ORF522 – Linear and Nonlinear Optimization

4. The simplex method

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Ed Forum

$$\min c^T x \quad \text{s.t. } Ax \leq b \;]\, m$$

INEQUALITY FORM

$$\min c^T x \quad \text{s.t. } Ax = b \;]\, m, \quad x \geq 0$$

STANDARD FORM

- Problem sizes in different formulations. What is m?

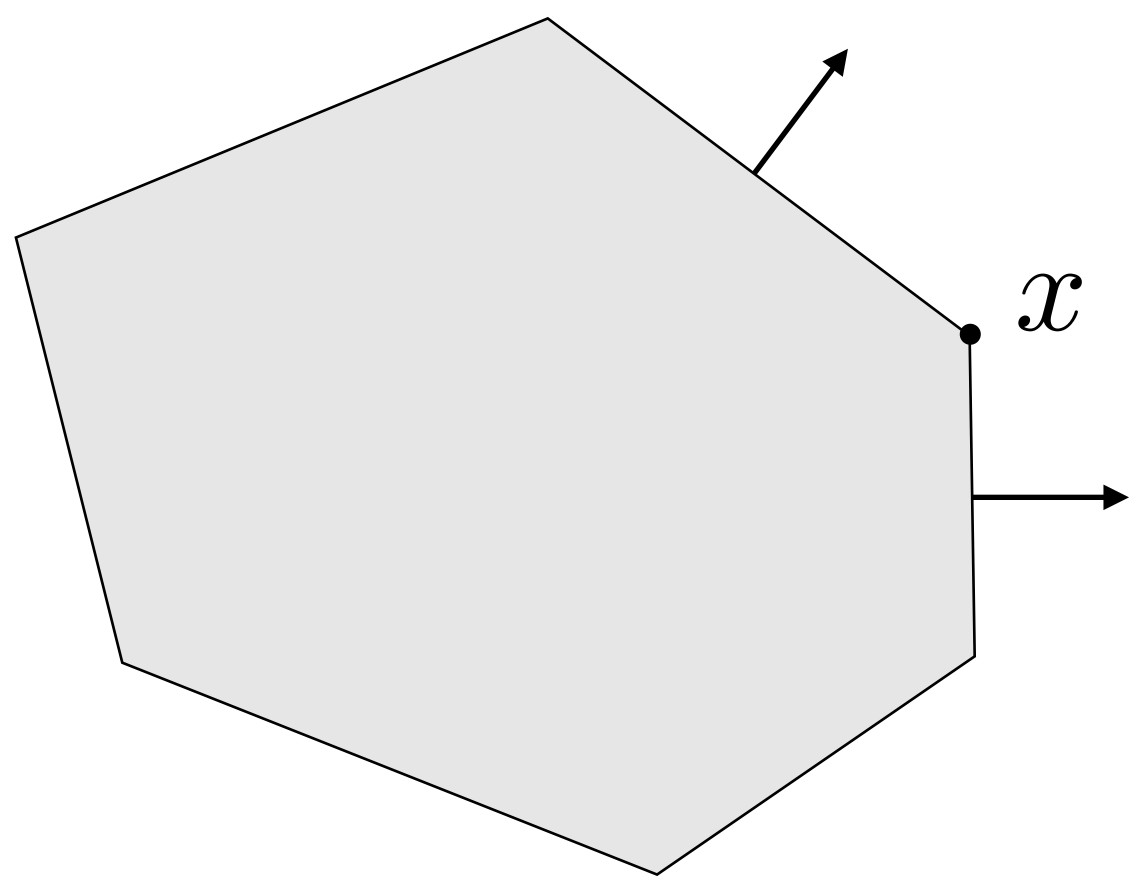
- Vector x is a basic solution if and only if there exists m columns of A being linearly independent. I'm just wondering what if there aren't m independent columns, what does it mean? Also, what does it mean if there're multiple sets of independent columns. Does it have any geometric meaning, like any relationship with the extreme points in polyhedron?

Recap

Equivalence

Theorem

Given a nonempty polyhedron $P = \{x \mid Ax \leq b\}$



Let $x \in P$

x is a **vertex** $\iff$ x is an **extreme point** $\iff$ x is a **basic feasible solution**

Basic feasible solution

$$P = \{x \mid a_i^T x \leq b_i, \quad i = 1, \ldots, m\}$$

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Active constraints at $\bar{x}$

$$\mathcal{I}(\bar{x}) = \{i \in \{1, \ldots, m\} \mid a_i^T \bar{x} = b_i\}$$

Index of all the constraints satisfied as **equality**

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Basic solution $\bar{x}$

- $\{a_i \mid i \in \mathcal{I}(\bar{x})\}$ has n linearly independent vectors

Basic feasible solution

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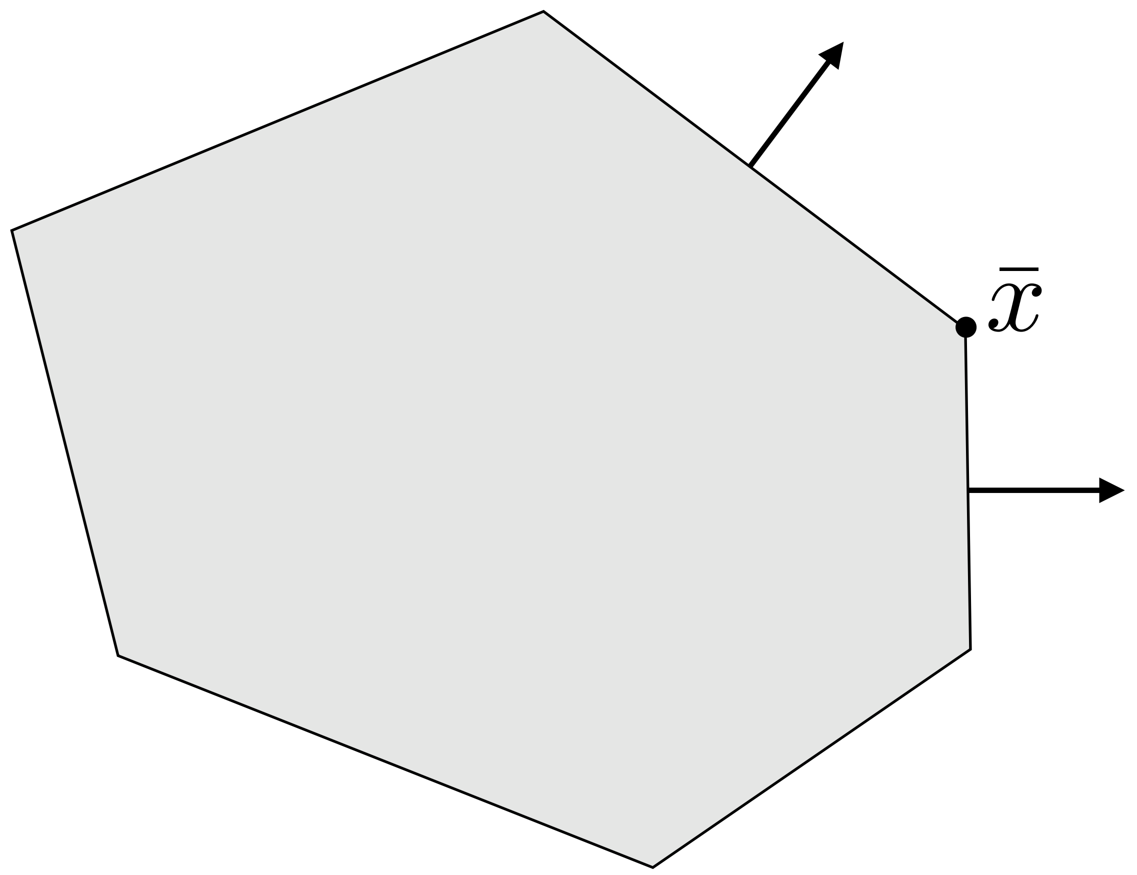
Index of all the constraints satisfied as **equality**

Basic solution $\bar{x}$

- $\{a_i \mid i \in \mathcal{I}(\bar{x})\}$ has n linearly independent vectors

Basic feasible solution $\bar{x}$

- $\bar{x} \in P$
- $\{a_i \mid i \in \mathcal{I}(\bar{x})\}$ has n linearly independent vectors



Standard form polyhedra

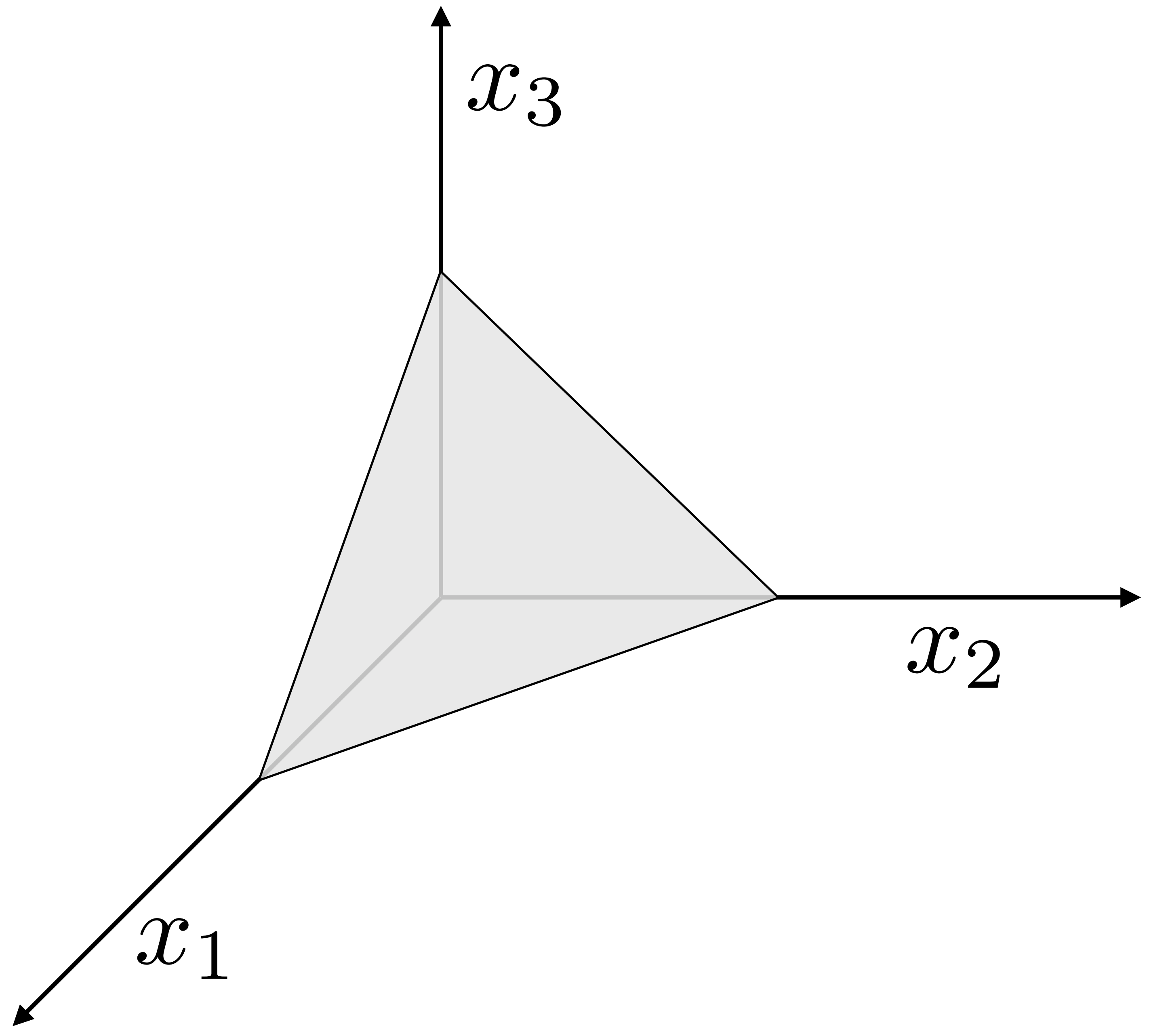
Definition

Standard form LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Standard form polyhedron

$$P = \{x \mid Ax = b,\ x \geq 0\}$$



Assumption

$A \in \mathbf{R}^{m \times n}$ has full row rank $m \leq n$

Interpretation

P lives in $(n - m)$-dimensional subspace

Basic solutions

Standard form polyhedra

$P = \{x \mid Ax = b,\ x \geq 0\}$ with $A \in \mathbf{R}^{m \times n}$ has full row rank $m \leq n$

x is a **basic solution** if and only if

- $Ax = b$
- There exist indices $B(1), \dots, B(m)$ such that
  - columns $A_{B(1)}, \dots, A_{B(m)}$ are linearly independent
  - $x_i = 0$ for $i \neq B(1), \dots, B(m)$

x is a **basic feasible solution** if x is a **basic solution** and $x \geq 0$

From geometry to standard form

minimize $\quad c^T x$

subject to $\quad Ax \leq b$

From geometry to standard form

minimize $c^T x$
subject to $Ax \leq b$

$\longrightarrow$

$$\begin{array}{ll} \text{minimize} & c^T(x^+ - x^-) \\ \text{subject to} & \begin{bmatrix} A & -A & I \end{bmatrix} \begin{bmatrix} x^+ \\ x^- \\ s \end{bmatrix} = b \\ & (x^+, x^-, s) \geq 0 \end{array}$$

From geometry to standard form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \underbrace{A}_{} x \le \overset{\mathbb{R}^m}{b} \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & c^T(x^+ - x^-) \\ \text{subject to} & \begin{bmatrix} A & -A & I \end{bmatrix} \begin{bmatrix} x^+ \\ x^- \\ s \end{bmatrix} = b \\ & (x^+, x^-, s) \ge 0 \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & \tilde{c}^T \tilde{x} \\ \text{subject to} & \tilde{A}\tilde{x} = b \\ & \tilde{x} \ge 0 \end{array}$$

$\hookrightarrow \mathbb{R}^n$

Variables: $\tilde{n} = 2n + m$

(Equality) constraints: $\tilde{m} = m \implies$ **active**

From geometry to standard form

minimize $c^T x$
subject to $Ax \leq b$ $\longrightarrow$

minimize $c^T(x^+ - x^-)$
subject to $\begin{bmatrix} A & -A & I \end{bmatrix} \begin{bmatrix} x^+ \\ x^- \\ s \end{bmatrix} = b$
$(x^+, x^-, s) \geq 0$

$\longrightarrow$

minimize $\tilde{c}^T \tilde{x}$
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$\tilde{x} \geq 0$

Variables: $\tilde{n} = 2n + m$
(Equality) constraints: $\tilde{m} = m \Longrightarrow$ **active**

For a **basic solution** $\longrightarrow$ We need $\tilde{n} - \tilde{m} = 2n$
active inequalities $\Rightarrow \tilde{x}_i = 0$ (non basic)

From geometry to standard form

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Variables: $\tilde{n} = 2n + m$

(Equality) constraints: $\tilde{m} = m \Longrightarrow$ **active**

For a **basic solution** $\longrightarrow$ We need $\tilde{n} - \tilde{m} = 2n$ active inequalities $\Rightarrow \tilde{x}_i = 0$ (non basic)

Which corresponds to m inequalities inactive $\Rightarrow \tilde{x}_i > 0$ (basic)

From geometry to standard form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \le b \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & c^T(x^+ - x^-) \\ \text{subject to} & \begin{bmatrix} A & -A & I \end{bmatrix} \begin{bmatrix} x^+ \\ x^- \\ s \end{bmatrix} = b \\ & (x^+, x^-, s) \ge 0 \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & \tilde{c}^T \tilde{x} \\ \text{subject to} & \tilde{A}\tilde{x} = b \\ & \tilde{x} \ge 0 \end{array}$$

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(Equality) constraints: $\tilde{m} = m \Longrightarrow$ **active**

Formal proof at Theorem 2.4 LO book

For a **basic solution** $\longrightarrow$ We need $\tilde{n} - \tilde{m} = 2n$ active inequalities $\Rightarrow \tilde{x}_i = 0$ (non basic)

Which corresponds to m inequalities inactive $\Rightarrow \tilde{x}_i > 0$ (basic)

Constructing basic solution

1. Choose any m independent columns of A: $A_{B(1)}, \ldots, A_{B(m)}$
2. Let $x_i = 0$ for all $i \neq B(1), \ldots, B(m)$
3. Solve $Ax = b$ for the remaining $x_{B(1)}, \ldots, x_{B(m)}$

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Basis matrix

Basis columns

Basic variables

$$A_B = \begin{bmatrix} | & | & & | \\ A_{B(1)} & A_{B(2)} & \dots & A_{B(m)} \\ | & | & & | \end{bmatrix}, \quad x_B = \begin{bmatrix} x_{B(1)} \\ \vdots \\ x_{B(m)} \end{bmatrix} \longrightarrow \text{Solve } A_B x_B = b$$

Constructing basic solution

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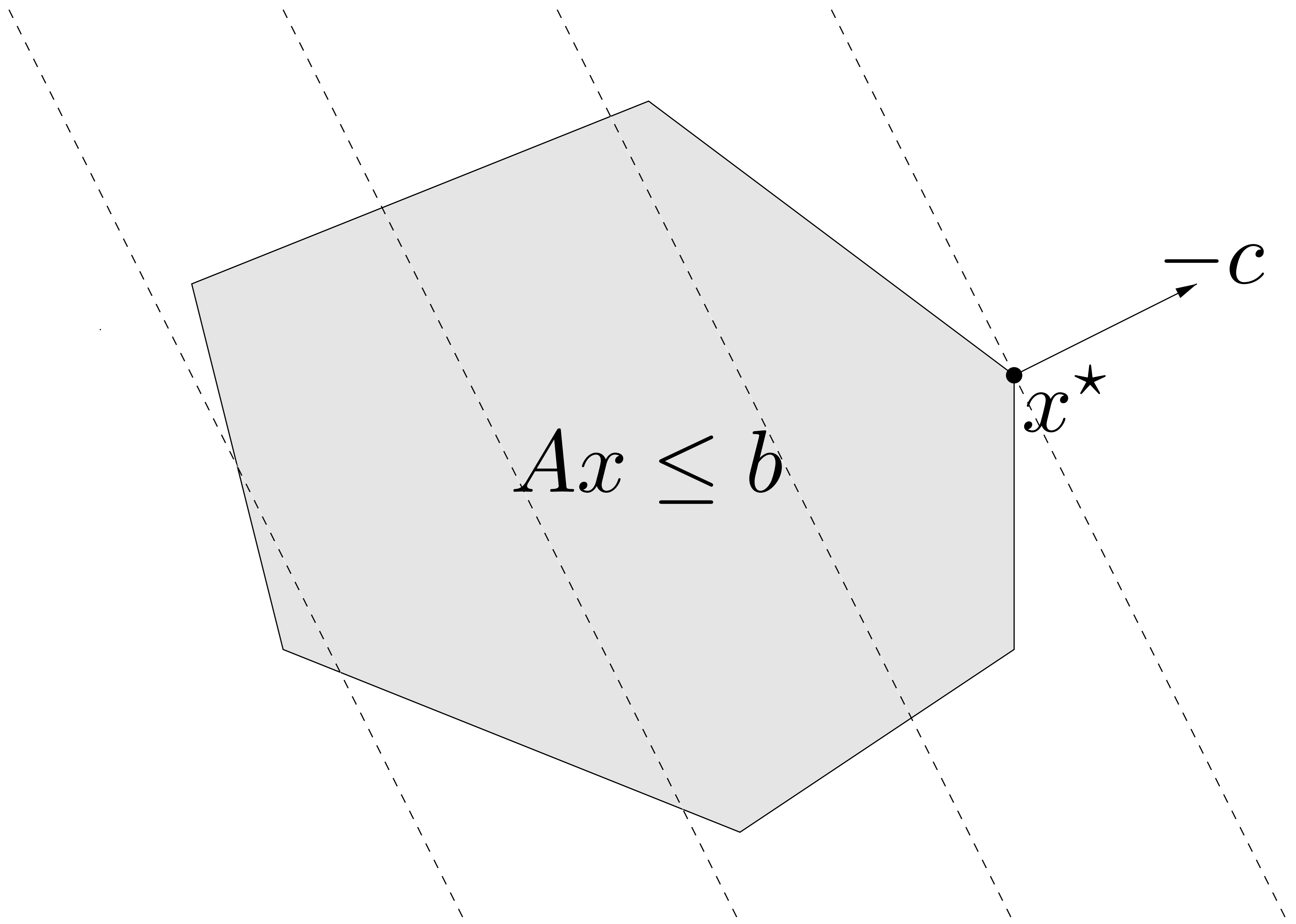
Solve $A_B x_B = b$

If $x_B \geq 0$, then x is a **basic feasible solution**

Optimality of extreme points



$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \le b \end{array}$$

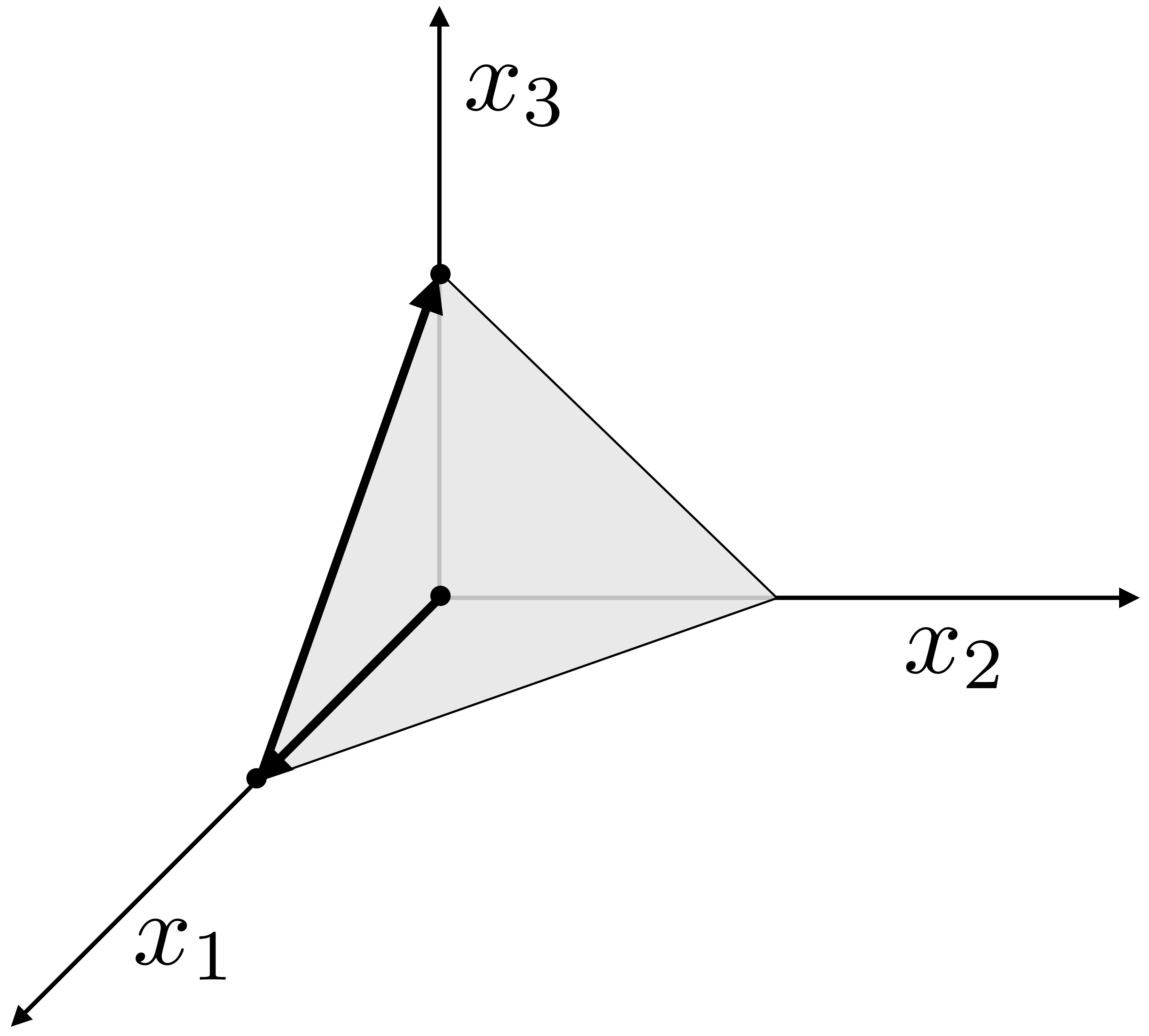
If
- P has at least one extreme point
- There exists an optimal solution $x^\star$

Then, there exists an optimal solution which is an **extreme point** of P

We only need to search between **extreme points**

Conceptual algorithm

- Start at corner
- Visit neighboring corner that improves the objective



Today's agenda
Readings: [Chapter 3, LO]

Simplex method

- Iterate between neighboring basic solutions
- Optimality conditions
- Simplex iterations

The simplex method

Top 10 algorithms of the 20th century

1946: Metropolis algorithm
1947: Simplex method
1950: Krylov subspace method
1951: The decompositional approach to matrix computations
1957: The Fortran optimizing compiler
1959: QR algorithm
1962: Quicksort
1965: Fast Fourier transform
1977: Integer relation detection
1987: Fast multipole method

[SIAM News (2000)]

The simplex method

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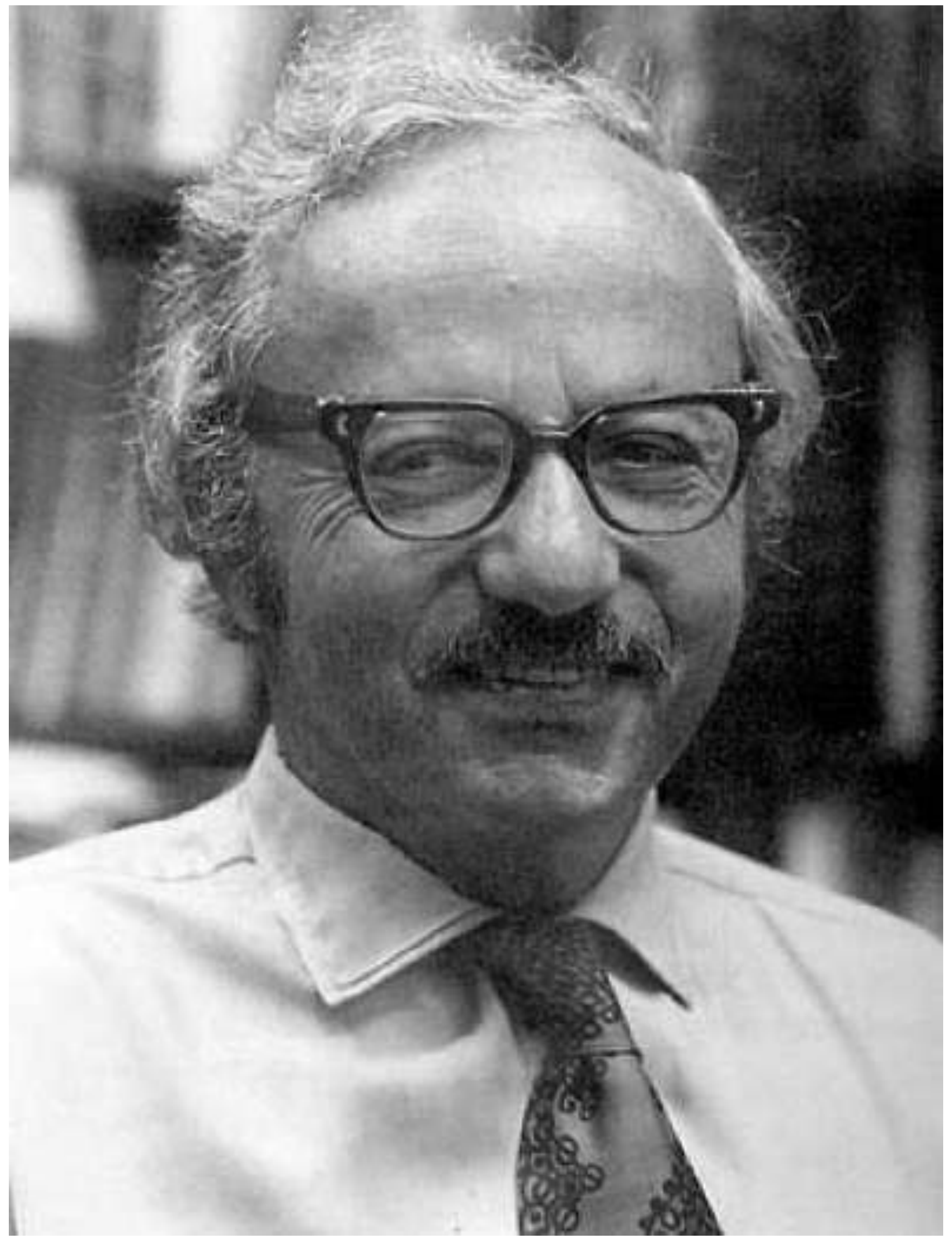
George Dantzig

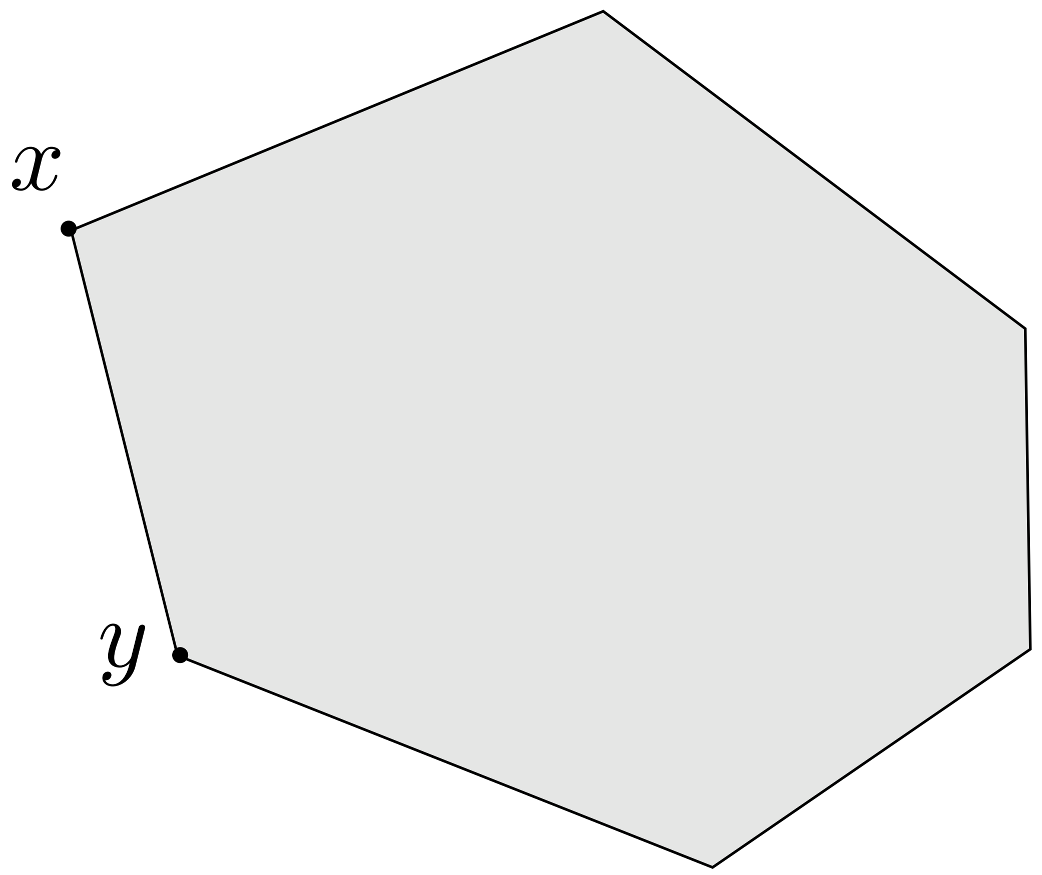


[SIAM News (2000)]

Neighboring basic solutions

Neighboring solutions

Two basic solutions are **neighboring** if their basic indices differ by exactly one variable



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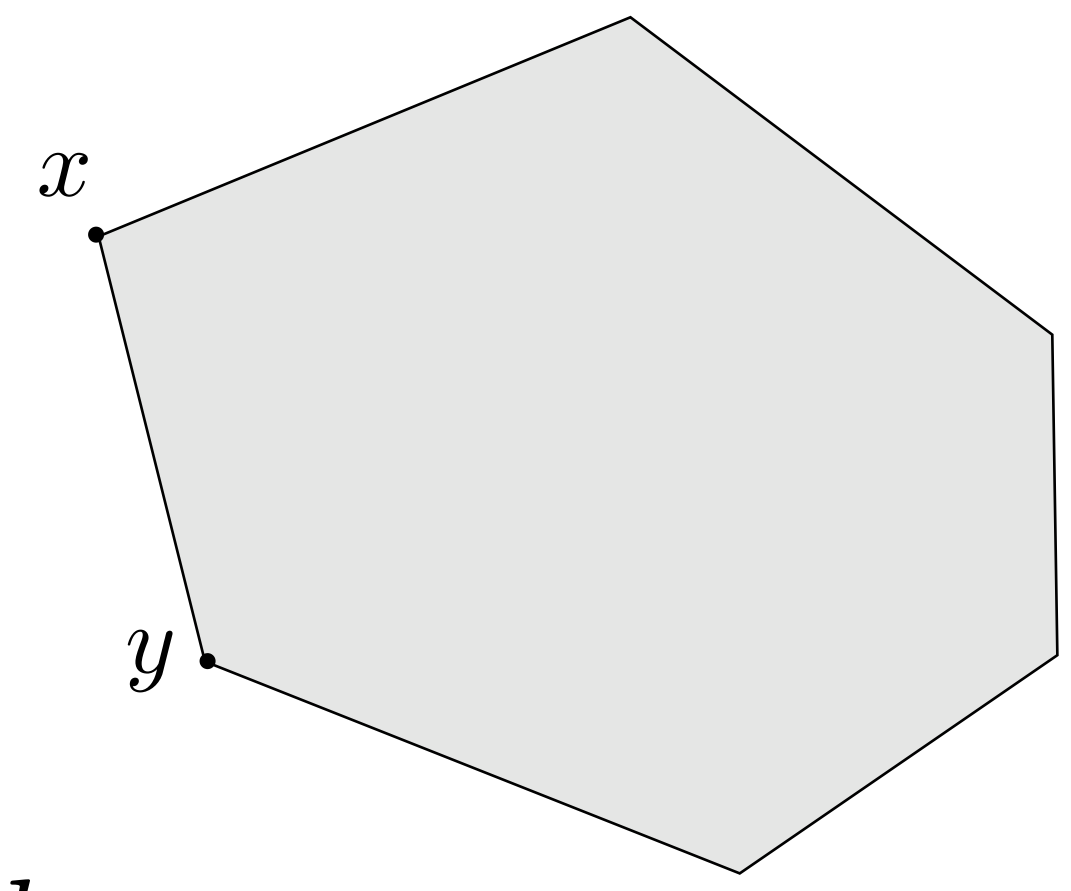

Example

$$\overset{A}{\begin{bmatrix} 1 & -1 & 0 & 3 & -2 \\ 2 & 0 & -1 & -1 & 0 \\ 0 & 2 & 4 & -1 & 4 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \overset{b}{\begin{bmatrix} -5 \\ -1 \\ 14 \end{bmatrix}}$$

Neighboring solutions

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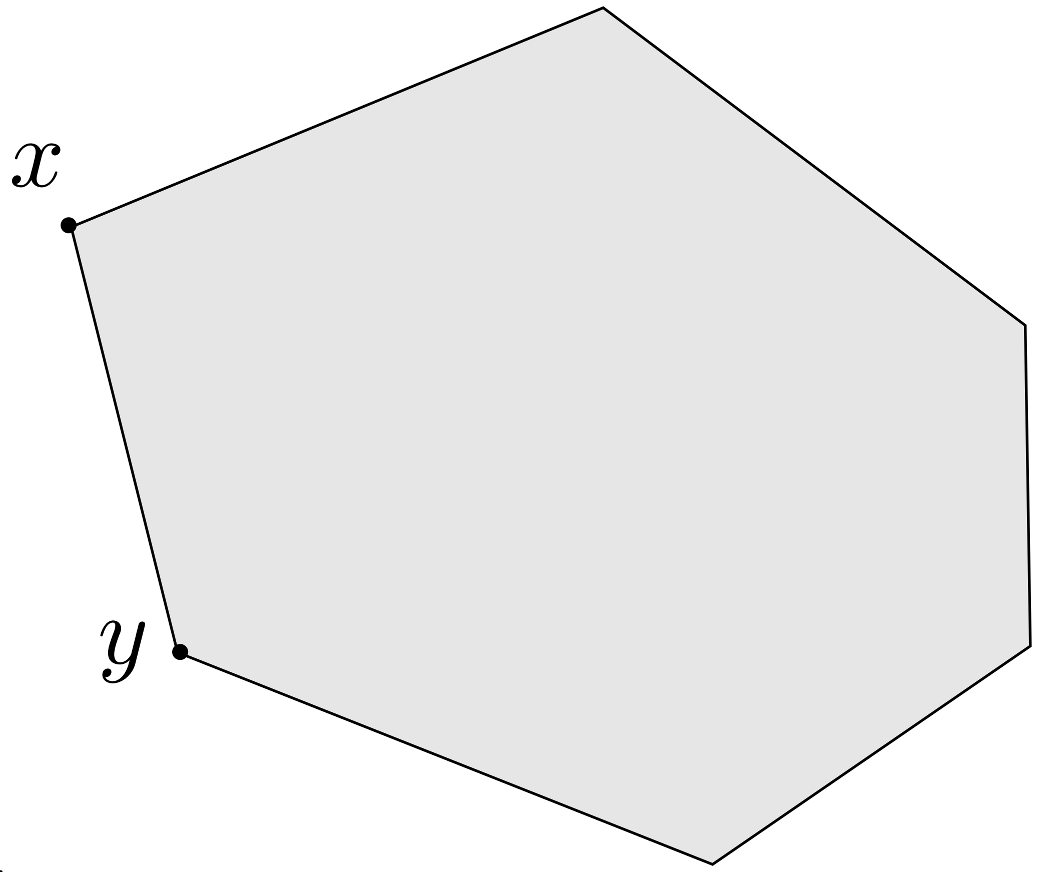

Example

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$$B = \{1, 3, 5\} \qquad x_2 = x_4 = 0$$

$$A_B x_B = b \longrightarrow x_B = \begin{bmatrix} x_1 \\ x_3 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2.5 \end{bmatrix}$$

Neighboring solutions

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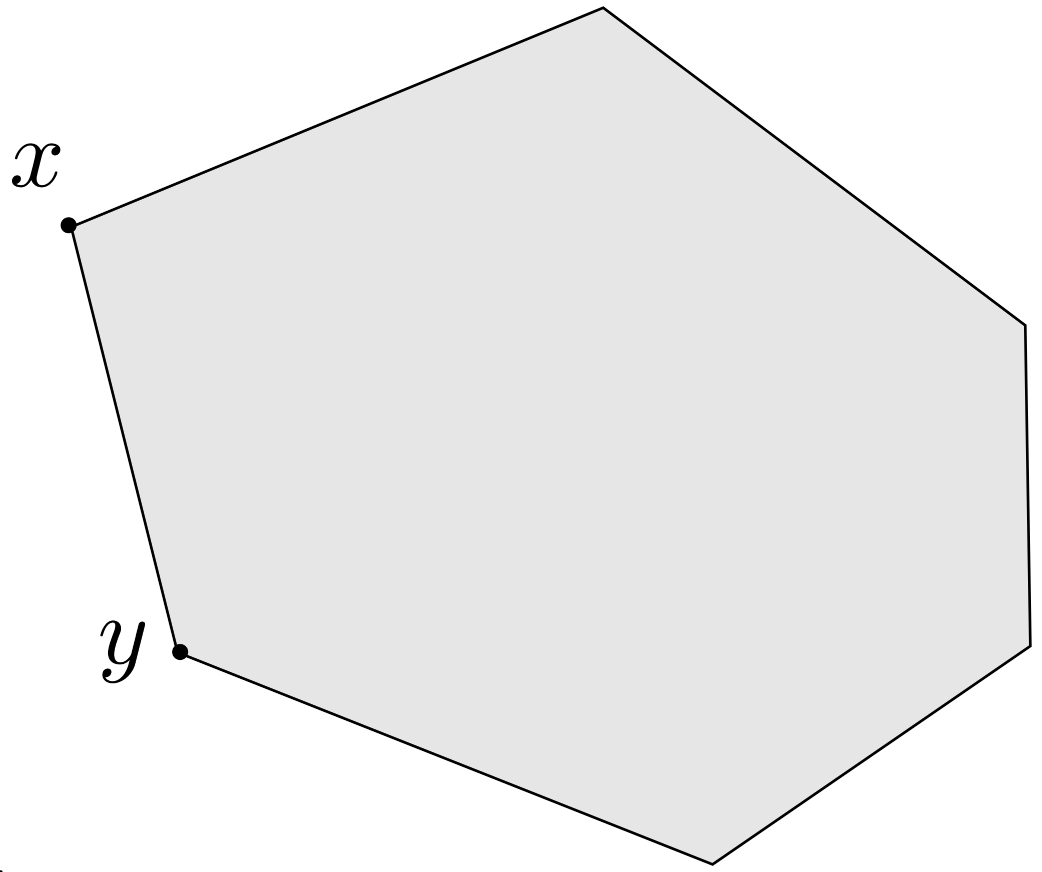

Example

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$$A_B x_B = b \longrightarrow x_B = \begin{bmatrix} x_1 \\ x_3 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2.5 \end{bmatrix}$$

$$\bar{B} = \{1, 3, 4\} \qquad y_2 = y_5 = 0$$

$$A_{\bar{B}} y_{\bar{B}} = b \longrightarrow y_{\bar{B}} = \begin{bmatrix} y_1 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 3.0 \\ \boxed{-1.7} \end{bmatrix}$$

NOT FEASIBLE

Feasible directions

Conditions

$$P = \{x \mid Ax = b,\ x \geq 0\}$$

Given a basis matrix $A_B = \begin{bmatrix} A_{B(1)} & \dots & A_{B(m)} \end{bmatrix}$

we have basic feasible solution x:

- x_B solves $A_B x_B = b$ & $x \geq 0$
- $x_i = 0,\ \forall i \neq B(1), \dots, B(m)$

Feasible directions

Conditions

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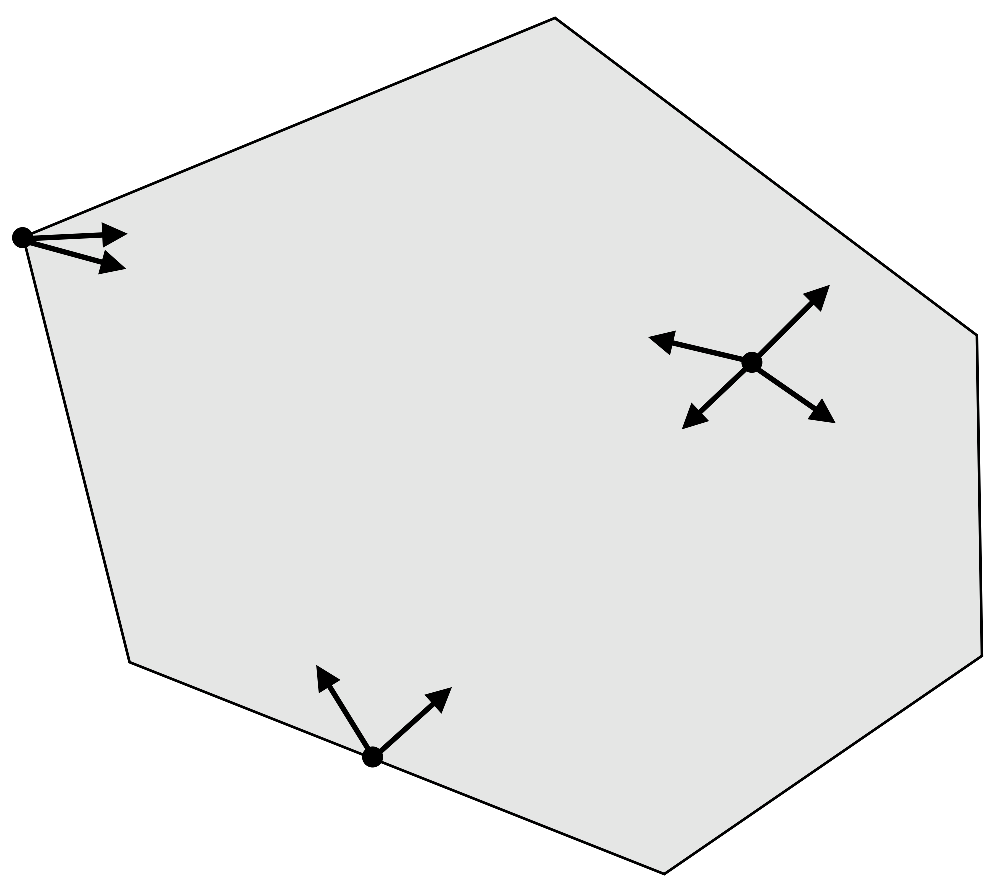
- x_B solves $A_B x_B = b$
- $x_i = 0,\ \forall i \neq B(1), \dots, B(m)$

& $x \geq 0$

Let $x \in P$, a vector d is a **feasible direction** at x if $\exists \theta > 0$ for which $x + \theta d \in P$



Feasible direction d

- $A(x + \theta d) = b \implies Ad = 0$
- $x + \theta d \geq 0$

Feasible directions

Computation

Feasible direction d

- $A(x + \theta d) = b \implies Ad = 0$
- $x + \theta d \geq 0$

Nonbasic indices $(x_i = 0)$

- $d_j = 1 \longrightarrow$ **Basic direction**
- $d_k = 0, \ \forall k \notin \{j, B(1), \ldots, B(m)\}$

Feasible directions

Computation

$d_N = (0 \dots 0\, 1\, 0 \dots 0)$

Feasible direction d

- $A(x + \theta d) = b \implies Ad = 0$
- $x + \theta d \geq 0$

Nonbasic indices

- $d_j = 1 \longrightarrow$ **Basic direction**
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Basic indices

$$Ad = 0 = \sum_{i=1}^{n} A_i d_i = A_B d_B + A_j = 0 \implies d_B = -A_B^{-1} A_j$$

Feasible directions

Computation

Feasible direction d

- $A(x + \theta d) = b \Longrightarrow Ad = 0$
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Basic indices

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Non-negativity (non-degenerate assumption) $(x \geqslant 0)$

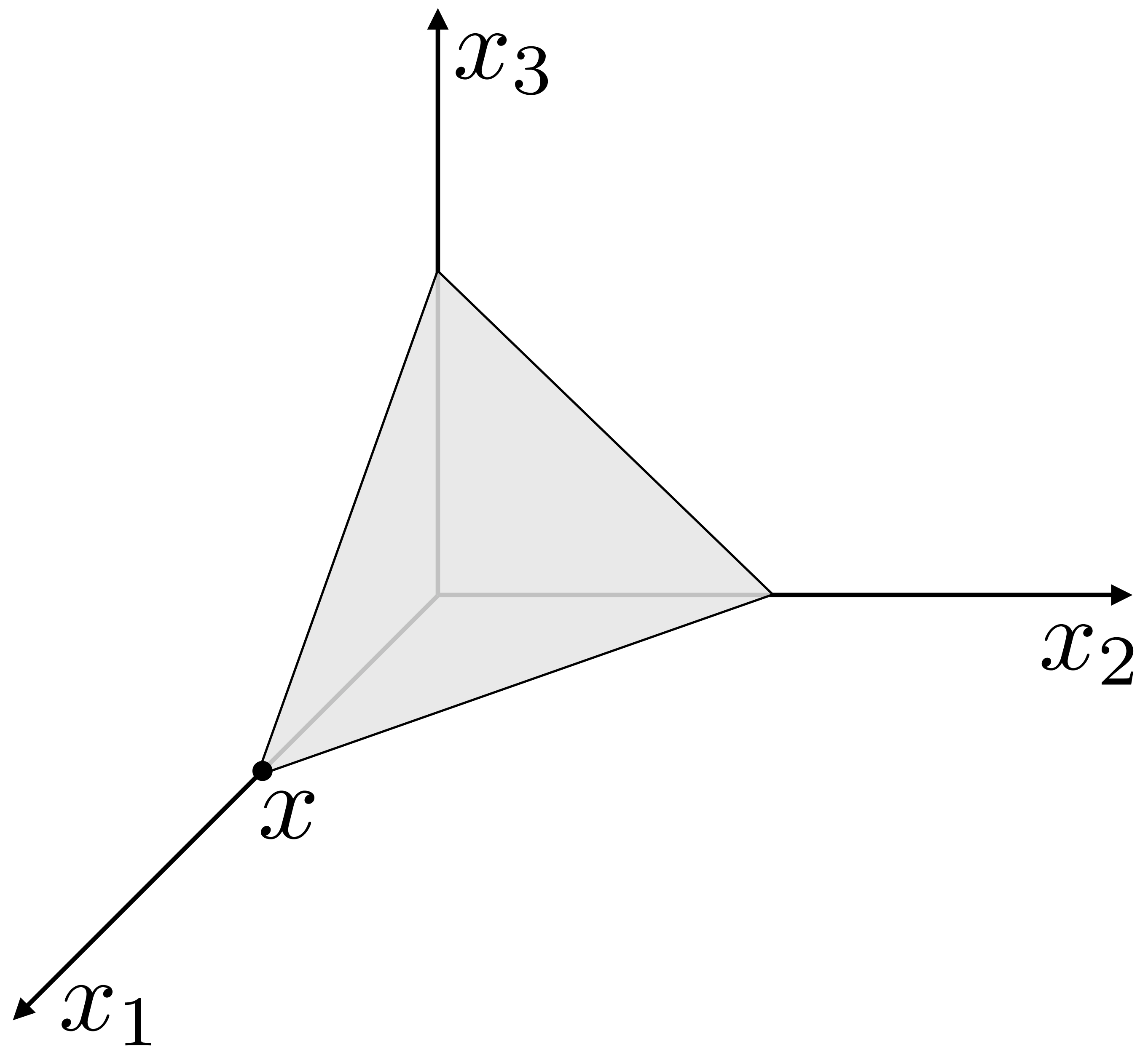
- Non-basic variables: $x_i = 0$. Nonnegative direction $d_i \geq 0$
- Basic variables: $x_B > 0$. Therefore $\exists \theta > 0$ such that $\underline{x_B + \theta d_B} \geq 0$

Feasible directions

Example

$$P = \{x \mid x_1 + x_2 + x_3 = 2, \quad x \geq 0\}$$

$$x = (2, 0, 0) \qquad\qquad B = \{1\}$$



Feasible directions

Example

$A = [1\ 1\ 1]$

$A_B = 1$

$A_j = 1$

$$P = \{x \mid x_1 + x_2 + x_3 = 2, \quad x \geq 0\}$$

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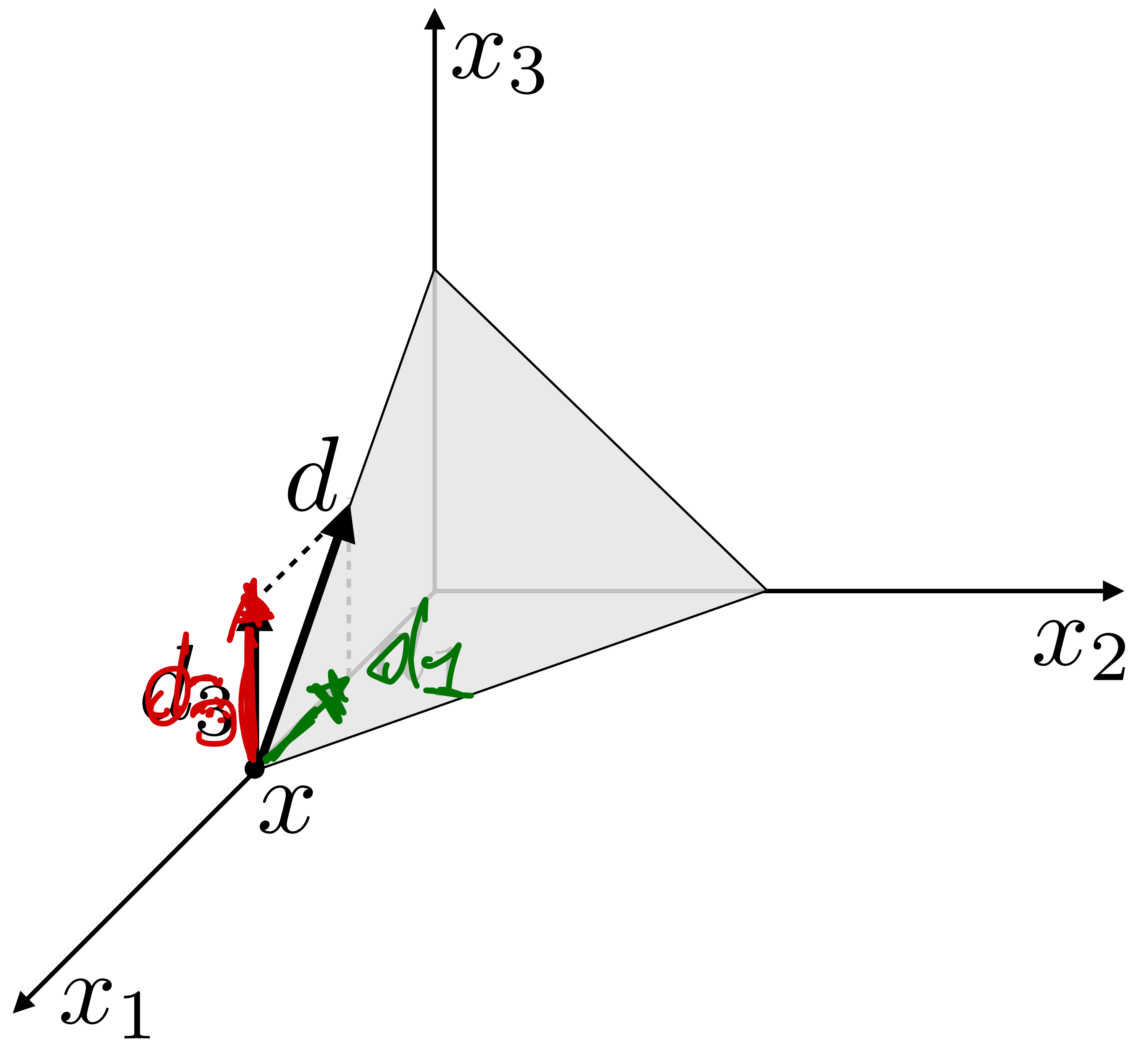

Nonbasic index $j = 3 \longrightarrow d = (-1, 0, 1)$

$$\boxed{d_j = 1}$$

$$A_B d_B = -A_j \quad \Rightarrow \quad \boxed{d_B = -1}$$

How does the cost change?

Cost improvement

$$c^T(x + \theta d) - c^T x = \theta c^T d$$

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New cost

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New cost

Old cost

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New cost **Old cost**

We call $\bar{c}_j$ the **reduced cost** of (introducing) variable x_j in the basis

$$\bar{c}_j = c^T d = \sum_{i=1}^{n} c_i d_i = c_j + c_B^T \boxed{d_B} = c_j - c_B^T A_B^{-1} A_j$$

$d_B = -A_B^{-1} A_j$

Reduced costs

Interpretation

Change in objective/marginal cost of adding x_j to the basis

$$\bar{c}_j = c_j - c_B^T A_B^{-1} A_j$$

- $\bar{c}_j > 0$: adding x_j will increase the objective (bad)
- $\bar{c}_j < 0$: adding x_j will decrease the objective (good)

Reduced costs

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Cost per-unit increase of variable x_j

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Cost per-unit increase of variable x_j

Cost to change other variables compensating for x_j to enforce $Ax = b$

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i

$e_i = (0, 0 \dots, 0 1, 0 \dots)$

Reduced costs for basic variables is 0

$$\bar{c}_{B(i)} = c_{B(i)} - c_B^T A_B^{-1} A_{B(i)} = c_{B(i)} - c_B^T (A_B^{-1} A_B) e_i$$
$$= c_{B(i)} - c_B^T e_i = c_{B(i)} - c_{B(i)} = 0$$

Vector of reduced costs

Reduced costs

$$\bar{c}_j = c_j - c_B^T A_B^{-1} A_j$$

Full vector in one shot?

$$\bar{c} = (\bar{c}_1, \dots, \bar{c}_n)$$

Vector of reduced costs

Reduced costs

$$\bar{c}_j = c_j - c_B^T A_B^{-1} A_j$$

Isolate basis B-related components p
(they are the same across j)

$$\bar{c}_j = c_j - A_j^T (A_B^{-1})^T c_B = c_j - A_j^T p$$

Full vector in one shot?

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Full vector in one shot?

$$\bar{c} = (\bar{c}_1, \ldots, \bar{c}_n)$$

Obtain p by solving linear system

$$p = (A_B^{-1})^T c_B \quad \Rightarrow \quad A_B^T p = c_B$$

Note: $(M^{-1})^T = (M^T)^{-1}$
for any square invertible M

Vector of reduced costs

Reduced costs

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Full vector in one shot?

$$\bar{c} = (\bar{c}_1, \dots, \bar{c}_n)$$

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NON BASIC
COMPONENTS

$$\boxed{\bar{c}_N = c_N - A_N^T p}$$

Computing reduced cost vector

1. Solve $A_B^T p = c_B$
2. $\bar{c} = c - A^T p$

Optimality conditions

Optimality conditions

Theorem

Let x be a basic feasible solution associated with basis matrix A_B
Let $\bar{c}$ be the vector of reduced costs.

If $\bar{c} \geq 0$, then x is **optimal**

Optimality conditions

Theorem

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Let $\bar{c}$ be the vector of reduced costs.

If $\bar{c} \geq 0$, then x is **optimal**

Remark

This is a **stopping criterion** for the simplex algorithm.

If the **neighboring solutions** do not improve the cost, we are done (because of convexity).

Optimality conditions

Proof

For a basic feasible solution x with basis B the reduced costs are $\bar{c} \geq 0$.

Optimality conditions

Proof

For a basic feasible solution x with basis B the reduced costs are $\bar{c} \geq 0$.

Consider any feasible solution y and define $d = y - x$

Optimality conditions

Proof

For a basic feasible solution x with basis B the reduced costs are $\bar{c} \geq 0$.

Consider any feasible solution y and define $d = y - x$

Since x and y are feasible, then $Ax = Ay = b$ and $Ad = 0$

$$Ad = A_B d_B + \sum_{i \in N} A_i d_i = 0 \quad \Rightarrow \quad d_B = -\sum_{i \in N} A_B^{-1} A_i d_i$$

N are the nonbasic indices

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The change in objective is

$$c^T d = c_B^T d_B + \sum_{i \in N} c_i d_i = \sum_{i \in N} (\overbrace{c_i - c_B^T A_B^{-1} A_i}^{\bar{c}_i}) d_i = \sum_{i \in N} \bar{c}_i d_i$$

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Since $y \geq 0$ and $x_i = 0,\ i \in N$, then $d_i = y_i - x_i \geq 0, i \in N$

$$c^T d = c^T (y - x) \geq 0 \quad \Rightarrow \quad c^T y \geq c^T x.$$

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Simplex iterations

Stepsize

What happens if some $\bar{c}_j < 0$?

We can decrease the cost by bringing x_j into the basis

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$\theta^\star = \max\{\theta \mid \theta \geq 0$ and $x + \theta d \geq 0\}$

d is the j-th basic direction

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Unbounded

If $d \geq 0$, then $\theta^\star = \infty$. The LP is unbounded.

Stepsize

$x_i + \theta d_i \geq 0$

$\theta d_i \geq -x_i$

$\theta \leq -x_i / d_i$

What happens if some $\bar{c}_j < 0$?

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How far can we go?

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d is the j-th basic direction

Unbounded

If $d \geq 0$, then $\theta^\star = \infty$. The LP is unbounded.

Bounded

If $d_i < 0$ for some i, then

$$\theta^\star = \min_{\{i | d_i < 0\}} \left(-\frac{x_i}{d_i}\right) = \min_{\{i \in B | d_i < 0\}} \left(-\frac{x_i}{d_i}\right)$$

(Since $d_i \geq 0,\ i \notin B$)

Moving to a new basis

Next feasible solution

$$x + \theta^\star d$$

Moving to a new basis

Next feasible solution

$$x + \theta^\star d$$

Let $B(\ell) \in \{B(1), \ldots, B(m)\}$ be the index such that $\theta^\star = -\dfrac{x_{B(\ell)}}{d_{B(\ell)}}$. Then,

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New solution

- $x_{B(\ell)}$ becomes 0 (exits)
- x_j becomes $\theta^\star$ (enters)

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New basis

$$A_{\bar{B}} = \begin{bmatrix} A_{B(1)} & \dots & A_{B(\ell-1)} & A_j & A_{B(\ell+1)} & \dots & A_{B(m)} \end{bmatrix}$$

An iteration of the simplex method

First part

We start with

- a basic feasible solution x
- a basis matrix $A_B = \begin{bmatrix} A_{B(1)} & \dots, A_{B(m)} \end{bmatrix}$

1. Compute the reduced costs $\bar{c}$
    - Solve $A_B^T p = c_B$
    - $\bar{c} = c - A^T p$
2. If $\bar{c} \geq 0$, x **optimal**. **break**
3. Choose j such that $\bar{c}_j < 0$

An iteration of the simplex method
Second part

4. Compute search direction d with $d_j = 1$ and $A_B d_B = -A_j$
5. If $d_B \geq 0$, the problem is **unbounded** and the optimal value is $-\infty$. **break**
6. Compute step length $\theta^\star = \min_{\{i \in B | d_i < 0\}} \left(-\frac{x_i}{d_i} \right)$
7. Define y such that $y = x + \theta^\star d$
8. Get new basis $\bar{B}$ (i exits and j enters)

Example

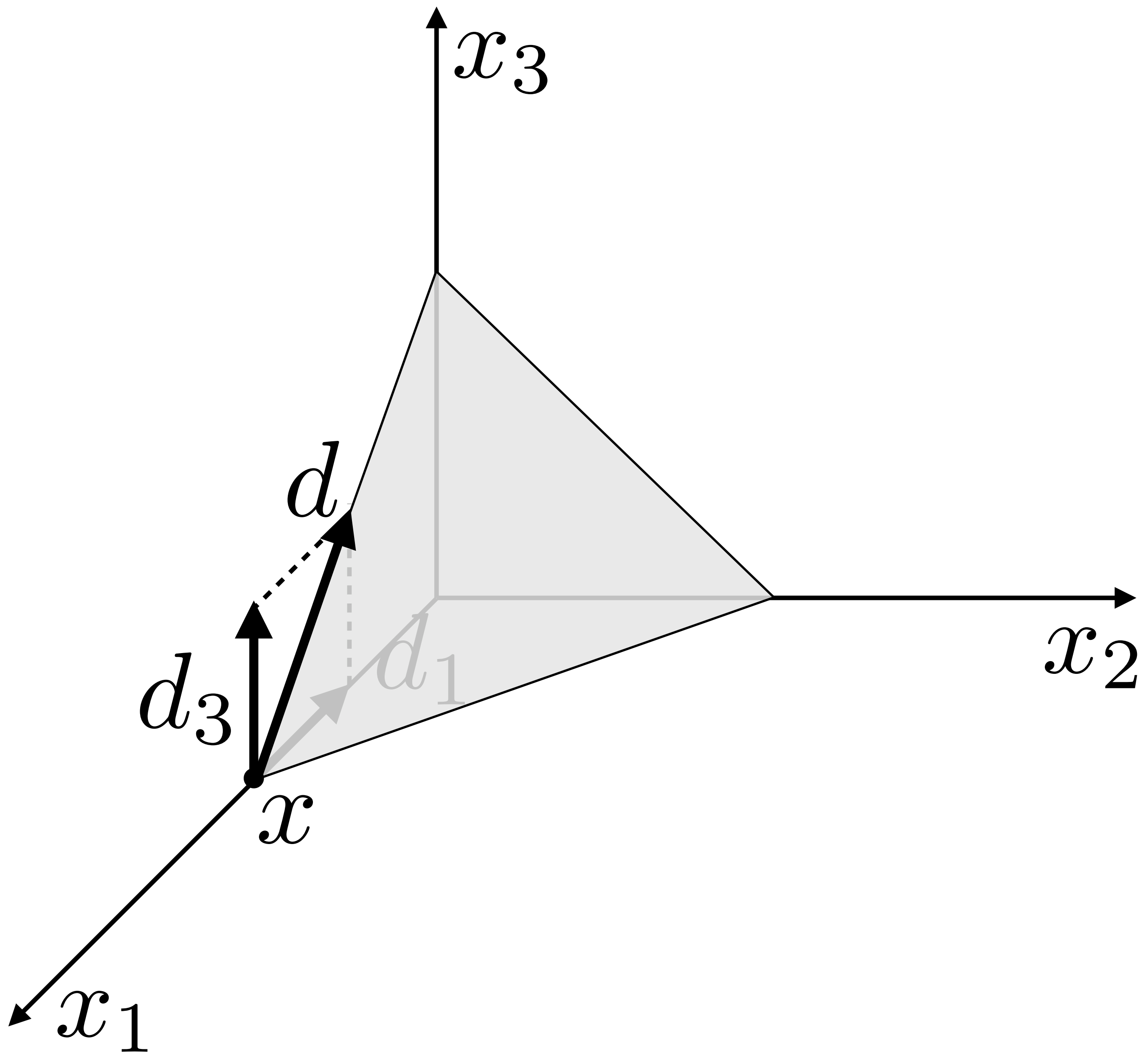
$$P = \{x \mid x_1 + x_2 + x_3 = 2, \quad x \geq 0\}$$

$$x = (2, 0, 0) \qquad\qquad B = \{1\}$$

Basic index $j = 3 \longrightarrow d = (-1, 0, 1)$

$$d_j = 1$$

$$A_B d_B = -A_j \quad \Rightarrow \quad d_B = -1$$



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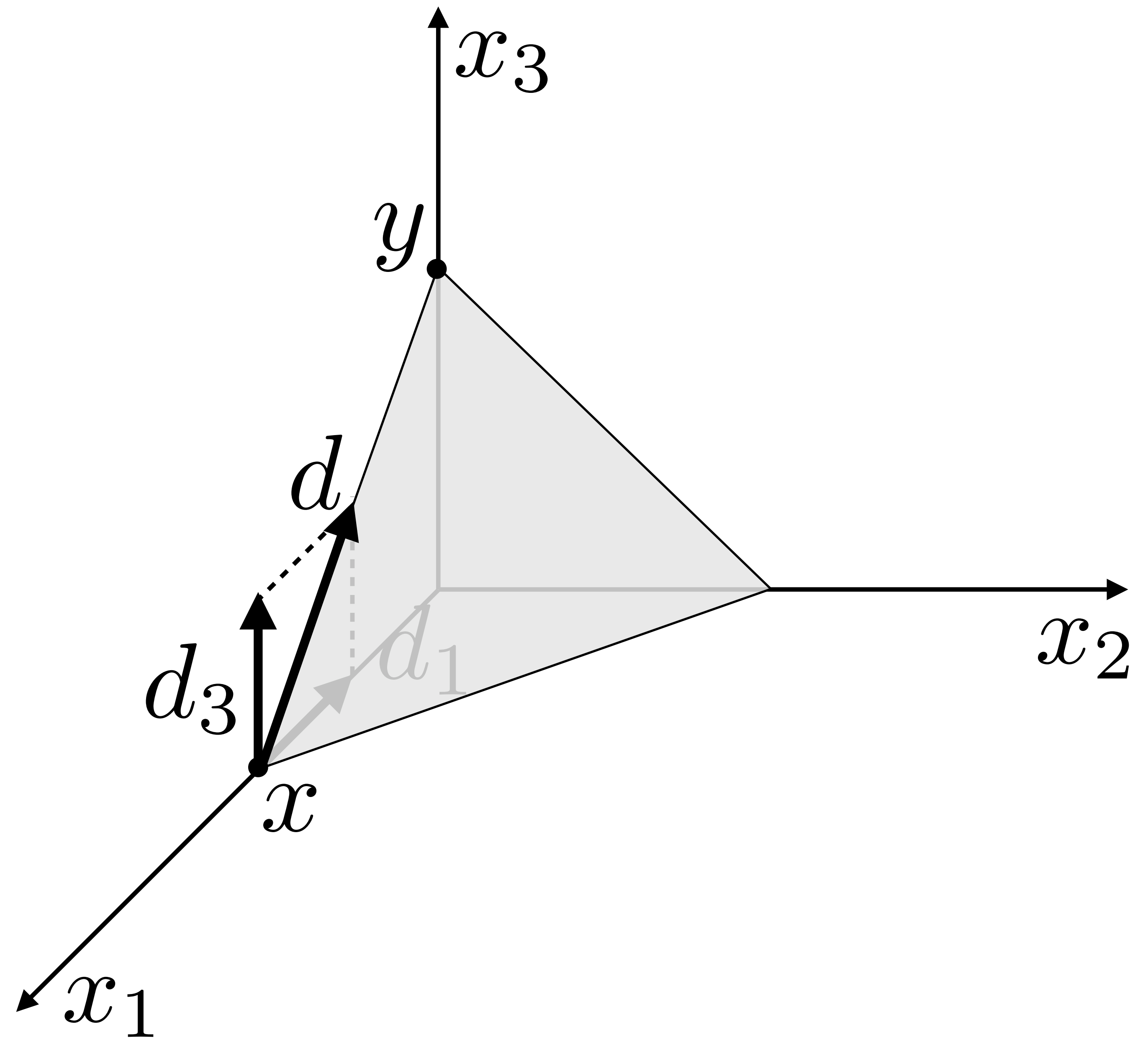

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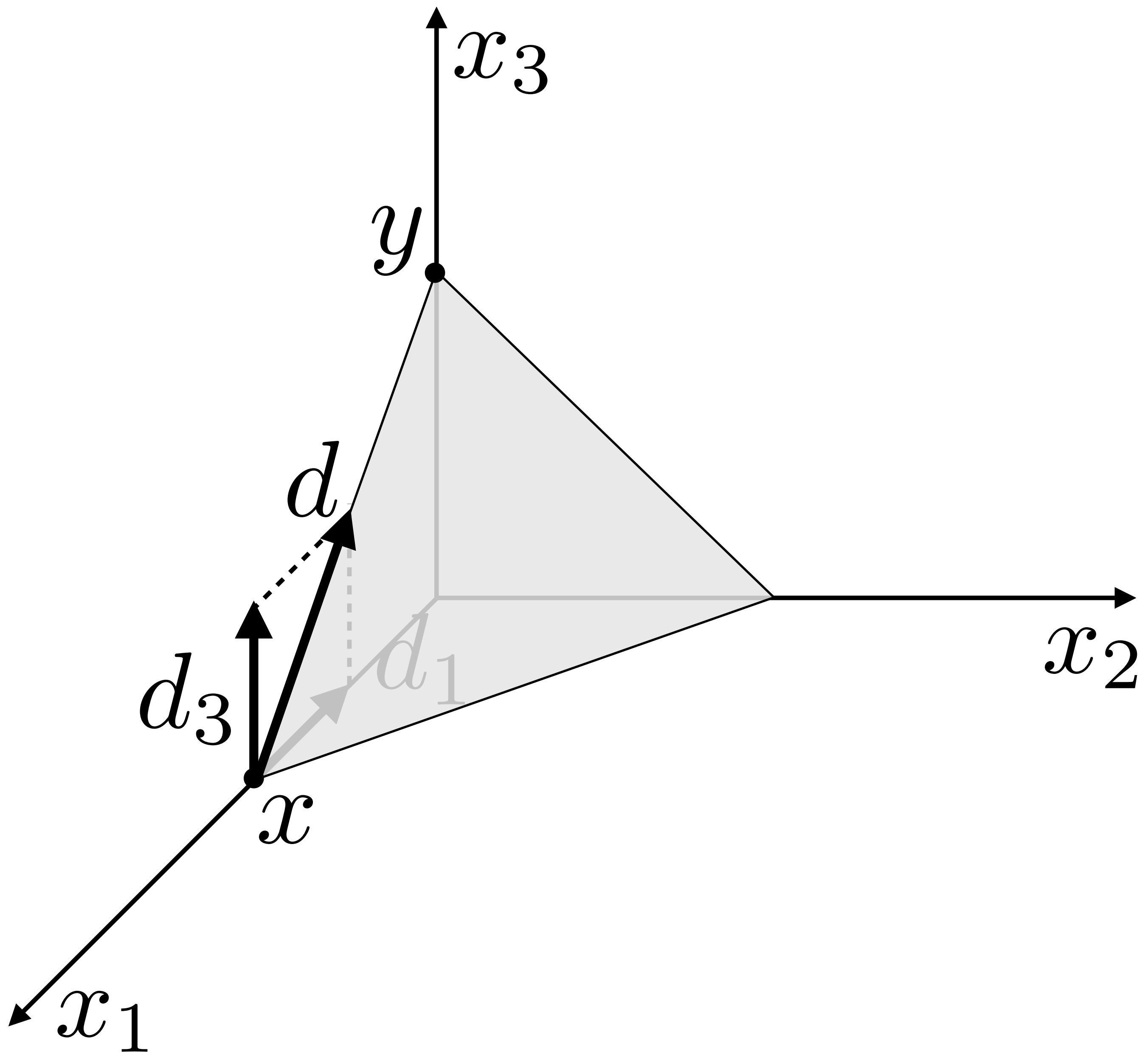
Stepsize $\theta^\star = -\dfrac{x_1}{d_1} = 2$

New solution $y = x + \theta^\star d = (0, 0, 2) \qquad \bar{B} = \{3\}$



Finite convergence

Assume that

- $P = \{x \mid Ax = b, x \geq 0\}$ not empty
- Every basic feasible solution **non degenerate**

Finite convergence

Assume that

- $P = \{x \mid Ax = b, x \geq 0\}$ not empty
- Every basic feasible solution **non degenerate**

Then

- The simplex method **terminates after a finite number of iterations**
- At termination we either have one of the following
  - an **optimal basis** B
  - a **direction** d such that $Ad = 0,\ d \geq 0,\ c^T d < 0$ and the optimal cost is $-\infty$

Finite convergence

Proof sketch

At each iteration the algorithm improves

- by a **positive** amount $\theta^\star$
- along the **direction** d such that $c^T d < 0$

Finite convergence

Proof sketch

At each iteration the algorithm improves

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Therefore

- The cost strictly decreases
- No basic feasible solution can be visited twice

Finite convergence
Proof sketch

At each iteration the algorithm improves

- by a **positive** amount $\theta^\star$
- along the **direction** d such that $c^T d < 0$

Therefore

- The cost strictly decreases
- No basic feasible solution can be visited twice

Since there is a **finite number of basic feasible solutions**

The algorithm **must eventually terminate**

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The simplex method

Today, we learned to:

- **Iterate** between basic feasible solutions
- **Verify** optimality and unboundedness conditions
- **Apply** a single iteration of the simplex method
- **Prove** finite convergence of the simplex method in the non-degenerate case

Next lecture

- Finding initial basic feasible solution
- Degeneracy
- Complexity