ORF522 – Linear and Nonlinear Optimization

9. Sensitivity analysis for linear optimization

Bartolomeo Stellato — Fall 2022

Ed Forum

X_B in BASIS

X_N NOT in the BASIS $\Rightarrow \bar{c}$?

- Dual simplex applications?

- In the dual simplex part, we talked about in primal simplex, $X_B > 0$ and $X_N = 0$. However, I can confused about why in dual problem $\bar{c}_B = 0$ and $\bar{c}_N > 0$. Is there any intuition behind this?

- In the illustration of dual simplex method, we used the fact that if y = -A_b^(-T)c_b, then A^Ty + c >=0 is equivalent to reduced cost >=0. From there (page 32), we seem to be constantly using A^Ty + c as the vector of reduced cost. However, I'm wondering why we can use this previous assumption in all our steps during the dual simplex. Why is y = -A_b^(-T)c_b satisfied at all such middle steps or is this something only satisfied at the optimal solution?

Recap

Reduced costs

Interpretation

Change in objective/marginal cost of adding x_j to the basis

$$\bar{c}_j = c_j - c_B^T A_B^{-1} A_j$$

- $\bar{c}_j > 0$: adding x_j will increase the objective (bad)
- $\bar{c}_j < 0$: adding x_j will decrease the objective (good)

Reduced costs

Interpretation

Change in objective/marginal cost of adding x_j to the basis

$$\bar{c}_j = \boxed{c_j} - c_B^T A_B^{-1} A_j$$

Cost per-unit increase of variable x_j

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Cost to change other variables compensating for x_j to enforce $Ax = b$

- $\bar{c}_j > 0$: adding x_j will increase the objective (bad)
- $\bar{c}_j < 0$: adding x_j will decrease the objective (good)

Reduced costs

$\min \; c^T x$
s.t. $\; Ax = b \;]\; m$
$x \geq 0 \quad x \in \mathbb{R}^n$

$x \; \begin{bmatrix} x_B > 0 \\ x_N = 0 \end{bmatrix} \longrightarrow \bar{c}_B = 0$

Interpretation

Change in objective/marginal cost of adding x_j to the basis

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Reduced costs for basic variables is 0

$$\begin{aligned}\bar{c}_{B(i)} &= c_{B(i)} - c_B^T A_B^{-1} A_{B(i)} = c_{B(i)} - c_B^T (A_B^{-1} A_B) e_i \\ &= c_{B(i)} - c_B^T e_i = c_{B(i)} - c_{B(i)} = 0\end{aligned}$$

BASIC FEASIBLE SOLUTION

$x \qquad x_B > 0 \longrightarrow \bar{c}_B = 0$

$\qquad x_N = 0$

$\downarrow$

$y = -A_B^{-T} c_B$

1) Primal Simplex

$\exists i : \quad \bar{c}_i < 0$

BEFORE CONVERGENCE

Dual Problem

$\max \; -b^T y$

$\text{s.t.} \quad A^T y + c \geq 0$

2) Dual Simplex

$\bar{c} := A^T y + c \geq 0$

$\rightarrow \bar{c}_N > 0$

ALWAYS DUAL FEASIBLE

Vector of reduced costs

Reduced costs

$$\bar{c}_j = c_j - c_B^T A_B^{-1} A_j$$

Full vector in one shot?

$$\bar{c} = (\bar{c}_1, \dots, \bar{c}_n)$$

Vector of reduced costs

Reduced costs

$$\bar{c}_j = c_j - c_B^T A_B^{-1} A_j$$

Isolate basis B-related components p
(they are the same across j)

$$\bar{c}_j = c_j - A_j^T (A_B^{-1})^T c_B = c_j - A_j^T p$$

Full vector in one shot?

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Vector of reduced costs

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Full vector in one shot?

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Obtain p by solving linear system

$$p = (A_B^{-1})^T c_B \quad \Rightarrow \quad A_B^T p = c_B$$

Note: $(M^{-1})^T = (M^T)^{-1}$
for any square invertible M

Vector of reduced costs

Reduced costs

$$\bar{c}_j = c_j - c_B^T A_B^{-1} A_j$$

Isolate basis B-related components p
(they are the same across j)

$$\bar{c}_j = c_j - A_j^T (A_B^{-1})^T c_B = c_j - A_j^T p$$

$\boxed{p = -y}$

Full vector in one shot?

$$\bar{c} = (\bar{c}_1, \dots, \bar{c}_n)$$

Obtain p by solving linear system

$$p = (A_B^{-1})^T c_B \quad \Rightarrow \quad A_B^T p = c_B$$

Note: $(M^{-1})^T = (M^T)^{-1}$
for any square invertible M

Computing reduced cost vector

1. Solve $A_B^T p = c_B$
2. $\bar{c} = c - A^T p$

Primal and dual basic feasible solutions

Primal problem

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Dual problem

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c \geq 0 \end{array}$$

Given a **basis** matrix A_B

Primal and dual basic feasible solutions

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Given a **basis** matrix A_B

Primal feasible: $Ax = b,\ x \geq 0 \quad \Rightarrow \quad x_B = A_B^{-1} b \geq 0$

Primal and dual basic feasible solutions

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minimize $c^T x$
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Dual problem

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Given a **basis** matrix A_B

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Dual feasible: $A^T y + c \geq 0$.

Primal and dual basic feasible solutions

Primal problem

minimize $c^T x$

subject to $Ax = b$

$x \geq 0$

Dual problem

maximize $-b^T y$

subject to $A^T y + c \geq 0$

Given a **basis** matrix A_B

Primal feasible: $Ax = b,\ x \geq 0 \quad \Rightarrow \quad x_B = A_B^{-1} b \geq 0$

Dual feasible: $A^T y + c \geq 0$. If $y = -A_B^{-T} c_B \quad \Rightarrow \quad c - A^T A_B^{-T} c_B \geq 0$

Primal and dual basic feasible solutions

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Reduced costs

Dual feasible: $A^T y + c \geq 0$. If $y = -A_B^{-T} c_B \quad \Rightarrow \quad c - A^T A_B^{-T} c_B \geq 0$

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Reduced costs

Dual feasible: $A^T y + c \geq 0$. If $y = -A_B^{-T} c_B \quad \Rightarrow \quad c - A^T A_B^{-T} c_B \geq 0$

Zero duality gap: $c^T x + b^T y = c_B^T x_B - b^T A_B^{-T} c_B = c_B^T x_B - c_B^T A_B^{-1} b = 0$

Primal and dual basic feasible solutions

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Given a **basis** matrix A_B

Reduced costs

Primal feasible: $Ax = b,\ x \geq 0 \quad \Rightarrow \quad x_B = A_B^{-1} b \geq 0$

Dual feasible: $A^T y + c \geq 0$. If $y = -A_B^{-T} c_B \quad \Rightarrow \quad c - A^T A_B^{-T} c_B \geq 0$

Zero duality gap: $c^T x + b^T y = c_B^T x_B - b^T A_B^{-T} c_B = c_B^T x_B - c_B^T A_B^{-1} b = 0$

(by construction)

Today's lecture
[Chapter 5, LO]

Sensitivity analysis in linear optimization

- Adding new constraints and variables
- Change problem data
- Differentiable optimization

Adding new constraints and variables

Adding new variables

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax = b \\
& x \geq 0
\end{array}
$$

Solution $x^\star, y^\star$

Adding new variables

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & c^T x + c_{n+1} x_{n+1} \\ \text{subject to} & Ax + A_{n+1} x_{n+1} = b \\ & x, x_{n+1} \geq 0 \end{array}$$

Solution $x^\star, y^\star$

Adding new variables

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & c^T x + c_{n+1} x_{n+1} \\ \text{subject to} & Ax + A_{n+1} x_{n+1} = b \\ & x, x_{n+1} \geq 0 \end{array}$$

Solution $x^\star, y^\star$

Solution $(x^\star, 0), y^\star$ **optimal** for the new problem?

Adding new variables

Optimality conditions

minimize $\quad c^T x + c_{n+1} x_{n+1}$

subject to $\quad Ax + \cancel{A_{n+1} x_{n+1}} = b \longrightarrow$ Solution $(x^\star, 0)$ is still **primal feasible**

$\quad x, \cancel{x_{n+1}} \geq 0$

Adding new variables

Optimality conditions

max. $-b^T y$

st. $A^T y + c \geq 0$

$A_{n+1}^T y + c_{n+1} \geq 0$ //

minimize	$c^T x + c_{n+1} x_{n+1}$
subject to	$Ax + A_{n+1} x_{n+1} = b$
	$x, x_{n+1} \geq 0$

$\longrightarrow$ Solution $(x^\star, 0)$ is still **primal feasible**

Is $y^\star$ still **dual feasible**?

$$A_{n+1}^T y^\star + c_{n+1} \geq 0$$

Adding new variables
Optimality conditions

minimize $\quad c^T x + c_{n+1} x_{n+1}$

subject to $\quad Ax + A_{n+1} x_{n+1} = b \quad \longrightarrow$ Solution $(x^\star, 0)$ is still **primal feasible**

$\quad x, x_{n+1} \geq 0$

Is $y^\star$ still **dual feasible**?

$$A_{n+1}^T y^\star + c_{n+1} \geq 0$$

Yes

$(x^\star, 0)$ still **optimal** for new problem

Otherwise

Primal simplex

Adding new variables

Example

$$
\begin{array}{ll}
\text{minimize} & -60x_1 - 30x_2 - 20x_3 \\
\text{subject to} & 8x_1 + 6x_2 + x_3 \leq 48 \\
& 4x_1 + 2x_2 + 1.5x_3 \leq 20 \\
& 2x_1 + 1.5x_2 + 0.5x_3 \leq 8 \\
& x \geq 0
\end{array}
$$

Adding new variables

Example

minimize $\quad -60x_1 - 30x_2 - 20x_3 \quad$ -profit

subject to

$$8x_1 + 6x_2 + x_3 \leq 48$$

$$4x_1 + 2x_2 + 1.5x_3 \leq 20$$

$$2x_1 + 1.5x_2 + 0.5x_3 \leq 8$$

$$x \geq 0$$

Adding new variables

Example

minimize	$-60x_1 - 30x_2 - 20x_3$	-profit
subject to	$8x_1 + 6x_2 + x_3 \leq 48$	material
	$4x_1 + 2x_2 + 1.5x_3 \leq 20$	
	$2x_1 + 1.5x_2 + 0.5x_3 \leq 8$	
	$x \geq 0$	

Adding new variables

Example

minimize	$-60x_1 - 30x_2 - 20x_3$	-profit
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Adding new variables

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	$2x_1 + 1.5x_2 + 0.5x_3 \leq 8$	quality control
	$x \geq 0$	

Adding new variables

Example

minimize	$-60x_1 - 30x_2 - 20x_3$	-profit
subject to	$8x_1 + 6x_2 + x_3 \leq 48$	material
	$4x_1 + 2x_2 + 1.5x_3 \leq 20$	production
	$2x_1 + 1.5x_2 + 0.5x_3 \leq 8$	quality control
	$x \geq 0$	

$$c = (-60, -30, -20, 0, 0, 0)$$

minimize	$c^T x$
subject to	$Ax = b$
	$x \geq 0$

$$A = \begin{bmatrix} 8 & 6 & 1 & 1 & 0 & 0 \\ 4 & 2 & 1.5 & 0 & 1 & 0 \\ 2 & 1.5 & 0.5 & 0 & 0 & 1 \end{bmatrix}$$

$$b = (48, 20, 8)$$

Adding new variables

Example

minimize	$-60x_1 - 30x_2 - 20x_3$	-profit
subject to	$8x_1 + 6x_2 + x_3 \leq 48$	material
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$$c = (-60, -30, -20, 0, 0, 0)$$

minimize $\quad c^T x$

subject to $\quad Ax = b$

$\qquad x \geq 0$

$$A = \begin{bmatrix} 8 & 6 & 1 & 1 & 0 & 0 \\ 4 & 2 & 1.5 & 0 & 1 & 0 \\ 2 & 1.5 & 0.5 & 0 & 0 & 1 \end{bmatrix}$$

$$b = (48, 20, 8)$$

$x^\star = (2, 0, 8, 24, 0, 0), \quad y^\star = (0, 10, 10), \quad c^T x^\star = -280,$ basis $\{1, 3, 4\}$

Adding new variables

Example: add new product?

minimize $\quad c^T x + \boxed{c_{n+1}} x_{n+1}$

subject to $\quad Ax + \boxed{A_{n+1}} x_{n+1} = b$

$\quad x, x_{n+1} \geq 0$

$$c = (-60, -30, -20, 0, 0, 0, -15)$$

$$A = \begin{bmatrix} 8 & 6 & 1 & 1 & 0 & 0 & 1 \\ 4 & 2 & 1.5 & 0 & 1 & 0 & 1 \\ 2 & 1.5 & 0.5 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$b = (48, 20, 8)$$

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$$b = (48, 20, 8)$$

Previous solution

$x^\star = (2, 0, 8, 24, 0, 0), \quad y^\star = (0, 10, 10), \quad c^T x^\star = -280,$ basis $\{1, 3, 4\}$

Adding new variables

Example: add new product?

$$c = (-60, -30, -20, 0, 0, 0, -15)$$

$$\begin{array}{ll} \text{minimize} & c^T x + c_{n+1} x_{n+1} \\ \text{subject to} & Ax + A_{n+1} x_{n+1} = b \\ & x, x_{n+1} \geq 0 \end{array}$$

$$A = \begin{bmatrix} 8 & 6 & 1 & 1 & 0 & 0 & 1 \\ 4 & 2 & 1.5 & 0 & 1 & 0 & 1 \\ 2 & 1.5 & 0.5 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$b = (48, 20, 8)$$

Previous solution

$x^\star = (2, 0, 8, 24, 0, 0), \quad y^\star = (0, 10, 10), \quad c^T x^\star = -280,$ basis $\{1, 3, 4\}$

Still optimal

$$A_{n+1}^T y^\star + c_{n+1} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 10 \\ 10 \end{bmatrix} - 15 = 5 \geq 0$$

$x = (x^*, 0), y^*$

$c^T x + b^T y = 0$

ADDED TO PREVIOUS COST

Adding new variables

Example: add new product?

$$c = (-60, -30, -20, 0, 0, 0, -15)$$

$$\begin{array}{lll} \text{minimize} & c^T x + \boxed{c_{n+1} x_{n+1}} & \\ \text{subject to} & Ax + A_{n+1} x_{n+1} = b \\ & x, x_{n+1} \geq 0 \end{array}$$

$$A = \begin{bmatrix} 8 & 6 & 1 & 1 & 0 & 0 & 1 \\ 4 & 2 & 1.5 & 0 & 1 & 0 & 1 \\ 2 & 1.5 & 0.5 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$b = (48, 20, 8)$$

Previous solution

$x^\star = (2, 0, 8, 24, 0, 0), \quad y^\star = (0, 10, 10), \quad c^T x^\star = -280,$ basis $\{1, 3, 4\}$

Still optimal

$$A_{n+1}^T y^\star + c_{n+1} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 10 \\ 10 \end{bmatrix} - 15 = 5 \geq 0$$

Shall we add a new product?

Adding new constraints

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Solution $x^\star, y^\star$

Adding new constraints

minimize	$c^T x$		minimize	$c^T x$
subject to	$Ax = b$	$\longrightarrow$	subject to	$Ax = b$
	$x \geq 0$			$a_{m+1}^T x = b_{m+1}$
				$x \geq 0$

Solution $x^\star, y^\star$

Adding new constraints

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & a_{m+1}^T x = b_{m+1} \\ & x \geq 0 \end{array}$$

Solution $x^\star, y^\star$

Dual

$$\begin{array}{ll} \text{maximize} & -b^T y - b_{m+1} y_{m+1} \\ \text{subject to} & A^T y + a_{m+1} y_{m+1} + c \geq 0 \end{array}$$

Adding new constraints

minimize	$c^T x$
subject to	$Ax = b$
	$x \geq 0$

$\longrightarrow$

minimize	$c^T x$
subject to	$Ax = b$
	$a_{m+1}^T x = b_{m+1}$
	$x \geq 0$

Solution $x^\star, y^\star$

Dual

maximize	$-b^T y - b_{m+1} y_{m+1}$
subject to	$A^T y + a_{m+1} y_{m+1} + c \geq 0$

Solution $x^\star, (y^\star, 0)$ **optimal** for the new problem?

Adding new constraints

Optimality conditions

$$
\begin{array}{ll}
\text{maximize} & -b^T y \; \cancel{- b_{m+1} y_{m+1}} \\
\text{subject to} & A^T y + \cancel{a_{m+1} y_{m+1}} + c \geq 0
\end{array}
\longrightarrow
$$

Solution $(y^\star, 0)$ is still **dual feasible**

Adding new constraints

Optimality conditions

maximize $\quad -b^T y$

subject to $\quad A^T y + a_{m+1} y_{m+1} + c \geq 0 \longrightarrow$ Solution $(y^\star, 0)$ is still **dual feasible**

Is $x^\star$ still **primal feasible**?

$$Ax = b$$
$$\boxed{a_{m+1}^T x = b_{m+1}}$$
$$x \geq 0$$

Adding new constraints

Optimality conditions

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + a_{m+1} y_{m+1} + c \geq 0 \end{array} \longrightarrow \text{Solution } (y^\star, 0) \text{ is still } \textbf{dual feasible}$$

Is $x^\star$ still **primal feasible**?

$$\begin{aligned} Ax &= b \\ a_{m+1}^T x &= b_{m+1} \\ x &\geq 0 \end{aligned}$$

Yes

$x^\star$ still **optimal** for new problem

Otherwise

Dual simplex

Adding new constraints

Example

Global sensitivity analysis

Information from primal-dual solution

Goal: extract information from $x^\star, y^\star$ about their sensitivity with respect to changes in problem data

Modified LP

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax = b + u \\
& x \geq 0
\end{array}
$$

Optimal cost $p^\star(u)$

Global sensitivity

Dual of modified LP

$$\begin{array}{ll} \text{maximize} & -(b+u)^T y \\ \text{subject to} & A^T y + c \geq 0 \end{array}$$

Global sensitivity

Dual of modified LP

$$\begin{array}{ll} \text{maximize} & -(b+u)^T y \\ \text{subject to} & A^T y + c \geq 0 \end{array}$$

DOES NOT DEPEND on u

Global lower bound

Given $y^\star$ a dual optimal solution for $u = 0$, then

$$\begin{aligned} p^\star(u) &\geq -(b+u)^T y^\star \\ &= p^\star(0) - u^T y^\star \end{aligned}$$

(from weak duality and dual feasibility)

Global sensitivity

Dual of modified LP

$$\begin{array}{ll} \text{maximize} & -(b+u)^T y \\ \text{subject to} & A^T y + c \geq 0 \end{array}$$

Global lower bound

Given $y^\star$ a dual optimal solution for $u = 0$, then

$$\begin{aligned} p^\star(u) &\geq -(b+u)^T y^\star \\ &= p^\star(0) - u^T y^\star \end{aligned}$$

(from weak duality and dual feasibility)

It holds for any u

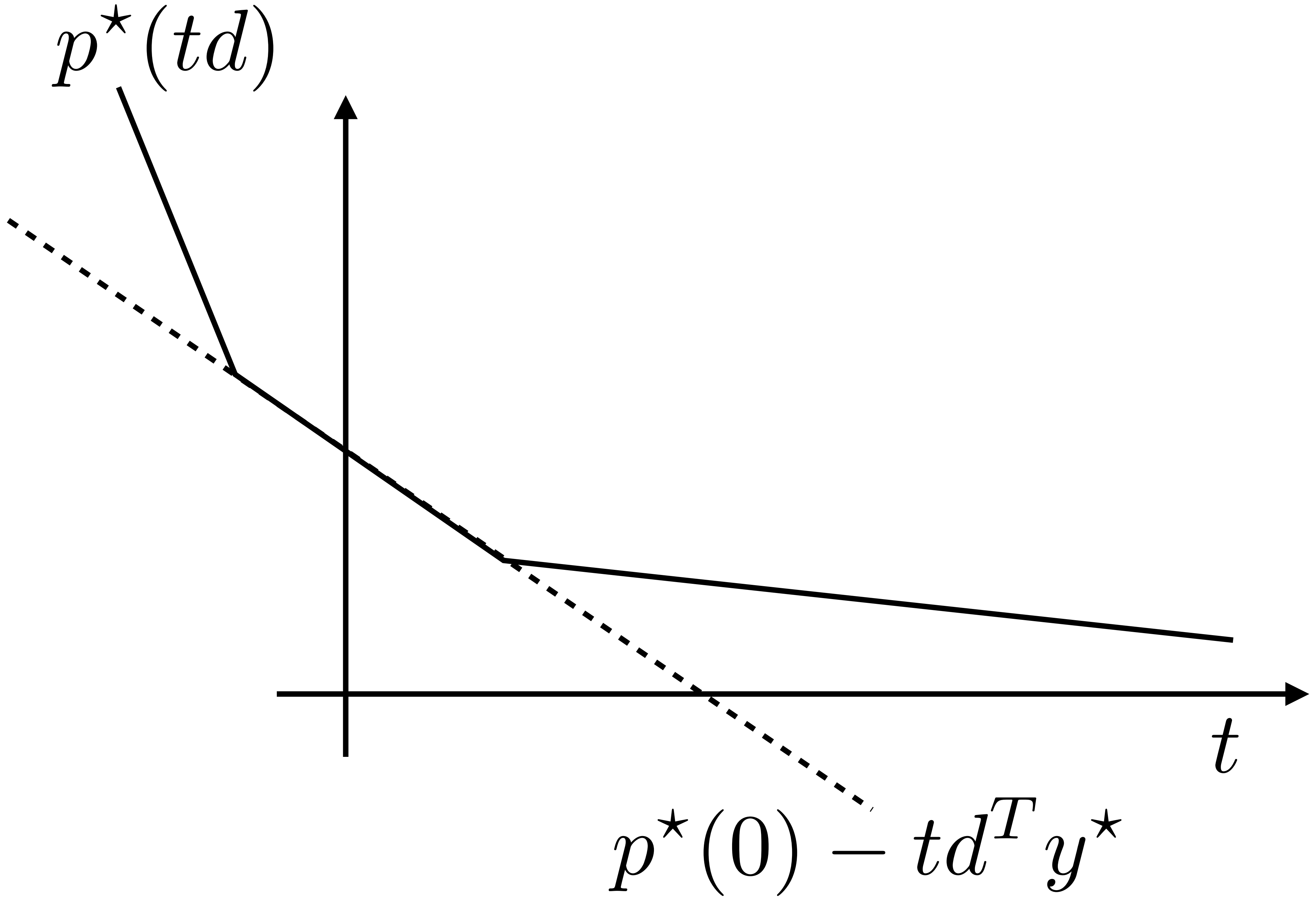
Global sensitivity

Example

Take $u = td$ with $d \in \mathbf{R}^m$ fixed

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b + td \\ & x \geq 0 \end{array}$$

$p^\star(td)$ is the optimal value as a function of t



Global sensitivity

Example

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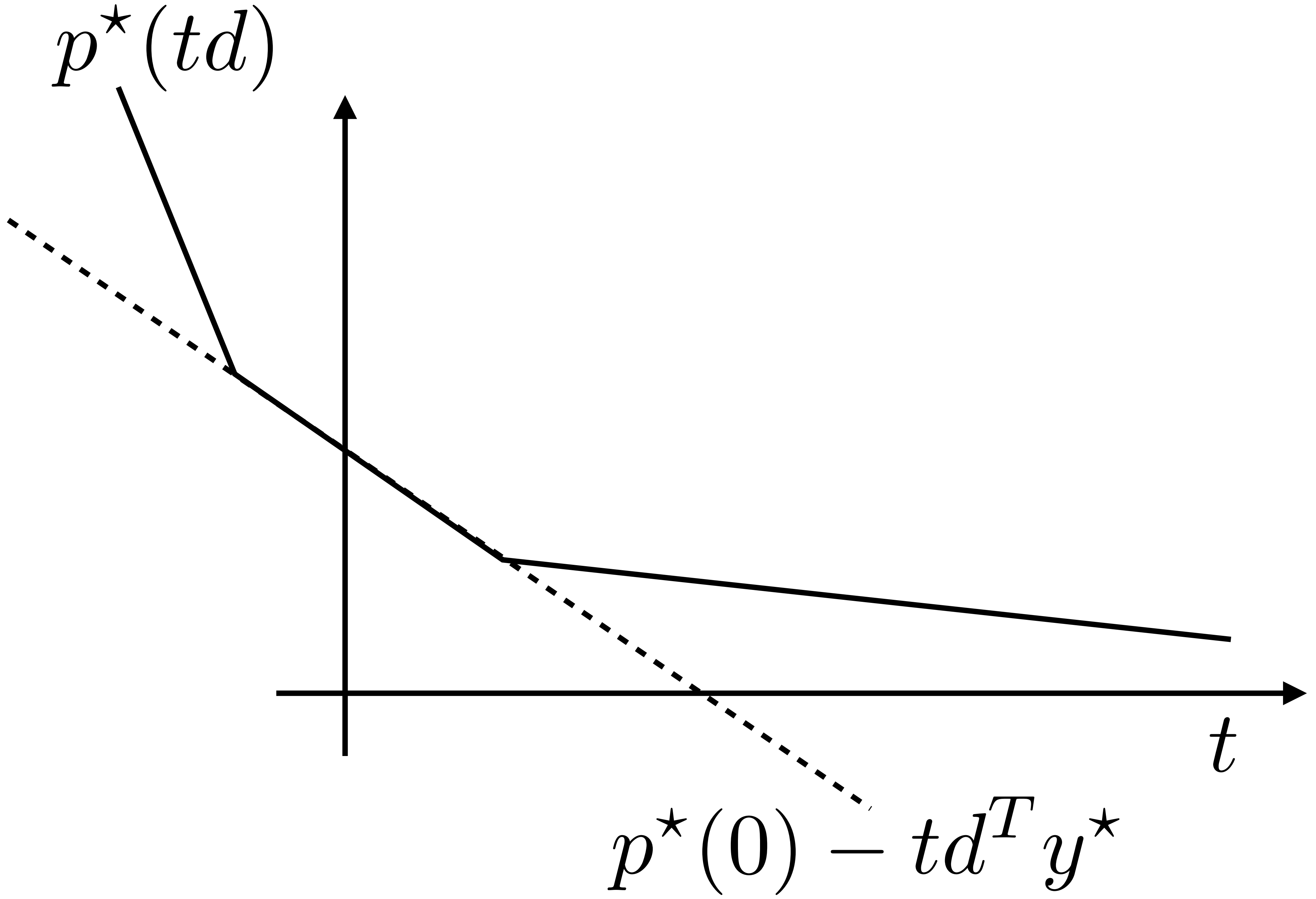

Sensitivity information (assuming $d^T y^\star \geq 0$)

- $t < 0$ the optimal value increases
- $t > 0$ the optimal value decreases (not so much if t is small)

Optimal value function

$$p^\star(u) = \min\{c^T x \mid Ax = b + u,\ x \geq 0\}$$

Assumption: $p^\star(0)$ is finite

Properties

- $p^\star(u) > -\infty$ everywhere (from global lower bound)
- the domain $\{u \mid p^\star(u) < +\infty\}$ is a polyhedron
- $p^\star(u)$ is piecewise-linear on its domain

Optimal value function is piecewise linear

Proof

$$p^\star(u) = \min\{c^T x \mid Ax = b + u,\ x \geq 0\}$$

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Dual feasible set

$$p^\star(u) = \min\{c^T x \mid Ax = b + u,\ x \geq 0\}$$

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Assumption: $p^\star(0)$ is finite

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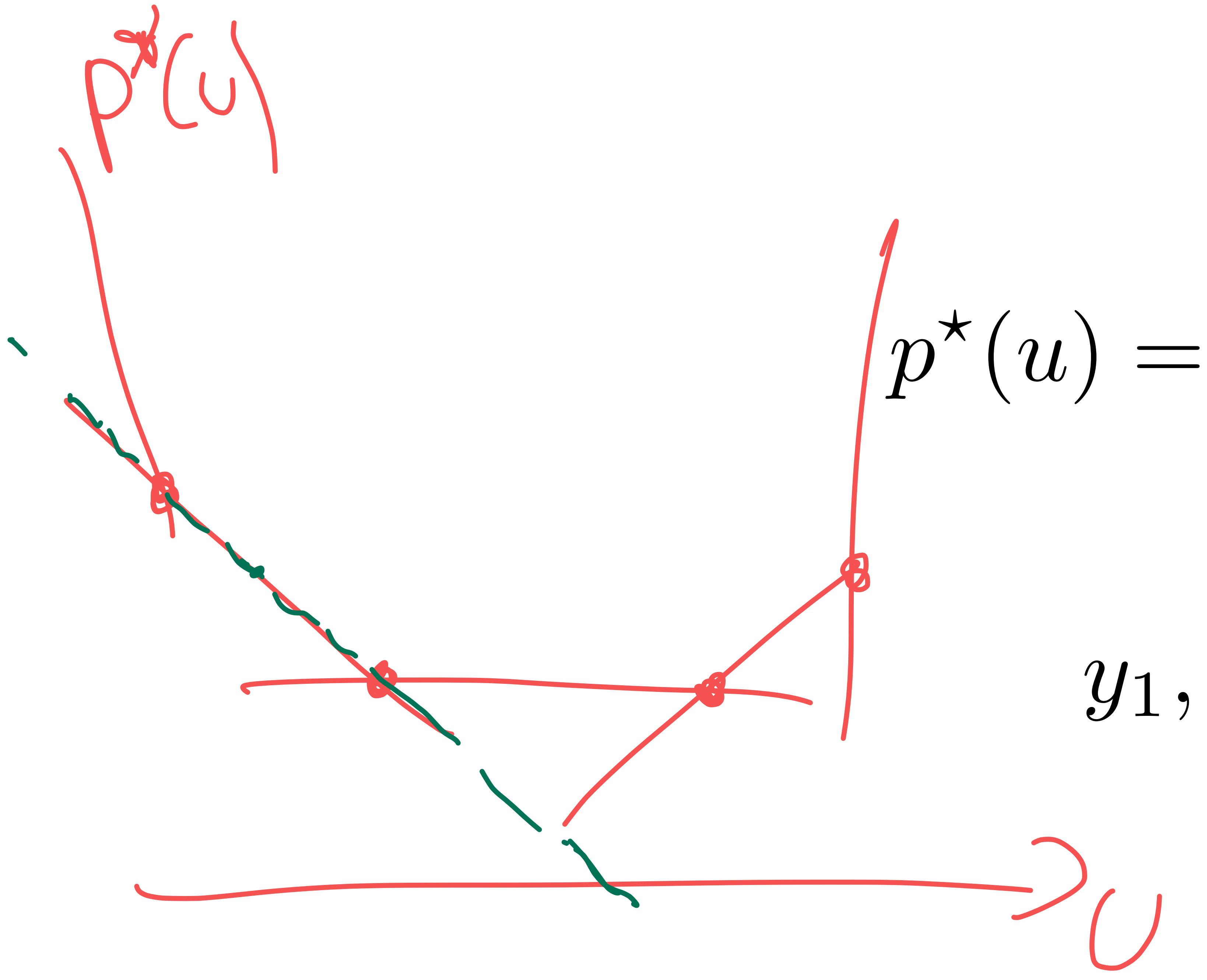

If $p^\star(u)$ finite

$$p^\star(u) = \max_{y \in D} -(b + u)^T y = \max_{k=1,\dots,r} -y_k^T u - b^T y_k$$

$y_1, \dots, y_r$ are the extreme points of D

Local sensitivity analysis

Local sensitivity

u in neighborhood of the origin

Original LP

minimize $\quad c^T x$

subject to $\quad Ax = b$

$\quad x \geq 0$

$\longrightarrow$

Optimal solution

Primal $\quad x_i = 0, \quad i \notin B$

$x_B^\star = A_B^{-1} b$

Dual $\quad y^\star = -A_B^{-T} c_B$

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Modified LP

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Modified dual

maximize $\quad -(b + u)^T y$

subject to $\quad A^T y + c \geq 0$

Optimal basis does not change

Local sensitivity

u in neighborhood of the origin

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Optimal solution

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$$x_i = 0, \quad i \notin B$$
$$x_B^\star = A_B^{-1} b$$

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Optimal basis does not change

Modified optimal solution

$$x_B^\star(u) = A_B^{-1}(b+u) = x_B^\star + \boxed{A_B^{-1} u}$$
$$y^\star(u) = y^\star$$

Derivative of the optimal value function

Modified optimal solution

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Optimal value function

$$\begin{aligned} p^\star(u) &= c^T x^\star(u) \\ &= c^T x^\star + \boxed{c_B^T A_B^{-1} u} \\ &= p^\star(0) - y^{\star T} u \end{aligned}$$

(affine for small u)

Derivative of the optimal value function

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(affine for small u)

Local derivative

$$\frac{\partial p^\star(u)}{\partial u} = -y^\star$$

($y^\star$ are the **shadow prices**)

Sensitivity example

$$
\begin{array}{ll}
\text{minimize} & -60x_1 - 30x_2 - 20x_3 \\
\text{subject to} & 8x_1 + 6x_2 + x_3 \leq 48 \\
& 4x_1 + 2x_2 + 1.5x_3 \leq 20 \\
& 2x_1 + 1.5x_2 + 0.5x_3 \leq 8 \\
& x \geq 0
\end{array}
$$

Sensitivity example

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-profit

Sensitivity example

minimize	$-60x_1 - 30x_2 - 20x_3$	-profit
subject to	$8x_1 + 6x_2 + x_3 \leq 48$	material
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$x^\star = (2, 0, 8, 24, 0, 0), \quad y^\star = (0, 10, 10), \quad c^T x^\star = -280,$ basis $\{1, 3, 4\}$

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$x^\star = (2, 0, 8, 24, 0, 0), \quad y^\star = (0, 10, 10), \quad c^T x^\star = -280,$ basis $\{1, 3, 4\}$

What does $y_3^\star = 10$ mean?

Sensitivity example

$$b = \begin{bmatrix} 48 \\ 20 \\ 8 \end{bmatrix}$$

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subject to	$8x_1 + 6x_2 + x_3 \leq 48$	material
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$$x^\star = (2, 0, 8, 24, 0, 0), \quad y^\star = (0, 10, 10), \quad c^T x^\star = -280, \quad \text{basis } \{1, 3, 4\}$$

What does $y_3^\star = 10$ mean?

Let's increase the quality control budget by 1, i.e., $u = (0, 0, 1)$ $b+u$

$$p^\star(u) = p^\star(0) - y^{\star T} u = -280 - 10 = -290$$

Differentiable optimization

Training a neural network

Single layer model

$$x = f(\theta)$$

Training a neural network

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Training

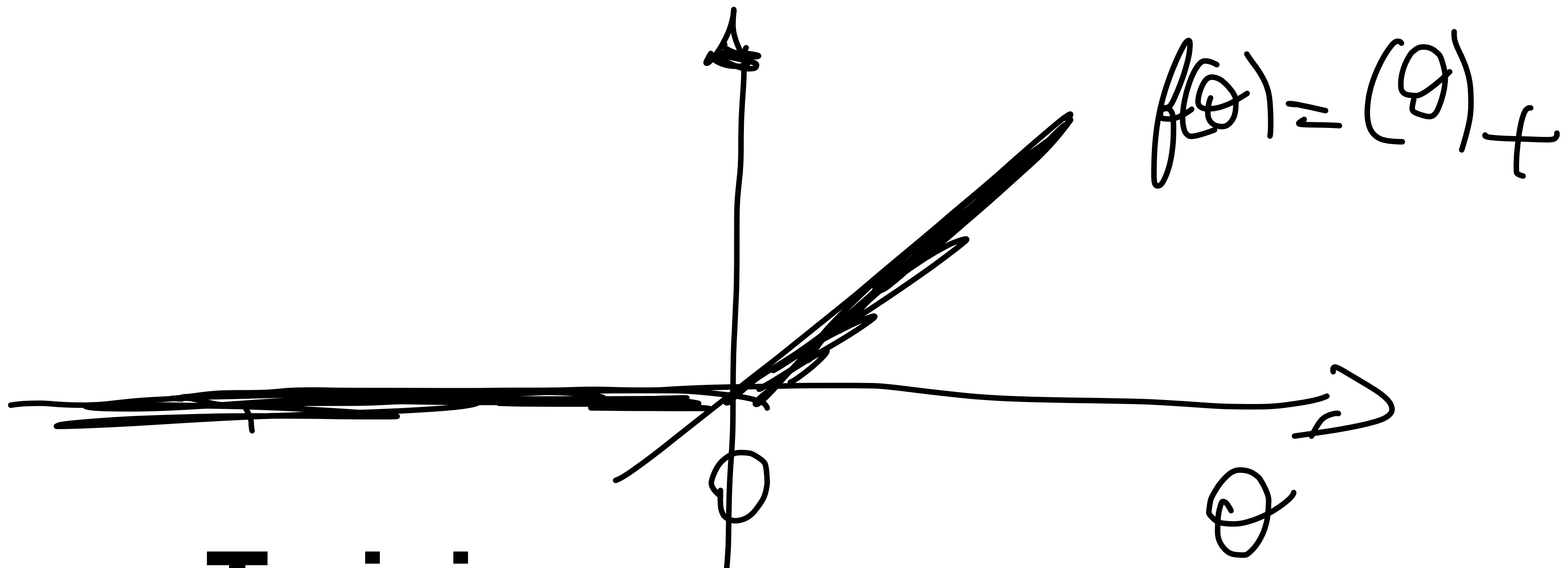
minimize $\quad \mathcal{L}(\theta)$

Gradient descent (more on this later)

$$\theta \leftarrow \theta - t\nabla_\theta \mathcal{L}(\theta)$$

Training a neural network

Single layer model



$$x = f(\theta)$$

$\theta \longrightarrow$ Layer $\longrightarrow x$

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minimize $\mathcal{L}(\theta)$

Gradient descent (more on this later)

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Sensitivity

$$\nabla_\theta \mathcal{L} = \left(\frac{\partial \mathcal{L}}{\partial \theta}\right)^T = \left(\frac{\partial \mathcal{L}}{\partial x}\frac{\partial x}{\partial \theta}\right)^T = \left(\frac{\partial x}{\partial \theta}\right)^T \nabla_x \mathcal{L}$$

Training a neural network

Single layer model

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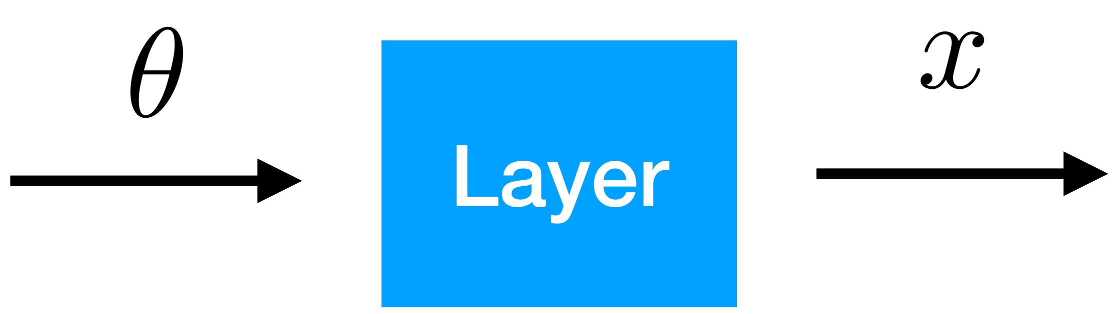

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Can f be an **optimization problem**?

Implicit layers

https://implicit-layers-tutorial.org/

find	$x(\theta)$
subject to	$r(\theta, x(\theta)) = 0$

($x(\theta)$ is implicitly defined by r)

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$$\frac{\partial x(\theta)}{\partial \theta}$$

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Implicit function theorem

Under mild assumptions (non-singularity),

$$\frac{\partial r(\theta, x(\theta))}{\partial x}\frac{\partial x(\theta)}{\partial \theta} + \frac{\partial r(\theta, x(\theta))}{\partial \theta} = 0 \longrightarrow \frac{\partial x(\theta)}{\partial \theta} = -\left(\frac{\partial r(\theta, x(\theta))}{\partial x}\right)^{-1} \frac{\partial r(\theta, x(\theta))}{\partial \theta}$$

[Theorem 1. B.1, Dontchev and Rockafellar 2009]

Optimization layers

$$x^\star(\theta) = \underset{x}{\text{argmin}} \quad c^T x \qquad \text{subject to} \quad Ax \leq b$$

Parameters: $\theta = \{c, A, b\}$
Solution $x^\star(\theta)$

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Features

- Add **domain knowledge** and **hard constraints**
- **End-to-end** training and optimization
- Nice theory and algorithms for general **convex optimization**
- **Applications** in RL, control, meta-learning, game theory, etc.

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Features

- Add **domain knowledge** and **hard constraints**
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Goal

Compute $\dfrac{\partial x^\star(\theta)}{\partial \theta}$

Optimality conditions

minimize $\quad c^T x$
subject to $\quad Ax \leq b$

Parameters: $\theta = \{c, A, b\}$
Solution $x^\star(\theta)$

Solve and obtain primal-dual pair $x^\star, y^\star$ (forward-pass)

Optimality conditions

$$\begin{aligned} A^T y + c &= 0 \\ \mathbf{diag}(y)(Ax - b) &= 0 \\ y \geq 0,\ b - Ax &\geq 0 \end{aligned}$$

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Mapping $r(\theta, x(\theta)) = 0$

Computing derivatives

Take differentials

$$\begin{aligned} A^T y^\star + c &= 0 \\ \mathbf{diag}(y^\star)(Ax - b) &= 0 \end{aligned} \quad \longrightarrow \quad \begin{aligned} \mathrm{d}A^T y^\star + A^T \mathrm{d}y &= 0 \\ \mathbf{diag}(Ax^\star - b)\mathrm{d}y + \mathbf{diag}(y^\star)(\mathrm{d}Ax^\star + A\mathrm{d}x - \mathrm{d}b) + \mathrm{d}c &= 0 \end{aligned}$$

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Linear system

$$\begin{bmatrix} 0 & A^T \\ \mathbf{diag}(y^\star)A & \mathbf{diag}(Ax^\star - b) \end{bmatrix} \begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \end{bmatrix} = - \begin{bmatrix} \mathrm{d}A^T y^\star + \mathrm{d}c \\ \mathbf{diag}(y^\star)(\mathrm{d}A x^\star - \mathrm{d}b) \end{bmatrix}$$

Computing derivatives

Take differentials

$$A^T y^\star + c = 0 \qquad \longrightarrow \qquad \mathrm{d}A^T y^\star + A^T \mathrm{d}y = 0$$

$$\mathbf{diag}(y^\star)(Ax - b) = 0 \qquad \mathbf{diag}(Ax - b)\mathrm{d}y + \mathbf{diag}(y^\star)(\mathrm{d}Ax^\star + A\mathrm{d}x - \mathrm{d}b) + \mathrm{d}c = 0$$

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Example: How does $x^\star$ change with b_1?

Set $\mathrm{d}b = e_1, \mathrm{d}A = 0, \mathrm{d}c = 0$ and solve the linear system.

The solution $\mathrm{d}x$ will correspond to $\dfrac{\partial x}{\partial b_1}$

Is it always differentiable?

The linear system matrix must be invertible
(the problem must have unique solution)

$$
\underset{M}{\begin{bmatrix} 0 & A^T \\ \mathbf{diag}(y^\star)A & \mathbf{diag}(Ax^\star - b) \end{bmatrix}} \begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \end{bmatrix} = - \underset{q}{\begin{bmatrix} \mathrm{d}A^T y^\star + \mathrm{d}c \\ \mathbf{diag}(y^\star)(\mathrm{d}Ax^\star - \mathrm{d}b) \end{bmatrix}}
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Remember. implicit function theorem

$$\frac{\partial x(\theta)}{\partial \theta} = -\left(\frac{\partial r(\theta, x(\theta))}{\partial x}\right)^{-1} \frac{\partial r(\theta, x(\theta))}{\partial \theta}$$

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Remember. implicit function theorem

$$\frac{\partial x(\theta)}{\partial \theta} = -\left(\frac{\partial r(\theta, x(\theta))}{\partial x}\right)^{-1} \frac{\partial r(\theta, x(\theta))}{\partial \theta}$$

If not, **least squares “**subdifferential”

$$\text{minimize} \quad \left\| M \begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \end{bmatrix} + q \right\|_2^2$$

Example

Learning to play Sudoku

			3
1			
		4	
4			1

2	4	1	3
1	3	2	4
3	1	4	2
4	2	3	1

Sudoku constraint satisfaction problem

$$\begin{array}{ll} \text{minimize} & 0 \\ \text{subject to} & Ax = b \\ & x \geq 0,\ x \in \mathbf{Z}^d \end{array}$$

[OptNet: Differentiable Optimization as a Layer in Neural Networks, B. Amos and J. Z. Kolter ICML 2017]

Example

Learning to play Sudoku

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Linear optimization layer (parameters $\theta = \{A, b\}$)

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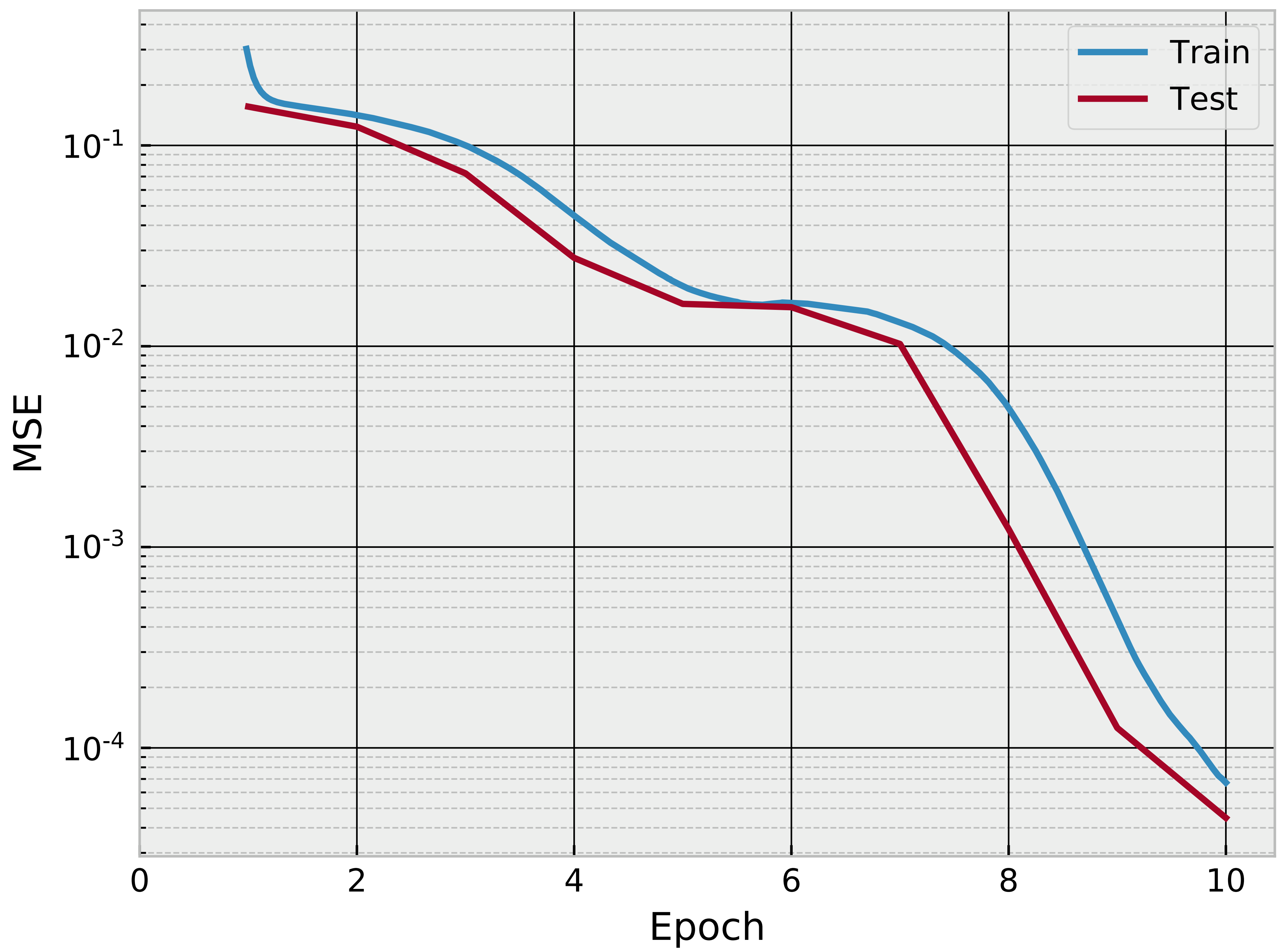

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Sensitivity analysis in linear optimization

Today, we learned to:

- **Use** the most appropriate primal/dual simplex algorithm when variables and/or constraints are added
- **Analyze** sensitivity of the cost with respect to change in the data
- **Apply** sensitivity analysis to differentiable linear optimization layers

Next lecture

- Barrier methods for linear optimization