

ORF522 – Linear and Nonlinear Optimization

7. Linear optimization duality

Recap

Linear optimization formulations

Standard form LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Inequality form LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

Today's agenda

[Chapter 4, LO][Chapter 5, LP]

- Obtaining lower bounds
- The dual problem
- Weak and strong duality

Obtaining lower bounds

Obtaining lower bounds

A simple example

$$\begin{array}{ll} \text{minimize} & x_1 + 3x_2 \\ \text{subject to} & x_1 + 3x_2 \geq 2 \end{array}$$

What is a **lower bound** on the optimal cost?

A lower bound is 2 because $x_1 + 3x_2 \geq 2$

Obtaining lower bounds

Another example

$$\begin{aligned} \text{minimize} \quad & x_1 + 3x_2 \\ \text{subject to} \quad & x_1 + x_2 \geq 2 \\ & x_2 \geq 1 \end{aligned}$$

What is a **lower bound** on the optimal cost?

Let's sum the constraints

$$\begin{aligned} & 1 \cdot (x_1 + x_2 \geq 2) \\ & + 2 \cdot (x_2 \geq 1) \\ & = x_1 + 3x_2 \geq 4 \end{aligned}$$

A lower bound is 4

Obtaining lower bounds

A more interesting example

$$\begin{aligned} \text{minimize} \quad & x_1 + 3x_2 \\ \text{subject to} \quad & x_1 + x_2 \geq 2 \\ & x_2 \geq 1 \\ & x_1 - x_2 \geq 3 \end{aligned}$$

How can we obtain a lower bound?

Add constraints

$$\begin{aligned} & y_1 \cdot (x_1 + x_2 \geq 2) \\ + & y_2 \cdot (x_2 \geq 1) \\ + & y_3 \cdot (x_1 - x_2 \geq 3) \\ = & x_1 + 3x_2 \geq 2y_1 + y_2 + 3y_3 \end{aligned}$$

Bound

Match cost coefficients

$$\begin{aligned} y_1 + y_3 &= 1 \\ y_1 + y_2 - y_3 &= 3 \\ y_1, y_2, y_3 &\geq 0 \end{aligned}$$

Many options

$$\begin{aligned} y &= (1, 2, 0) \Rightarrow \text{Bound 4} \\ y &= (0, 4, 1) \Rightarrow \text{Bound 7} \end{aligned}$$

How can we get the **best one**?

Obtaining lower bounds

A more interesting example – Best lower bound

We can obtain the **best lower bound** by solving the following problem

$$\begin{aligned} \text{maximize} \quad & 2y_1 + y_2 + 3y_3 \\ \text{subject to} \quad & y_1 + y_3 = 1 \\ & y_1 + y_2 - y_3 = 3 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

This linear optimization problem is called the **dual problem**

The dual problem

Lagrange multipliers

Consider the LP in standard form

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax = b \\ &&& x \geq 0 \end{aligned}$$

Lower bound

$$g(y) \leq c^T x^* + y^T (Ax^* - b) = c^T x^*$$

Relax the constraint

$$g(y) = \begin{aligned} &\text{minimize} && c^T x + y^T (Ax - b) \\ &&& \text{subject to} && x \geq 0 \end{aligned}$$

Best lower bound

$$\text{maximize}_y g(y)$$

The dual

Dual function

$$\begin{aligned}g(y) &= \underset{x \geq 0}{\text{minimize}} (c^T x + y^T (Ax - b)) \\ &= -b^T y + \underset{x \geq 0}{\text{minimize}} (c + A^T y)^T x\end{aligned}$$

$$g(y) = \begin{cases} -b^T y & \text{if } c + A^T y \geq 0 \\ -\infty & \text{otherwise} \end{cases}$$

Dual problem (find the best bound)

$$\begin{aligned}\underset{y}{\text{maximize}} \quad g(y) &= \underset{y}{\text{maximize}} \quad -b^T y \\ &\text{subject to} \quad A^T y + c \geq 0\end{aligned}$$

Primal and dual problems

Primal problem

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax = b \\ &&& x \geq 0 \end{aligned}$$

Primal variable $x \in \mathbf{R}^n$

Dual problem

$$\begin{aligned} &\text{maximize} && -b^T y \\ &\text{subject to} && A^T y + c \geq 0 \end{aligned}$$

Dual variable $y \in \mathbf{R}^m$

The dual problem carries **useful information** for the primal problem

Duality is useful also to **solve** optimization problems

Dual of inequality form LP

What if you find an LP with inequalities?

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

1. We could first transform it to standard form
2. We can compute the dual function (same procedure as before)

Relax the constraint

$$g(y) = \underset{x}{\text{minimize}} \quad c^T x + y^T (Ax - b)$$

Lower bound

$$g(y) \leq c^T x^* + y^T (Ax^* - b) \leq c^T x^*$$

we must have $y \geq 0$

Dual of LP with inequalities

Derivation

Dual function

$$\begin{aligned} g(y) &= \underset{x}{\text{minimize}} (c^T x + y^T (Ax - b)) \\ &= -b^T y + \underset{x}{\text{minimize}} (c + A^T y)^T x \end{aligned}$$

$$g(y) = \begin{cases} -b^T y & \text{if } c + A^T y = 0 \text{ (and } y \geq 0) \\ -\infty & \text{otherwise} \end{cases}$$

Dual problem (find the best bound)

$$\begin{aligned} \underset{y}{\text{maximize}} \quad g(y) &= \text{maximize} \quad -b^T y \\ &\text{subject to} \quad A^T y + c = 0 \\ &\quad \quad \quad y \geq 0 \end{aligned}$$

General forms

Primal		Standard form LP	Dual	
minimize	$c^T x$		maximize	$-b^T y$
subject to	$Ax = b$		subject to	$A^T y + c \geq 0$
	$x \geq 0$			

Primal		Inequality form LP	Dual	
minimize	$c^T x$		maximize	$-b^T y$
subject to	$Ax \leq b$		subject to	$A^T y + c = 0$
				$y \geq 0$

LP with inequalities and equalities				
Primal		Dual		
minimize	$c^T x$	maximize	$-b^T y - d^T z$	
subject to	$Ax \leq b$	subject to	$A^T y + C^T z + c = 0$	
	$Cx = d$		$y \geq 0$	

Example from before

$$\begin{array}{ll} \text{minimize} & x_1 + 3x_2 \\ \text{subject to} & x_1 + x_2 \geq 2 \\ & x_2 \geq 1 \\ & x_1 - x_2 \geq 3 \end{array}$$



Inequality form LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

$$c = (1, 3)$$

$$A = \begin{bmatrix} -1 & -1 \\ 0 & -1 \\ -1 & 1 \end{bmatrix}$$

$$b = (-2, -1, -3)$$

Dual

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c = 0 \\ & y \geq 0 \end{array}$$



$$\begin{array}{ll} \text{maximize} & 2y_1 + y_2 + 3y_3 \\ \text{subject to} & -y_1 - y_3 = -1 \\ & -y_1 - y_2 + y_3 = -3 \\ & y_1, y_2, y_3 \geq 0 \end{array}$$

To memorize

Ways to get the dual

- Derive dual function directly
- Transform the problem in inequality form LP and dualize

Sanity-checks and signs convention

- Consider constraints as $g(x) \leq 0$ or $g(x) = 0$
- Each dual variable is associated to a primal constraint
- y free for primal equalities and $y \geq 0$ for primal inequalities

Dual of the dual

Theorem

If we transform the primal into its dual and then transform the dual to its dual, we obtain a problem equivalent to the original problem. In other words, the **dual of the dual is the primal**.

Exercise

Derive dual and dualize again

Primal		Dual	
minimize	$c^T x$	maximize	$-b^T y - d^T z$
subject to	$Ax \leq b$	subject to	$A^T y + C^T z + c = 0$
	$Cx = d$		$y \geq 0$

Theorem

If we **transform a linear optimization problem to another form** (inequality form, standard form, inequality and equality form), **the dual of the two problems will be equivalent**.

Weak and strong duality

Optimal objective values

Primal

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax \leq b \end{aligned}$$

p^* is the primal optimal value

Primal infeasible: $p^* = +\infty$

Primal unbounded: $p^* = -\infty$

Dual

$$\begin{aligned} &\text{maximize} && -b^T y \\ &\text{subject to} && A^T y + c = 0 \\ &&& y \geq 0 \end{aligned}$$

d^* is the dual optimal value

Dual infeasible: $d^* = -\infty$

Dual unbounded: $d^* = +\infty$

Weak duality

Theorem

If x, y satisfy:

- x is a feasible solution to the primal problem
 - y is a feasible solution to the dual problem
- $\longrightarrow -b^T y \leq c^T x$

Proof

We know that $Ax \leq b$, $A^T y + c = 0$ and $y \geq 0$. Therefore,

$$0 \leq y^T (b - Ax) = b^T y - y^T Ax = c^T x + b^T y \quad \blacksquare$$

Remark

- Any dual feasible y gives a **lower bound** on the primal optimal value
- Any primal feasible x gives an **upper bound** on the dual optimal value
- $c^T x + b^T y$ is the **duality gap**

Weak duality

Corollaries

Unboundedness vs feasibility

- Primal unbounded ($p^* = -\infty$) \Rightarrow dual infeasible ($d^* = -\infty$)
- Dual unbounded ($d^* = +\infty$) \Rightarrow primal infeasible ($p^* = +\infty$)

Optimality condition

If x, y satisfy:

- x is a feasible solution to the primal problem
- y is a feasible solution to the dual problem
- The duality gap is zero, *i.e.*, $c^T x + b^T y = 0$

Then x and y are **optimal solutions** to the primal and dual problem respectively

Strong duality

Theorem

If a linear optimization problem has an optimal solution, so does its dual, and the optimal value of primal and dual are equal

$$d^* = p^*$$

Strong duality

Constructive proof

Given a primal optimal solution x^* we will construct a dual optimal solution y^*

Apply simplex to problem in **standard form**

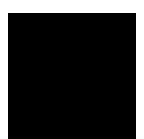
$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \longrightarrow \begin{array}{l} \bullet \text{ optimal basis } B \\ \bullet \text{ optimal solution } x^* \text{ with } A_B x_B^* = b \\ \bullet \text{ reduced costs } \bar{c} = c - A^T A_B^{-T} c_B \geq 0 \end{array}$$

Define y^* such that $y^* = -A_B^{-T} c_B$. Therefore, $A^T y^* + c \geq 0$ (y^* dual feasible).

$$-b^T y^* = -b^T (-A_B^{-T} c_B) = c_B^T (A_B^{-1} b) = c_B^T x_B^* = c^T x^*$$

By weak duality theorem corollary, y^* is an optimal solution of the dual.

Therefore, $d^* = p^*$.



Exception to strong duality

Primal

$$\begin{array}{ll} \text{minimize} & x \\ \text{subject to} & 0 \cdot x \leq -1 \end{array}$$

Optimal value is $p^* = +\infty$

Dual

$$\begin{array}{ll} \text{maximize} & y \\ \text{subject to} & 0 \cdot y + 1 = 0 \\ & y \geq 0 \end{array}$$

Optimal value is $d^* = -\infty$

Both **primal** and **dual infeasible**

Relationship between primal and dual

	$p^* = +\infty$	p^* finite	$p^* = -\infty$
$d^* = +\infty$	primal inf. dual unb.		
d^* finite		optimal values equal	
$d^* = -\infty$	exception		primal unb. dual inf

- Upper-right excluded by **weak duality**
- (1, 1) and (3, 3) proven by **weak duality**
- (3, 1) and (2, 2) proven by **strong duality**

Example

Production problem

maximize $x_1 + 2x_2$ ← Profits
subject to $x_1 \leq 100$
 $2x_2 \leq 200$ ← Resources
 $x_1 + x_2 \leq 150$
 $x_1, x_2 \geq 0$

Dualize

1. Transform in inequality form

minimize $c^T x$
subject to $Ax \leq b$

2. Derive dual

maximize $-b^T y$
subject to $A^T y + c = 0$
 $y \geq 0$

$$c = (-1, -2)$$
$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$b = (100, 200, 150, 0, 0)$$

Production problem

The dual

$$\text{minimize } 100y_1 + 200y_2 + 150y_3$$

$$\text{subject to } y_1 + y_3 \geq 1$$

$$2y_2 + y_3 \geq 2$$

$$y_1, y_2, y_3 \geq 0$$

Interpretation

- **Sell all your resources** at a fair (minimum) price
- Selling must be **more convenient than producing**:
 - Product 1 (price 1, needs 1× resource 1 and 3): $y_1 + y_3 \geq 1$
 - Product 2 (price 2, needs 2× resource 2 and 1× resource 3): $2y_2 + y_3 \geq 2$

Linear optimization duality

Today, we learned to:

- **Dualize** linear optimization problems
- **Prove** weak and strong duality conditions
- **Interpret** simple dual optimization problems

Next lecture

More on duality:

- Game theoretic interpretation
- Complementary slackness
- Alternative systems