ORF522 – Linear and Nonlinear Optimization

7. Linear optimization duality
Ed Forum

- How do we pick a permutation matrix?
  Optimal way (intractable). In practice, we heuristics. A famous one is Approximate Minimum Degree ordering (AMD)

- Why $O(n^3)$ complexity for LU factorization?
  It can be computed with an algorithm called: Gaussian Elimination with Partial pivoting. *Its complexity is $O(n^3)$*

- Is there a standard number of times the same $A_B$ can be used before needing to refactored?
  Not standard because it depends on the problem dimensions. In practice, around 100 iterations.
Recap
Linear optimization formulations

**Standard form LP**

- minimize $c^T x$
- subject to $Ax = b$
- $x \geq 0$

**Inequality form LP**

- minimize $c^T x$
- subject to $Ax \leq b$
Today’s agenda
Readings: [Chapter 4, Bertsimas, Tsitsiklis][Chapter 5, Vanderbei]

• Obtaining lower bounds
• The dual problem
• Weak and strong duality
Obtaining lower bounds
Obtaining lower bounds

A simple example

minimize \( x_1 + 3x_2 \)
subject to \( x_1 + 3x_2 \geq 2 \)

What is a lower bound on the optimal cost?

A lower bound is 2 because \( x_1 + 3x_2 \geq 2 \)
Obtaining lower bounds
Another example

minimize \( x_1 + 3x_2 \)
subject to \( x_1 + x_2 \geq 2 \)
\( x_2 \geq 1 \)

What is a lower bound on the optimal cost?

Let’s sum the constraints
\[
1 \cdot (x_1 + x_2 \geq 2) \\
+ 2 \cdot (x_2 \geq 1) \\
= x_1 + 3x_2 \geq 4
\]
A lower bound is 4
Obtaining lower bounds
A more interesting example

minimize \( x_1 + 3x_2 \)
subject to \( x_1 + x_2 \geq 2 \)
\( x_2 \geq 1 \)
\( x_1 - x_2 \geq 3 \)

How can we obtain a lower bound?

Add constraints
\[
\begin{align*}
y_1 \cdot (x_1 + x_2 \geq 2) \\
y_2 \cdot (x_2 \geq 1) \\
y_3 \cdot (x_1 - x_2 \geq 3)
\end{align*}
\]
\[
= x_1 + 3x_2 \geq 2y_1 + y_2 + 3y_3
\]

Match cost coefficients
\[
\begin{align*}
y_1 + y_3 &= 1 \\
y_1 + y_2 - y_3 &= 3 \\
y_1, y_2, y_3 &\geq 0
\end{align*}
\]

Many options
\[
\begin{align*}
y &= (1, 2, 0) \Rightarrow \text{Bound 4} \\
y &= (0, 4, 1) \Rightarrow \text{Bound 7}
\end{align*}
\]

How can we get the best one?
Obtaining lower bounds
A more interesting example — Best lower bound

We can obtain the best lower bound by solving the following problem

maximize \[ 2y_1 + y_2 + 3y_3 \]
subject to \[ y_1 + y_3 = 1 \]
\[ y_1 + y_2 - y_3 = 3 \]
\[ y_1, y_2, y_3 \geq 0 \]

This linear optimization problem is called the dual problem
The dual problem
Lagrange multipliers

Consider the LP in standard form
\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\]

Relax the constraint
\[
\begin{align*}
g(y) = \quad \text{minimize} & \quad c^T x + y^T (Ax - b) \\
\text{subject to} & \quad x \geq 0
\end{align*}
\]

Lower bound
\[
g(y) \leq c^T x^* + y^T (Ax^* - b) = c^T x^*
\]

Best lower bound
\[
\text{maximize } y \quad g(y)
\]
The dual

Dual function

\[ g(y) = \min_{x \geq 0} \left( c^T x + y^T (Ax - b) \right) \]

\[ = -b^T y + \min_{x \geq 0} \left( c + A^T y \right)^T x \]

\[ g(y) = \begin{cases} 
-b^T y & \text{if } c + A^T y \geq 0 \\
-\infty & \text{otherwise} 
\end{cases} \]

Dual problem (find the best bound)

\[ \max_y g(y) = \max \quad -b^T y \]

subject to \[ A^T y + c \geq 0 \]
# Primal and dual problems

<table>
<thead>
<tr>
<th>Primal problem</th>
<th>Dual problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>minimize $c^T x$</td>
<td>maximize $-b^T y$</td>
</tr>
<tr>
<td>subject to $Ax = b$</td>
<td>subject to $A^T y + c \geq 0$</td>
</tr>
<tr>
<td>$x \geq 0$</td>
<td>$y \in \mathbb{R}^m$</td>
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</tbody>
</table>

Primal variable $x \in \mathbb{R}^n$  
Dual variable $y \in \mathbb{R}^m$

The dual problem carries **useful information** for the primal problem.

Duality is useful also to **solve** optimization problems.
Dual of inequality form LP

What if you find an LP with inequalities?

\[
\begin{align*}
& \text{minimize} & & c^T x \\
& \text{subject to} & & Ax \leq b
\end{align*}
\]

1. We could first transform it to standard form
2. We can compute the dual function (same procedure as before)

Relax the constraint
\[
g(y) = \min_x c^T x + y^T (Ax - b)
\]

Lower bound
\[
g(y) \leq c^T x^* + y^T (Ax^* - b) \leq c^T x^*
\]
we must have \( y \geq 0 \)
Dual of LP with inequalities

Derivation

**Dual function**

\[ g(y) = \min_x (c^T x + y^T (Ax - b)) \]

\[ = -b^T y + \min_x (c + A^T y)^T x \]

\[ g(y) = \begin{cases} 
-b^T y & \text{if } c + A^T y = 0 \ (\text{and } y \geq 0) \\
-\infty & \text{otherwise}
\end{cases} \]

**Dual problem** (find the best bound)

\[ \max_y g(y) = \max_y -b^T y \]

subject to \[ A^T y + c = 0 \]

\[ y \geq 0 \]
# General forms

<table>
<thead>
<tr>
<th>Primal</th>
<th>Standard form LP</th>
<th>Dual</th>
</tr>
</thead>
<tbody>
<tr>
<td>minimize $c^T x$</td>
<td>maximize $-b^T y$</td>
<td>subject to $A^T y + c \geq 0$</td>
</tr>
<tr>
<td>subject to $Ax = b$</td>
<td>subject to $Ax = b$</td>
<td>$x \geq 0$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Primal</th>
<th>Inequality form LP</th>
<th>Dual</th>
</tr>
</thead>
<tbody>
<tr>
<td>minimize $c^T x$</td>
<td>maximize $-b^T y$</td>
<td>subject to $A^T y + c = 0$</td>
</tr>
<tr>
<td>subject to $Ax \leq b$</td>
<td>subject to $Ax \leq b$</td>
<td>$y \geq 0$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Primal</th>
<th>LP with inequalities and equalities</th>
<th>Dual</th>
</tr>
</thead>
<tbody>
<tr>
<td>minimize $c^T x$</td>
<td>maximize $-b^T y - d^T z$</td>
<td>subject to $Ax \leq b$</td>
</tr>
<tr>
<td>subject to $Ax \leq b$</td>
<td>subject to $Ax \leq b$</td>
<td>$C x = d$</td>
</tr>
<tr>
<td></td>
<td>subject to $A^T y + C^T z + c = 0$</td>
<td>$y \geq 0$</td>
</tr>
</tbody>
</table>
Example from before

minimize \( x_1 + 3x_2 \)
subject to \( x_1 + x_2 \geq 2 \)
\( x_2 \geq 1 \)
\( x_1 - x_2 \geq 3 \)

\[ \text{Dual} \]
maximize \(-b^T y\)
subject to \( A^T y + c = 0 \)
\( y \geq 0 \)

\[ \text{Inequality form LP} \]
minimize \( c^T x \)
subject to \( Ax \leq b \)

\[ c = (1, 3) \]
\[ A = \begin{bmatrix} -1 & -1 \\ 0 & -1 \\ -1 & 1 \end{bmatrix} \]
\[ b = (-2, -1, -3) \]

maximize \( 2y_1 + y_2 + 3y_3 \)
subject to \(-y_1 - y_3 = -1 \)
\(-y_1 - y_2 + y_3 = -3 \)
\( y_1, y_2, y_3 \geq 0 \)
To memorize

Ways to get the dual
• Derive dual function directly
• Transform the problem in inequality form LP and dualize

Sanity-checks and signs convention
• Consider constraints as $g(x) \leq 0$ or $g(x) = 0$
• Each dual variable is associated to a primal constraint
• $y$ free for primal equalities and $y \geq 0$ for primal inequalities
Dual of the dual

**Theorem**
If we transform the primal into its dual and then transform the dual to its dual, we obtain a problem equivalent to the original problem. In other words, the dual of the dual is the primal.

**Exercise**
Derive dual and dualize again

<table>
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<tr>
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<td>minimize $c^T x$</td>
<td>maximize $-b^T y - d^T z$</td>
</tr>
<tr>
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<td>subject to $A^T y + C^T z + c = 0$</td>
</tr>
<tr>
<td>$C x = d$</td>
<td>$y \geq 0$</td>
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**Theorem**
If we transform a linear optimization problem to another form (inequality form, standard form, inequality and equality form), the dual of the two problems will be equivalent.
Weak and strong duality
Optimal objective values

**Primal**
- minimize \( c^T x \)
- subject to \( Ax \leq b \)

\( p^* \) is the primal optimal value

Primal infeasible: \( p^* = +\infty \)
Primal unbounded: \( p^* = -\infty \)

**Dual**
- maximize \( -b^T y \)
- subject to \( A^T y + c = 0 \)
  - \( y \geq 0 \)

\( d^* \) is the dual optimal value

Dual infeasible: \( d^* = -\infty \)
Dual unbounded: \( d^* = +\infty \)
Weak duality

Theorem
If $x, y$ satisfy:

- $x$ is a feasible solution to the primal problem
- $y$ is a feasible solution to the dual problem

\[ -b^T y \leq c^T x \]

Proof
We know that $Ax \leq b$, $A^T y + c = 0$ and $y \geq 0$. Therefore,

\[ 0 \leq y^T (b - Ax) = b^T y - y^T Ax = c^T x + b^T y \]

Remark
- Any dual feasible $y$ gives a lower bound on the primal optimal value
- Any primal feasible $x$ gives an upper bound on the dual optimal value
- $c^T x + b^T y$ is the duality gap
Weak duality
Corollaries

Unboundedness vs feasibility
- Primal unbounded \((p^* = -\infty)\) \(\Rightarrow\) dual infeasible \((d^* = -\infty)\)
- Dual unbounded \((d^* = +\infty)\) \(\Rightarrow\) primal infeasible \((p^* = +\infty)\)

Optimality condition
If \(x, y\) satisfy:
- \(x\) is a feasible solution to the primal problem
- \(y\) is a feasible solution to the dual problem
- The duality gap is zero, i.e., \(c^T x + b^T y = 0\)

Then \(x\) and \(y\) are optimal solutions to the primal and dual problem respectively
Strong duality

Theorem
If a linear optimization problem has an optimal solution, so does its dual, and the optimal value of primal and dual are equal

\[ d^* = p^* \]
Strong duality

Constructive proof

Given a primal optimal solution \( x^* \) we will construct a dual optimal solution \( y^* \).

Apply simplex to problem in **standard form**

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\]

- optimal basis \( B \)
- optimal solution \( x^* \) with \( A_B x_B^* = b \)
- reduced costs \( \bar{c} = c - A^T A_B^{-T} c_B \geq 0 \)

Define \( y^* \) such that \( y^* = -A_B^{-T} c_B \). Therefore, \( A^T y^* + c \geq 0 \) (\( y^* \) dual feasible).

\[
-b^T y^* = -b^T (-A_B^{-T} c_B) = c_B^T (A_B^{-1} b) = c_B^T x_B^* = c^T x^*
\]

By weak duality theorem corollary, \( y^* \) is an optimal solution of the dual.
Therefore, \( d^* = p^* \).
## Exception to strong duality

<table>
<thead>
<tr>
<th>Primal</th>
<th>Dual</th>
</tr>
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<tbody>
<tr>
<td>minimize $x$</td>
<td>maximize $y$</td>
</tr>
</tbody>
</table>
| subject to $0 \cdot x \leq -1$ | subject to $0 \cdot y + 1 = 0$
|                 | $y \geq 0$    |

Optimal value is $p^* = +\infty$

Optimal value is $d^* = -\infty$

Both **primal** and **dual infeasible**
## Relationship between primal and dual

<table>
<thead>
<tr>
<th></th>
<th>$p^* = +\infty$</th>
<th>$p^*$ finite</th>
<th>$p^* = -\infty$</th>
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</thead>
<tbody>
<tr>
<td>$d^* = +\infty$</td>
<td>primal inf. dual unb.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$d^*$ finite</td>
<td>optinal values equal</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$d^* = -\infty$</td>
<td>exception</td>
<td></td>
<td>primal unbound dual inf</td>
</tr>
</tbody>
</table>

- Upper-right excluded by weak duality
- $(1, 1)$ and $(3, 3)$ proven by weak duality
- $(3, 1)$ and $(2, 2)$ proven by strong duality
Example
Production problem

maximize \( x_1 + 2x_2 \) \hfill \text{Profits}
subject to
\[
\begin{align*}
  & x_1 \leq 100 \\
  & 2x_2 \leq 200 \\
  & x_1 + x_2 \leq 150 \\
  & x_1, x_2 \geq 0
\end{align*}
\]

\text{Resources}

Dualize

1. Transform in inequality form
2. Derive dual

minimize \( c^T x \)
subject to
\[
\begin{align*}
  & Ax \leq b \\
  & A^T y + c = 0 \\
  & y \geq 0
\end{align*}
\]

\( c = (-1, -2) \)
\[
A = \begin{bmatrix}
  1 & 0 \\
  0 & 2 \\
  1 & 1 \\
  -1 & 0 \\
  0 & -1
\end{bmatrix}
\]
\( b = (100, 200, 150, 0, 0) \)
Production problem

The dual

minimize \[ 100y_1 + 200y_2 + 150y_3 \]
subject to \[
\begin{align*}
y_1 + y_3 & \geq 1 \\
2y_2 + y_3 & \geq 2 \\
y_1, y_2, y_3 & \geq 0
\end{align*}
\]

Interpretation

• Sell all your resources at a fair (minimum) price
• Selling must be more convenient than producing:
  – Product 1 (price 1, needs 1× resource 1 and 3): \[ y_1 + y_3 \geq 1 \]
  – Product 2 (price 2, needs 2× resource 2 and 1× resource 3): \[ 2y_2 + y_3 \geq 2 \]
Linear optimization duality

Today, we learned to:

• **Dualize** linear optimization problems

• **Prove** weak and strong duality conditions

• **Interpret** simple dual optimization problems
Next lecture

More on duality:

• Game theoretic interpretation
• Complementary slackness
• Alternative systems