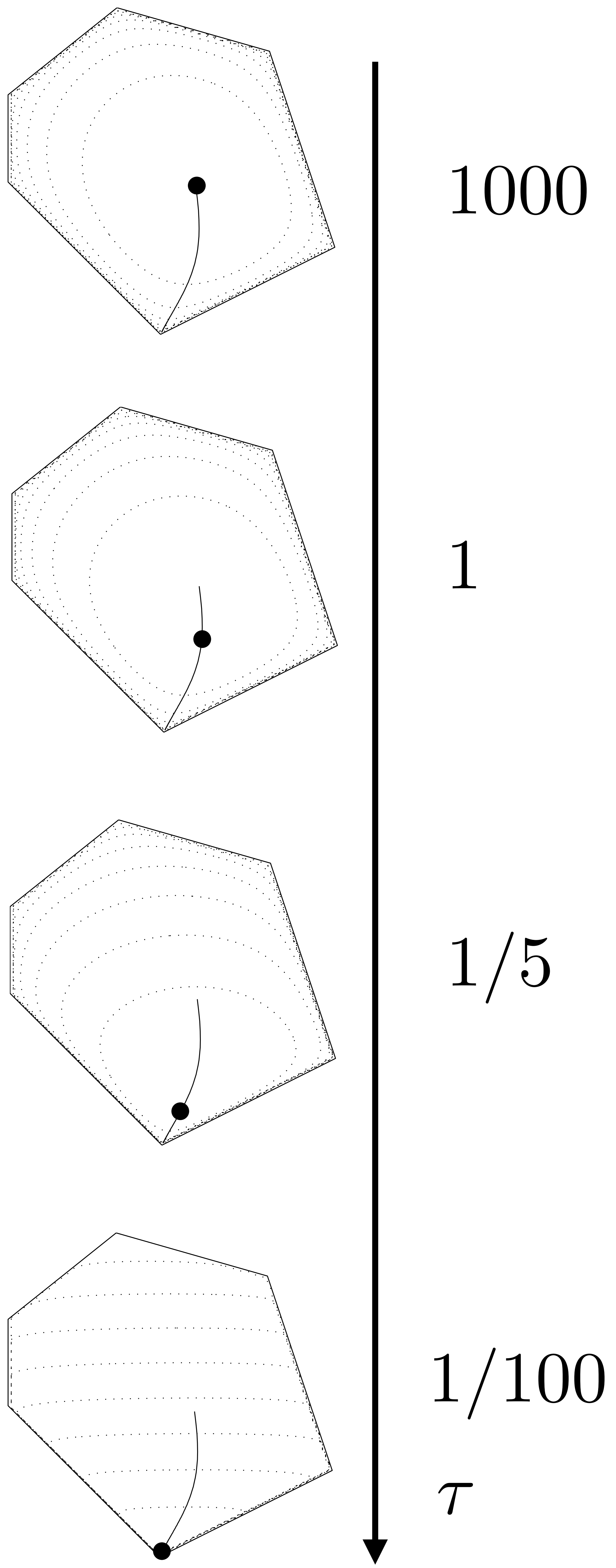
Analytic Center
$\tau \to \infty$



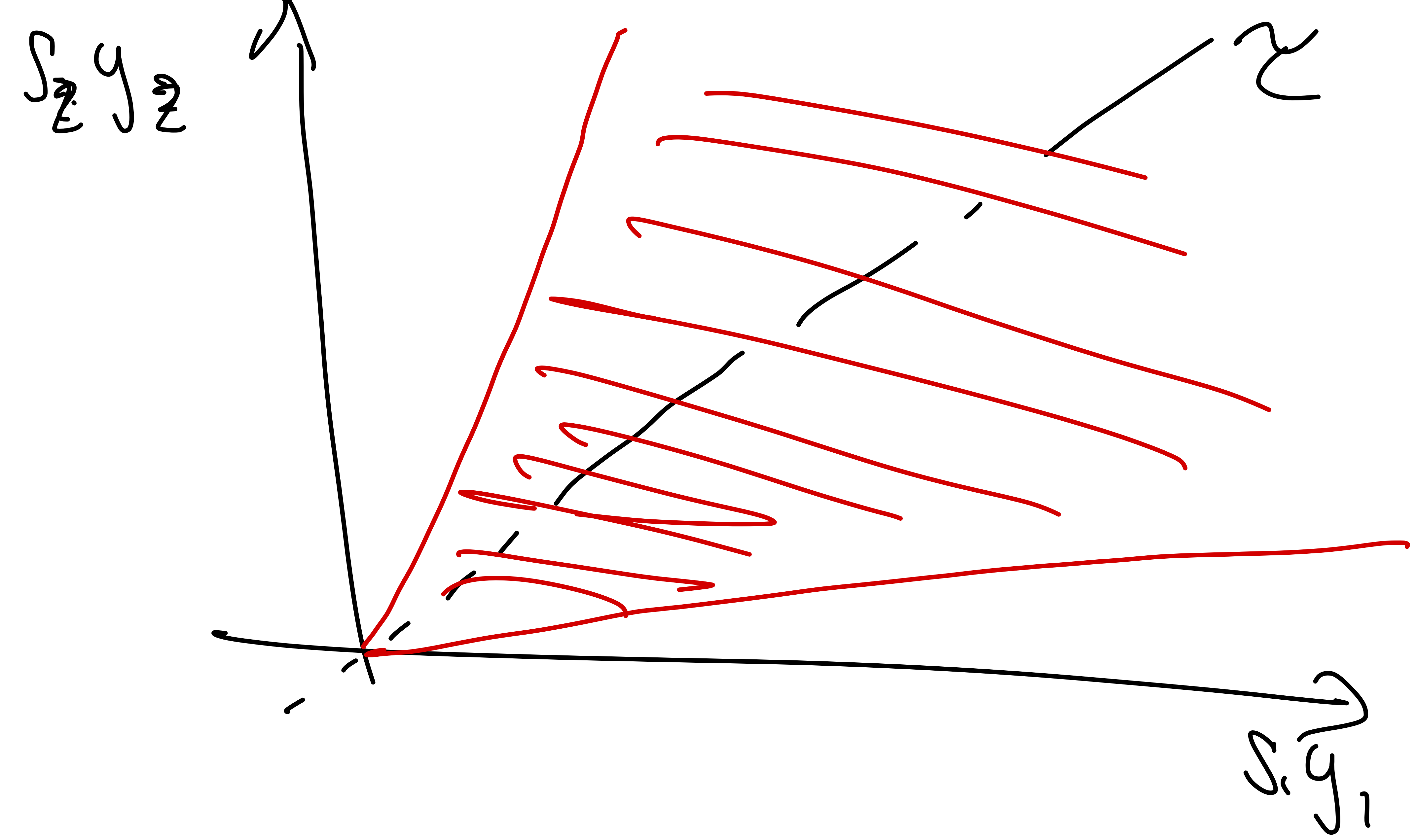
Main idea

Follow central path as $\tau \to 0$

Convergence

Definitions

$\mu_k = \frac{s_k^T y_k}{m}$



Primal-dual strictly feasible set

$$\mathcal{F}^\circ = \{(x, s, y) \mid Ax + s = b,\ A^T y + c = 0,\ s, y > 0\}$$

Central path neighborhood

$$\mathcal{N}(\gamma) = \{(x, s, y) \in \mathcal{F}^\circ \mid s_i y_i \geq \gamma \mu\} \quad \text{with } \gamma \in (0, 1] \quad \text{(almost all the feasible region)}$$

Theorem

[Page 402-406, NO]

[Theorem 14.3] Smallest decrement

$$\mu_{k+1} \leq (1 - \delta/n)\, \mu_k \qquad \text{with constant } \delta > 0$$

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Iteration complexity

Given $(x_0, s_0, y_0) \in \mathcal{N}(\gamma)$, there exists $K = O(n \log(1/\epsilon))$ such that

$$\mu_k \leq \epsilon \mu_0 \qquad \text{for all } k \geq K$$

Theorem

[Page 402-406, NO]

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Given $(x_0, s_0, y_0) \in \mathcal{N}(\gamma)$, there exists $K = O(n \log(1/\epsilon))$ such that

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Remark Modified versions achieve $O(\sqrt{n} \log(1/\epsilon))$

Iteration complexity proof

[Page 402-406, NO]

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[Page 402-406, NO]

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$$\log \mu_{k+1} \leq \log\left(1 - \delta/n\right) + \log \mu_k$$

Iteration complexity proof

[Page 402-406, NO]

$$\mu_{k+1} \leq (1 - \delta/n)\,\mu_k$$

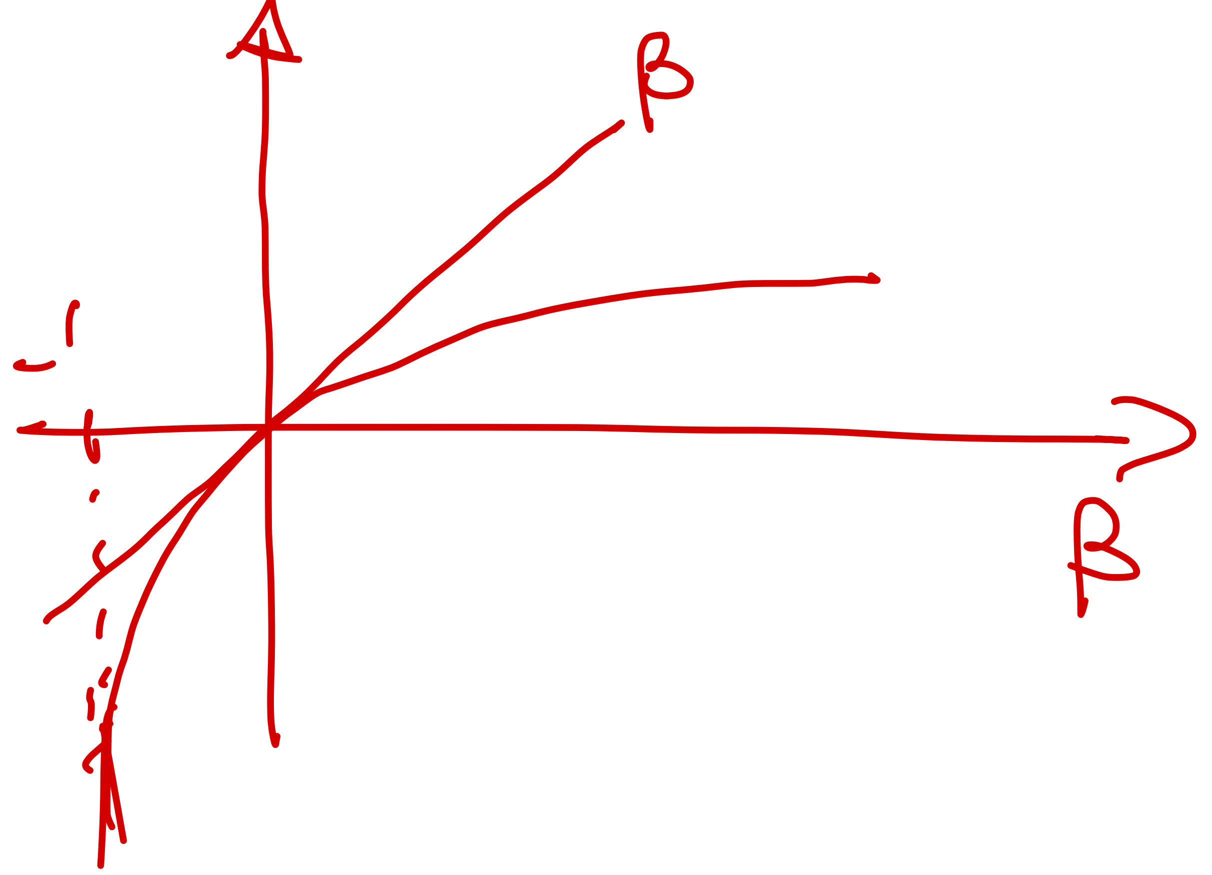
(take logarithm)

$$\log \mu_{k+1} \leq \log(1 - \delta/n) + \log \mu_k \longrightarrow \log \mu_k \leq k \log(1 - \delta/n) + \log \mu_0$$

(apply iteratively)

Iteration complexity proof

[Page 402-406, NO]



$$\mu_{k+1} \leq (1 - \delta/n)\,\mu_k$$

(take logarithm) (apply iteratively)

$$\log \mu_{k+1} \leq \log(1 - \delta/n) + \log \mu_k \longrightarrow \log \mu_k \leq k \log(1 \boxed{-\delta/n}) + \log \mu_0$$

Since $\log(1+\beta) \leq \beta, \quad \forall \beta > -1 \longrightarrow \log(\mu_k/\mu_0) \leq k(-\delta/n)$

Iteration complexity proof

[Page 402-406, NO]

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If $k(-\delta/n) \leq \log(\epsilon)$

GOAL

then $\log(\mu_k/\mu_0) \leq \log(\epsilon)$. Therefore, $\boxed{\mu_k/\mu_0 \leq \epsilon}$

Iteration complexity proof

[Page 402-406, NO]

$$\mu_{k+1} \leq (1 - \delta/n)\, \mu_k$$

(take logarithm)

$$\log \mu_{k+1} \leq \log (1 - \delta/n) + \log \mu_k \longrightarrow \log \mu_k \leq k \log (1 - \delta/n) + \log \mu_0$$

(apply iteratively)

Since $\log(1 + \beta) \leq \beta, \quad \forall \beta > -1$ $\longrightarrow$ $\log(\mu_k / \mu_0) \leq k(-\delta/n)$

If $k(-\delta/n) \leq \log(\epsilon)$ $\rightsquigarrow$ $k \geq -(n/\delta) \log(\epsilon)$

then $\log(\mu_k/\mu_0) \leq \log(\epsilon)$. Therefore, $\mu_k/\mu_0 \leq \epsilon$

Rewriting the inequality: $k \geq (n/\delta) \log(1/\epsilon)$ $\blacksquare$

Interior-point methods for linear optimization

Today, we learned to:

- **Apply** Newton's method to solve optimality conditions
- **Analyze** the central path and the smoothed optimality conditions
- **Develop** a prototype primal-dual path-following algorithm

Next lecture

- Practical interior-point method (Mehrotra predictor-corrector algorithm)
- Linear algebra implementation details
- Linear optimization recap