ORF522 – Linear and Nonlinear Optimization

8. Linear optimization duality

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Ed Forum

- Why do we need to solve dual problem instead of the primal problem? When we have a LP problem, in what scenario does solving dual problem more efficient than primal problem?
- How does the definition of y imply nonnegative reduced costs?

Recap

Optimal objective values

Primal

minimize $c^T x$

subject to $Ax \leq b$

Dual

maximize $-b^T y$

subject to $A^T y + c = 0$

$y \geq 0$

$p^\star$ is the primal optimal value

$d^\star$ is the dual optimal value

Primal infeasible: $p^\star = +\infty$
Primal unbounded: $p^\star = -\infty$

Dual infeasible: $d^\star = -\infty$
Dual unbounded: $d^\star = +\infty$

Relationship between primal and dual

	$p^\star = +\infty$	$p^\star$ finite	$p^\star = -\infty$
$d^\star = +\infty$	primal inf. dual unb.		
$d^\star$ finite		optimal values equal	
$d^\star = -\infty$	exception		primal unb. dual inf

- Upper-right excluded by **weak duality**
- $(1, 1)$ and $(3, 3)$ proven by **weak duality**
- $(3, 1)$ and $(2, 2)$ proven by **strong duality**

Today's agenda

Readings: [Chapter 4, LO][Chapter 11, LP]

- Two-person zero-sum games
- Farkas lemma
- Complementary slackness
- Dual simplex method

Two-person zero-sum games

Rock paper scissors

Rules

At count to three declare one of: Rock, Paper, or Scissors

Winners

Identical selection is a draw, otherwise:

- Rock beats (“dulls”) scissors
- Scissors beats (“cuts”) paper
- Paper beats (“covers”) rock

Extremely popular: world RPS society, USA RPS league, etc.

Two-person zero-sum game

- Player 1 (P1) chooses a number $i \in \{1, \ldots, m\}$ (one of m actions)
- Player 2 (P2) chooses a number $j \in \{1, \ldots, n\}$ (one of n actions)

Two players make their choice independently

Rule

Player 1 pays A_{ij} to player 2

$A \in \mathbf{R}^{m \times n}$ is the **payoff matrix**

Rock, Paper, Scissors

$$A = \begin{array}{c} \\ \mathrm{R} \\ \mathrm{P} \\ \mathrm{S} \end{array} \begin{array}{c} \begin{array}{ccc} \mathrm{R} & \mathrm{P} & \mathrm{S} \end{array} \\ \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \end{array}$$

Mixed (randomized) strategies

Deterministic strategies can be **systematically defeated**

Randomized strategies

- P1 chooses randomly according to distribution x:

  $x_i =$ probability that P1 selects action i

- P2 chooses randomly according to distribution y:

  $y_i =$ probability that P2 selects action j

Expected payoff (from P1 P2), if they use mixed-strategies x and y,

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j A_{ij} = x^T A y$$

Mixed strategies and probability simplex

Probability simplex in $\mathbf{R}^k$

$P_k = \{p \in \mathbf{R}^k \mid p \geq 0, \quad \mathbf{1}^T p = 1\}$

Mixed strategy

For a game player, a mixed strategy is a distribution over all possible deterministic strategies.

The **set of all mixed strategies** is the probability simplex $\longrightarrow$ $x \in P_m, \quad y \in P_n$

Optimal mixed strategies

P1: optimal strategy $x^\star$ is the solution of

$$\begin{array}{ll} \text{minimize} & \max_{y \in P_n} x^T A y \\ \text{subject to} & x \in P_m \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & \max_{j=1,\dots,n} (A^T x)_j \\ \text{subject to} & x \in P_m \end{array}$$

Inner problem over deterministic strategies (**vertices**)

P2: optimal strategy $y^\star$ is the solution of

$$\begin{array}{ll} \text{maximize} & \min_{x \in P_m} x^T A y \\ \text{subject to} & y \in P_n \end{array} \longrightarrow \begin{array}{ll} \text{maximize} & \min_{i=1,\dots,m} (A y)_i \\ \text{subject to} & y \in P_n \end{array}$$

Optimal strategies $x^\star$ and $y^\star$ can be computed using **linear optimization**

Minmax theorem

Theorem

$$\max_{y \in P_n} \min_{x \in P_m} x^T A y = \min_{x \in P_m} \max_{y \in P_n} x^T A y$$

Proof

The optimal $x^\star$ is the solution of

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & A^T x \leq t\mathbf{1} \\ & \mathbf{1}^T x = 1 \\ & x \geq 0 \end{array}$$

The optimal $y^\star$ is the solution of

$$\begin{array}{ll} \text{maximize} & w \\ \text{subject to} & Ay \geq w\mathbf{1} \\ & \mathbf{1}^T y = 1 \\ & y \geq 0 \end{array}$$

The two LPs are **duals** and by **strong duality** the equality follows. ■

Nash equilibrium

Theorem

$$\max_{y \in P_n} \min_{x \in P_m} x^T A y = \min_{x \in P_m} \max_{y \in P_n} x^T A y$$

Consequence

The pair of mixed strategies $(x^\star, y^\star)$ attains the **Nash equilibrium** of the two-person matrix game, i.e.,

$$x^T A y^\star \geq x^{\star T} A y^\star \geq x^{\star T} A y, \quad \forall x \in P_m,\ \forall y \in P_n$$

Example

$$A = \begin{bmatrix} 4 & 2 & 0 & -3 \\ -2 & -4 & -3 & 3 \\ -2 & -3 & 4 & 1 \end{bmatrix}$$

$$\min_i \max_j A_{ij} = 3 > -2 = \max_j \min_i A_{ij}$$

Optimal mixed strategies

$$x^\star = (0.37, 0.33, 0.3), \qquad y^\star = (0.4, 0, 0.13, 0.47)$$

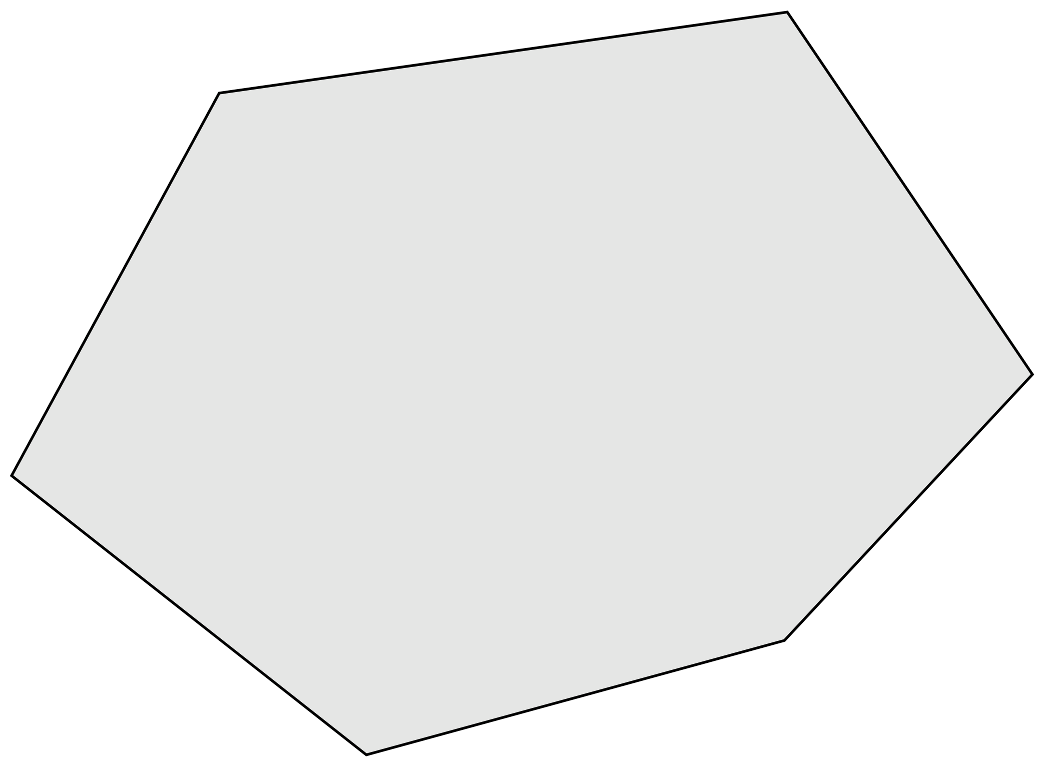
Expected payoff

$$x^{\star T} A y^\star = 0.2$$

Farkas lemma

Feasibility of polyhedra

$$P = \{x \mid Ax = b, \quad x \geq 0\}$$



How to show that P is **feasible**?

Easy: we just need to provide an $x \in P$, i.e., a **certificate**

How to show that P is **infeasible**?

Farkas lemma

Theorem

Given A and b, exactly one of the following statements is true:

1. There exists an x with $Ax = b$, $x \geq 0$
2. There exists a y with $A^T y \geq 0$, $b^T y < 0$

Farkas lemma

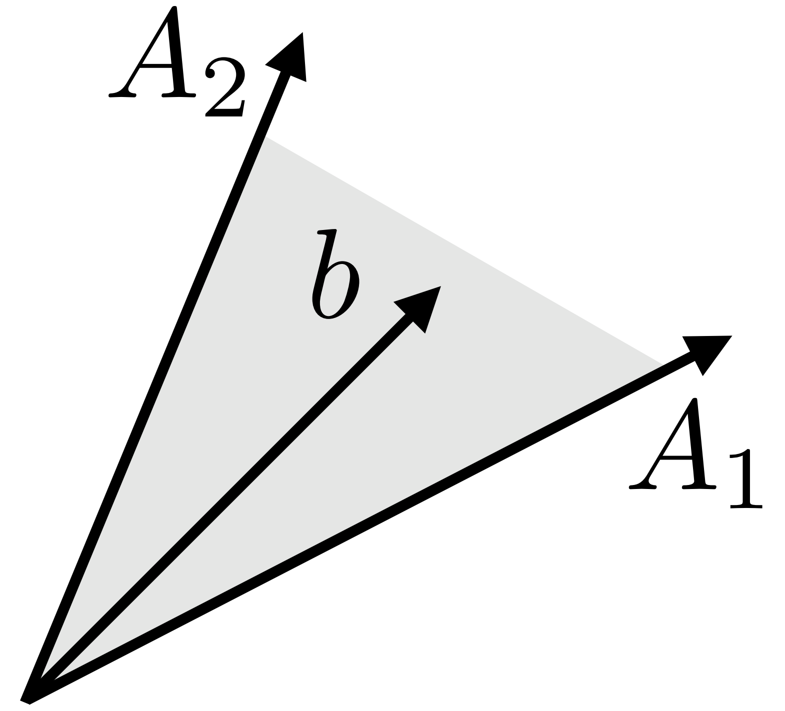
Geometric interpretation

1. First alternative

There exists an x with $Ax = b$, $x \geq 0$

$$b = \sum_{i=1}^{n} x_i A_i, \quad x_i \geq 0,\ i = 1, \ldots, n$$

b is in the cone generated by the columns of A

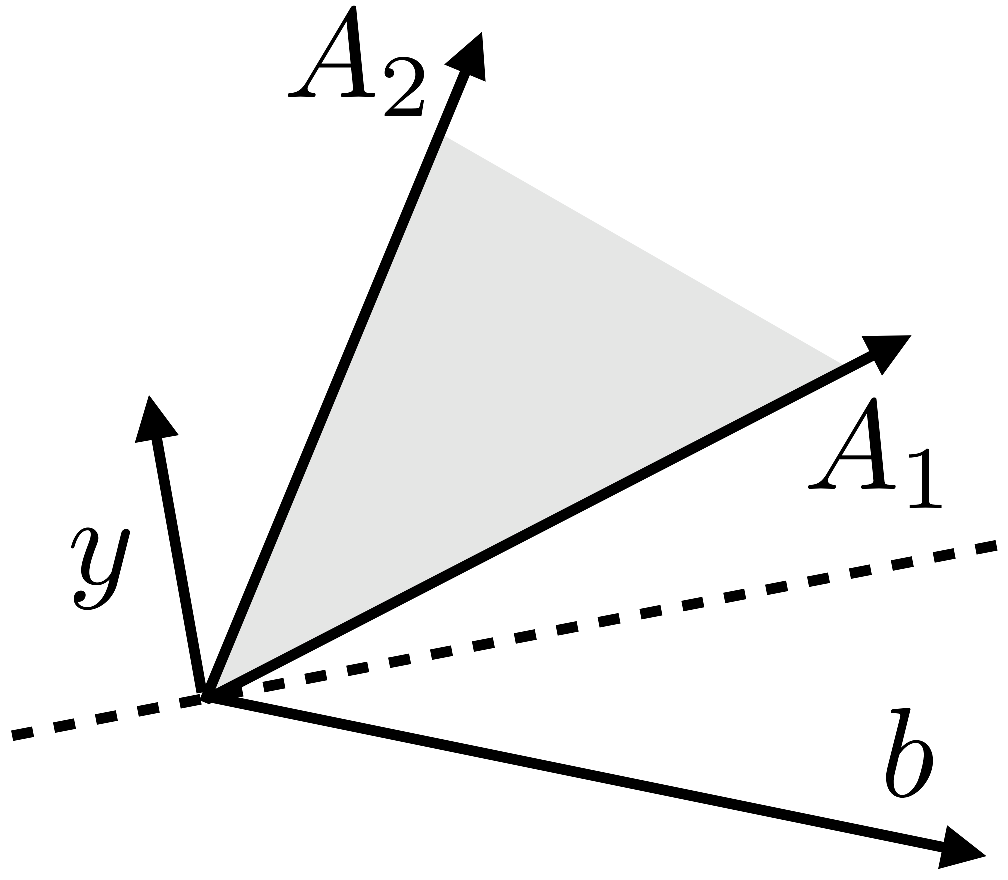


2. Second alternative

There exists a y with $A^T y \geq 0$, $b^T y < 0$

$$y^T A_i \geq 0, \quad i = 1, \ldots, m, \qquad y^T b < 0$$

The hyperplane $y^T z = 0$ separates b from $A_1, \ldots, A_n$



Farkas lemma

There exists x with $Ax = b,\ x \geq 0$ **OR** There exists y with $A^T y \geq 0,\ b^T y < 0$

Proof

1 and 2 cannot be both true (easy)

$x \geq 0$, $Ax = b$ and $y^T A \geq 0$ ⟶ $y^T b = y^T Ax \geq 0$

Farkas lemma

There exists x with $Ax = b,\ x \geq 0$ **OR** There exists y with $A^T y \geq 0,\ b^T y < 0$

Proof

1 and 2 cannot be both false (duality)

Primal

$$\begin{array}{ll} \text{minimize} & 0 \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Dual

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y \geq 0 \end{array}$$

$y = 0$ always feasible

Strong duality holds

$$d^\star \neq -\infty, \quad p^\star = d^\star$$

Farkas lemma

There exists x with $Ax = b,\ x \geq 0$ **OR** There exists y with $A^T y \geq 0,\ b^T y < 0$

Proof

1 and 2 cannot be both false (duality)

Primal		**Dual**	
minimize	0	maximize	$-b^T y$
subject to	$Ax = b$	subject to	$A^T y \geq 0$
	$x \geq 0$		

Alternative 1: primal feasible $p^\star = d^\star = 0$

$b^T y \geq 0$ for all y such that $A^T y \geq 0$

Farkas lemma

There exists x with $Ax = b,\ x \geq 0$ **OR** There exists y with $A^T y \geq 0,\ b^T y < 0$

Proof

1 and 2 cannot be both false (duality)

Primal

minimize 0
subject to $Ax = b$
$x \geq 0$

Dual

maximize $-b^T y$
subject to $A^T y \geq 0$

Alternative 2: primal infeasible $p^\star = d^\star = +\infty$

There exists y such that $A^T y \geq 0$ and $b^T y < 0$

y is an **infeasibility certificate**

Farkas lemma

Many variations

There exists x with $Ax = b,\ x \geq 0$

OR

There exists y with $A^T y \geq 0,\ b^T y < 0$

There exists x with $Ax \leq b,\ x \geq 0$

OR

There exists y with $A^T y \geq 0,\ b^T y < 0,\ y \geq 0$

There exists x with $Ax \leq b$

OR

There exists y with $A^T y = 0,\ b^T y < 0,\ y \geq 0$

Complementary slackness

Optimality conditions

Primal

minimize $c^T x$

subject to $Ax \leq b$

Dual

maximize $-b^T y$

subject to $A^T y + c = 0$

$y \geq 0$

x and y are **primal** and **dual** optimal if and only if

- x is **primal feasible**: $Ax \leq b$
- y is **dual feasible:** $A^T y + c = 0$ and $y \geq 0$
- The **duality gap** is zero: $c^T x + b^T y = 0$

Can we **relate** x and y (not only the objective)?

Complementary slackness

Primal

minimize $\quad c^T x$

subject to $\quad Ax \leq b$

Dual

maximize $\quad -b^T y$

subject to $\quad A^T y + c = 0$

$\quad y \geq 0$

Theorem

Primal,dual feasible x, y are optimal if and only if

$$y_i(b_i - a_i^T x) = 0, \quad i = 1, \ldots, m$$

i.e., at optimum, $b - Ax$ and y have a **complementary sparsity** pattern:

$$y_i > 0 \quad \Rightarrow \quad a_i^T x = b_i$$
$$a_i^T x < b_i \quad \Rightarrow \quad y_i = 0$$

Complementary slackness

Primal

minimize $c^T x$

subject to $Ax \leq b$

Dual

maximize $-b^T y$

subject to $A^T y + c = 0$

$y \geq 0$

Proof

The duality gap at primal feasible x and dual feasible y can be written as

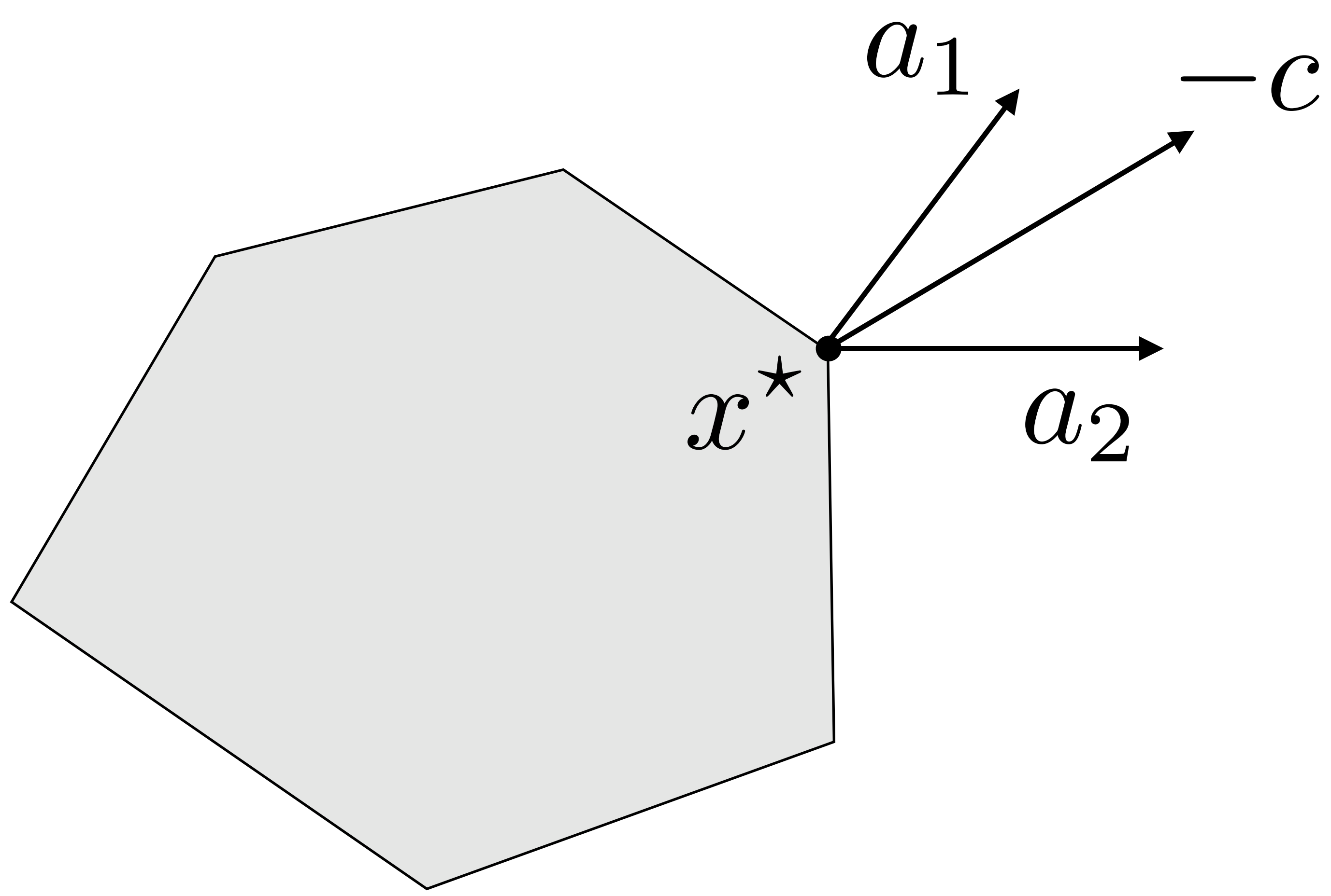
$$c^T x + b^T y = (-A^T y)^T x + b^T y = (b - Ax)^T y = \sum_{i=1}^m y_i (b_i - a_i^T x) = 0$$

Since all the elements of the sum are nonnegative, they must all be 0 ■

For **feasible** x and y **complementary slackness** = **zero duality gap**

Geometric interpretation

Example in $\mathbb{R}^2$



Two active constraints at optimum: $a_1^T x^\star = b_1, \quad a_2^T x^\star = b_2$

Optimal dual solution y satisfies:

$$A^T y + c = 0, \quad y \geq 0, \quad y_i = 0 \text{ for } i \neq \{1, 2\}$$

In other words, $-c = a_1 y_1 + a_2 y_2$ with $y_1, y_2 \geq 0$

Geometric interpretation: $-c$ lies in the cone generated by a_1 and a_2

Example

$$\begin{array}{ll} \text{minimize} & -4x_1 - 5x_2 \\ \text{subject to} & \begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 3 \\ 0 \\ 3 \end{bmatrix} \end{array}$$

Let's **show** that feasible $x = (1, 1)$ is optimal

Second and fourth constraints are active at x $\longrightarrow$ $y = (0, y_2, 0, y_4)$

$$A^T y = -c \quad \Rightarrow \quad \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_2 \\ y_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

and

$$y_2 \geq 0, \quad y_4 \geq 0$$

$y = (0, 1, 0, 2)$ satisfies these conditions and proves that x is optimal

Complementary slackness is useful to recover $y^\star$ from $x^\star$

The dual simplex

Primal and dual basic feasible solutions

Primal problem

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Dual problem

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c \geq 0 \end{array}$$

Given a **basis** matrix A_B

Primal feasible: $Ax = b,\ x \geq 0 \quad \Rightarrow \quad x_B = A_B^{-1} b \geq 0$

Reduced costs

Dual feasible: $A^T y + c \geq 0$. If $y = -A_B^{-T} c_B \quad \Rightarrow \quad c - A^T A_B^{-T} c_B \geq 0$

Zero duality gap: $c^T x + b^T y = c_B^T x_B - b^T A_B^{-T} c_B = c_B^T x_B - c_B^T A_B^{-1} b = 0$

(by construction)

The primal (dual) simplex method

Primal problem

minimize $c^T x$

subject to $Ax = b$

$x \geq 0$

Dual problem

maximize $-b^T y$

subject to $A^T y + c \geq 0$

Primal simplex

- Primal feasibility
- Zero duality gap

↓

Dual feasibility

Dual simplex

- Dual feasibility
- Zero duality gap

↓

Primal feasibility

Feasible dual directions

Conditions

$$P = \{y \mid A^T y + c \geq 0\}$$

Given a basis matrix $A_B = \begin{bmatrix} A_{B(1)} & \dots & A_{B(m)} \end{bmatrix}$
we have dual feasible solution y:

$$\bar{c} = A^T y + c \geq 0$$

Feasible direction d

$$y + \theta d$$

Reduced cost change

$$c + A^T(y + \theta d) \geq 0 \quad \Rightarrow \quad \bar{c} + \theta z \geq 0$$

$A^T d = z$ (subspace restriction)

Feasible directions

Computation

Subspace restriction

$$\bar{c} + \theta z \geq 0$$

$$A^T d = z \longrightarrow \begin{array}{l} A_B^T d = z_B \\ A_N^T d = z_N \end{array}$$

Basic indices

$z_B = e_i \longrightarrow B(\ell) = i$ **exits the basis**

Get d by solving $A_B^T d = z_B$

Nonbasic indices

$$z_N = A_N^T d = A_N^T A_B^{-T} e_i$$

Non-negativity of reduced costs (non-degenerate assumption)

- Basic variables: $\bar{c}_B = 0$. Nonnegative direction $z_B \geq 0$.
- Nonbasic variables: $\bar{c}_N > 0$. Therefore $\exists \theta > 0$ such that $\bar{c}_N + \theta z_N \geq 0$

Stepsize

How far can we go?

$$\theta^\star = \max\{\theta \mid \theta \geq 0 \text{ and } \bar{c} + \theta z \geq 0\}$$

Unbounded

If $z \geq 0$, then $\theta^\star = \infty$. The dual problem is unbounded (primal infeasible).

Bounded

If $z_j < 0$ for some j, then $\theta^\star = \min_{\{j \mid z_j < 0\}} \left(-\frac{\bar{c}_j}{z_j}\right) = \min_{\{j \in N \mid z_j < 0\}} \left(-\frac{\bar{c}_j}{z_j}\right)$

(Since $z_j \geq 0,\ j \in B$)

Moving to a new basis

Next reduced cost

$$\bar{c} + \theta^\star z$$

Let $j \notin \{B(1), \ldots, B(m)\}$ be the index such that $\theta^\star = -\dfrac{\bar{c}_j}{z_j}$. Then,

$$\bar{c}_j + \theta^\star z_j = 0$$

New basis

$$A_{\bar{B}} = \begin{bmatrix} A_{B(1)} & \ldots & A_{B(\ell-1)} & A_j & A_{B(\ell+1)} & \ldots & A_{B(m)} \end{bmatrix}$$

New solution

$$A_{\bar{B}} x_{\bar{B}} = b$$

An iteration of the dual simplex method

Remark

Reduced costs nonnegative

↓

objective non-decreasing

Initialization

- a basic dual feasible solution y, i.e. $A^T y + c \geq 0$
- a basis matrix $A_B = \begin{bmatrix} A_{B(1)} & \ldots, A_{B(m)} \end{bmatrix}$

Iteration steps

1. Get x
    - Solve $A_B x_B = b$ ($O(m^2)$)
    - Set $x_i = 0$ if $i \notin B$
2. If $x \geq 0$, x **feasible. break**
3. Choose i such that $x_i < 0$
4. Compute each direction z with $z_i = 1$, $A_B^T d = e_i$ and $z_N = A_N^T d$ ($O(m^2)$)
5. If $z_N \geq 0$, the dual problem is **unbounded** and the optimal value is $+\infty$. **break**
6. Compute step length $\theta^\star = \min_{\{j \in N | z_j < 0\}} \left(-\frac{\bar{c}_j}{z_j} \right)$
7. Compute new point $y + \theta^\star d$
8. Get new basis $A_{\bar{B}} = A_B + (A_j - A_i)e_\ell^T$ perform rank-1 factor update (j enters, i exists) $O(m^2)$

Example

From lecture 6

$$c = (-10, -12, -12, 0, 0, 0)$$

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$b = (20, 20, 20)$$

minimize $c^T x$

subject to $Ax = b$

$x \geq 0$

Dual problem

maximize $-b^T y$

subject to $A^T y + c \geq 0$

Initialize

$y = (10, 0, 0) \qquad B = \{1, 5, 6\}$

$$c + A^T y = (0, 8, 8, 10, 0, 0) \geq 0$$

Example

Iteration 1

$$y = (10, 0, 0)$$
$$-b^T y = -200$$
$$c + A^T y = (0, 8, 8, 10, 0, 0)$$
$$B = \{1, 5, 6\}$$
$$A_B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$c = (-10, -12, -12, 0, 0, 0)$$
$$A = \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{bmatrix}$$
$$b = (20, 20, 20)$$

Primal solution $x = (20, 0, 0, 0, -20, -20)$

Solve $Ax_B = b \quad \Rightarrow \quad x_B = (20, -20, -20)$

Direction $z = (0, -3, -2, -2, 1, 0), \quad i = 5$

Solve $A_B^T d = e_i \quad \Rightarrow \quad d = (-2, 1, 0)$

Get $z_N = A_N^T d = (-3, -2, -2)$

Step $\theta^\star = 2.66, \quad j = 2$

$$\theta^\star = \min_{\{j | z_j < 0\}} (-\bar{c}_j / z_j) = \{2.66, 4, 5\}$$

New $y \leftarrow y + \theta^\star d = (4.66, 2.66, 0)$

Example
Iteration 2

$$y = (4.66, 2.66, 0)$$
$$-b^T y = -146.66$$
$$c + A^T y = (0, 0, 2.66, 4.66, 2.66, 0)$$
$$B = \{1, 2, 6\}$$
$$A_B = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

$$c = (-10, -12, -12, 0, 0, 0)$$
$$A = \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{bmatrix}$$
$$b = (20, 20, 20)$$

Primal solution $x = (6.66, 6.66, 0, 0, 0, -6.66)$

Solve $Ax_B = b \quad \Rightarrow \quad x_B = (6.66, 6.66, -6.66)$

Direction $z = (0, 0, -1.66, -0.66, -0.66, 1), \quad i = 6$

Solve $A_B^T d = e_i \quad \Rightarrow \quad d = (-0.66, -0.66, 1)$

Get $z_N = A_N^T d = (-1.66, -0.66, -0.66)$

Step $\theta^\star = 1.6, \quad j = 3$

$$\theta^\star = \min_{\{j | z_j < 0\}} (-\bar{c}_j / z_j) = \{1.6, 7, 4\}$$

New $y \leftarrow y + \theta^\star d = (3.6, 1.6, 1.6)$

Example

Iteration 3

$$y = (3.6, 1.6, 1.6)$$
$$-b^T y = -136$$
$$c + A^T y = (0, 0, 0, 3.6, 1.6, 1.6)$$
$$B = \{1, 2, 3\}$$
$$A_B = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$c = (-10, -12, -12, 0, 0, 0)$$
$$A = \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{bmatrix}$$
$$b = (20, 20, 20)$$

Primal solution $x = (4, 4, 4, 0, 0, 0)$

Solve $Ax_B = b \quad \Rightarrow \quad x_B = (4, 4, 4)$

$x \geq 0 \longrightarrow$

Optimal solution

$$x^\star = (4, 4, 4, 0, 0, 0)$$

Same as primal simplex!

Equivalence and symmetry

The dual simplex is equivalent to the primal simplex applied to the dual problem.

Dual problem

maximize $-b^T y$

subject to $A^T y + c \geq 0$

$\longrightarrow$

Standard form

minimize $\begin{bmatrix} b & -b & 0 \end{bmatrix} w$

subject to $\begin{bmatrix} A^T & -A^T & -I \end{bmatrix} w = -c$

$w \geq 0$

$$w = (y^+, y^-, s)$$

Dual simplex efficiency

Sequence of problems with varying feasible region

previous y still dual feasible $\longrightarrow$ **warm-start**

Applied in many different contexts, for example:

1. **sequential decision-making**

2. **mixed-integer optimization** to **solve subproblems**

(more later in the course…)

Linear optimization duality

Today, we learned to:

- **Interpret** linear optimization duality using game theory
- **Prove** Farkas lemma using duality
- **Geometrically link** primal and dual solutions with complementary slackness
- **Implement** the dual simplex method

Next lecture

- Sensitivity analysis