ORF522 – Linear and Nonlinear Optimization

5. The simplex method

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Ed Forum

- Can neighboring basic solutions be infeasible?

  Yes!

- Is there a chance that as we move from our starting basic feasible point and check all the neighboring solutions and find none of them to be more optimal, that we miss another point (that isn't neighboring) that could be better? Is this an issue of identifying local vs. global optima?

  “More optimal” does not exist! There is no way to get better solutions there. Proof of this in previous lecture. Yes, this is due to global optimality for LPs.

- I was under the impression that solvers used a standard step size for each problem and that they did not iteratively calculate one every single step. Would this not increase computational time in a significant manner..?
  Standard step size is not a thing for simplex and interior-point methods. It always changes.

- I'm not exactly sure why d_j is always equal to one, and how do the equations and the picture correspond exactly?

  Directions can be rescaled as we please (and change theta accordingly). We set d_j=1 to simplify the math instead of having, e.g., d_j=1.947 (which would allow us to derive the same things.

Recap

Standard form polyhedra

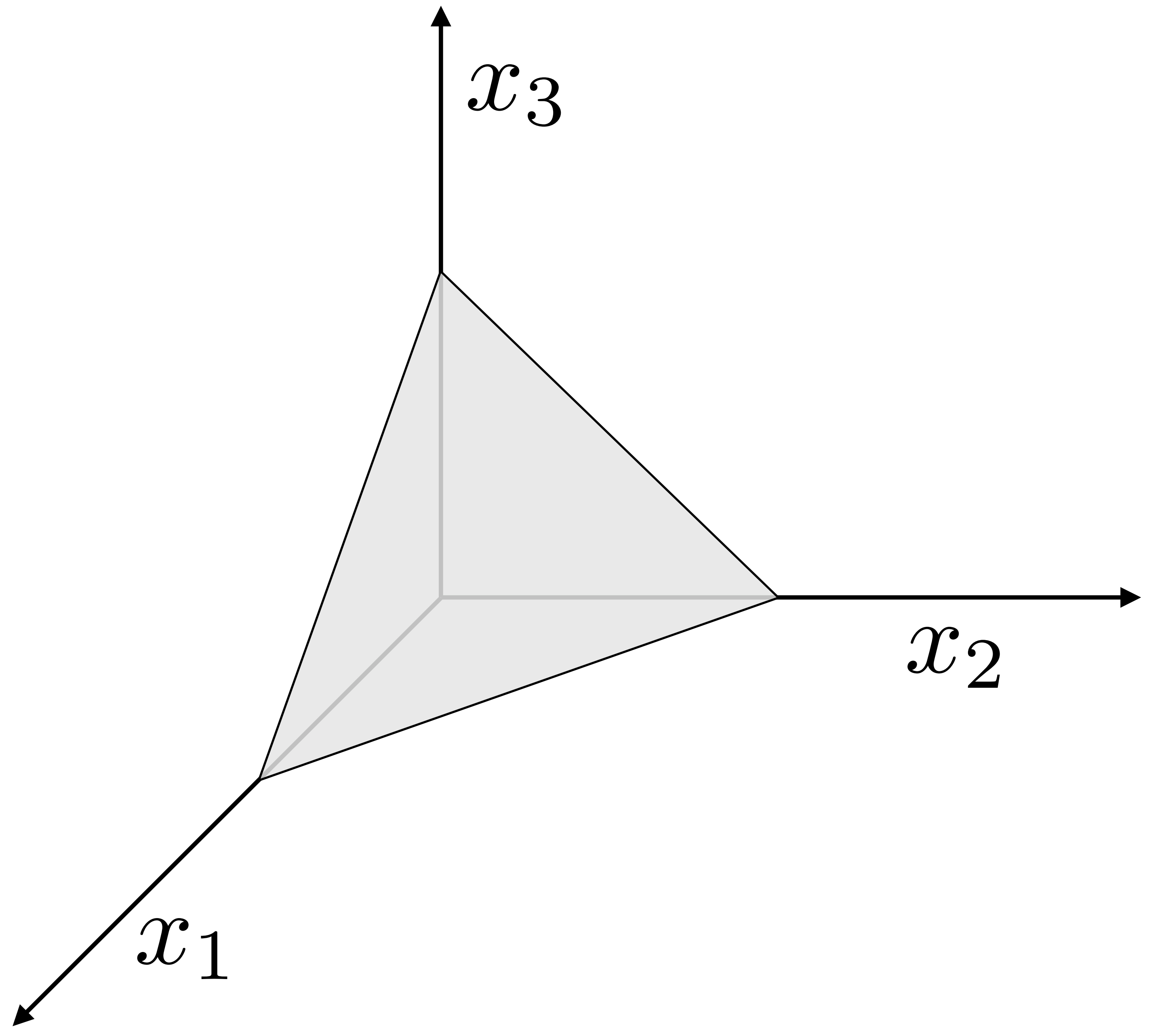
Definition

Standard form LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Standard form polyhedron

$$P = \{x \mid Ax = b,\ x \geq 0\}$$



Assumption

$A \in \mathbf{R}^{m \times n}$ has full row rank $m \leq n$

Interpretation

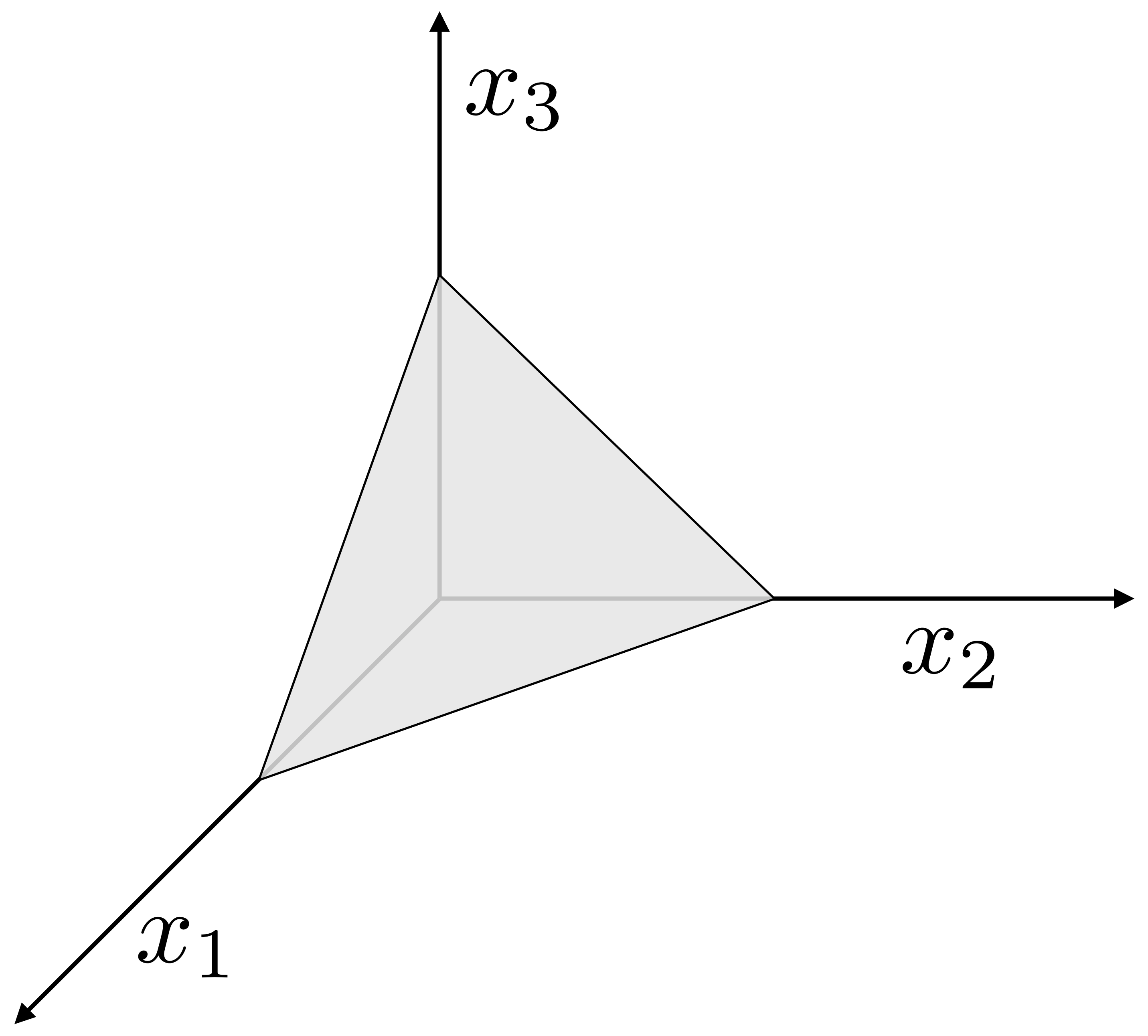
P lives in $(n-m)$-dimensional subspace

Standard form polyhedra

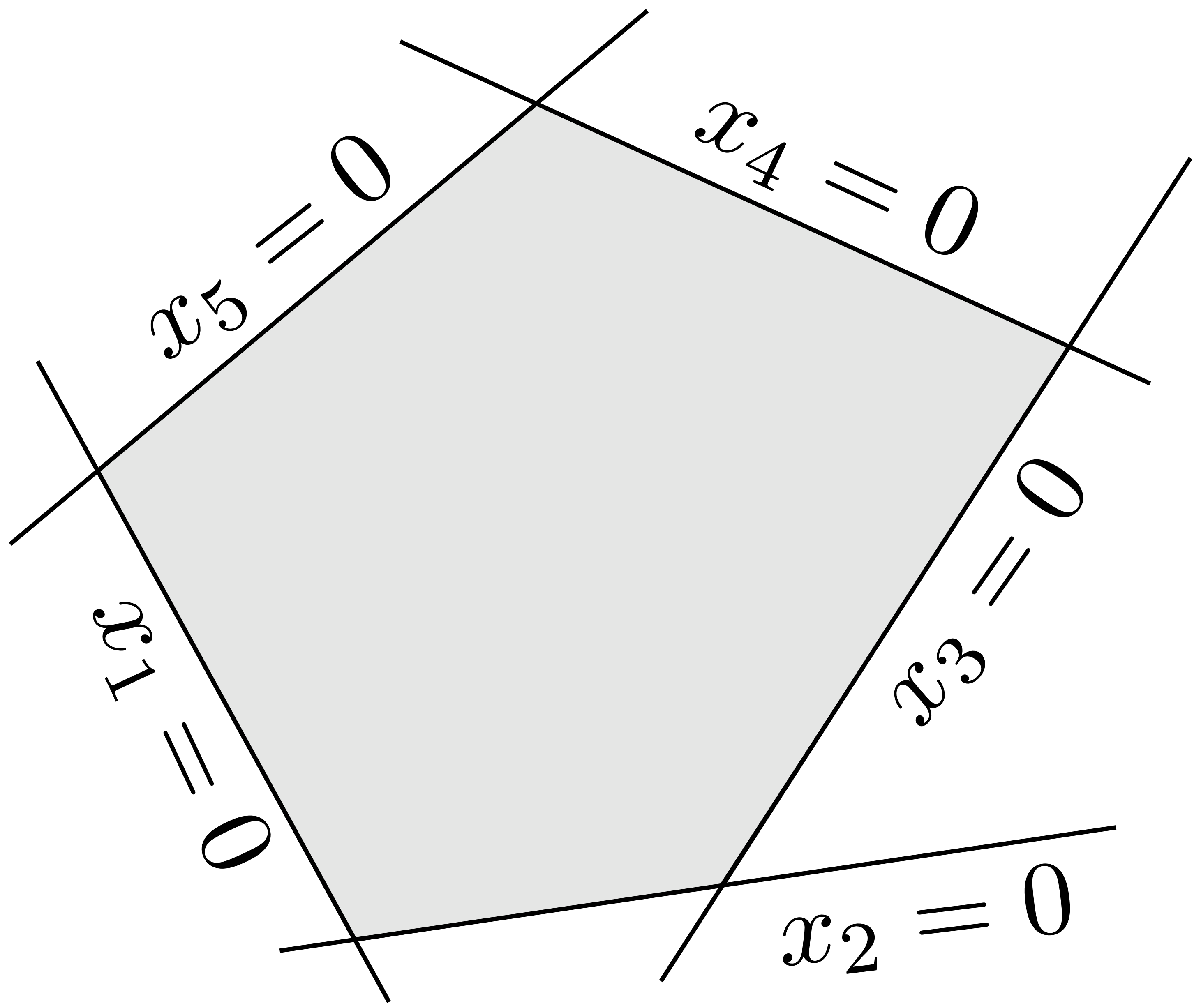
Visualization

$$P = \{x \mid Ax = b,\ x \geq 0\}, \quad n - m = 2$$

Three dimensions



Higher dimensions



Constructing basic solution

1. Choose any m independent columns of A: $A_{B(1)}, \ldots, A_{B(m)}$
2. Let $x_i = 0$ for all $i \neq B(1), \ldots, B(m)$
3. Solve $Ax = b$ for the remaining $x_{B(1)}, \ldots, x_{B(m)}$

Basis matrix

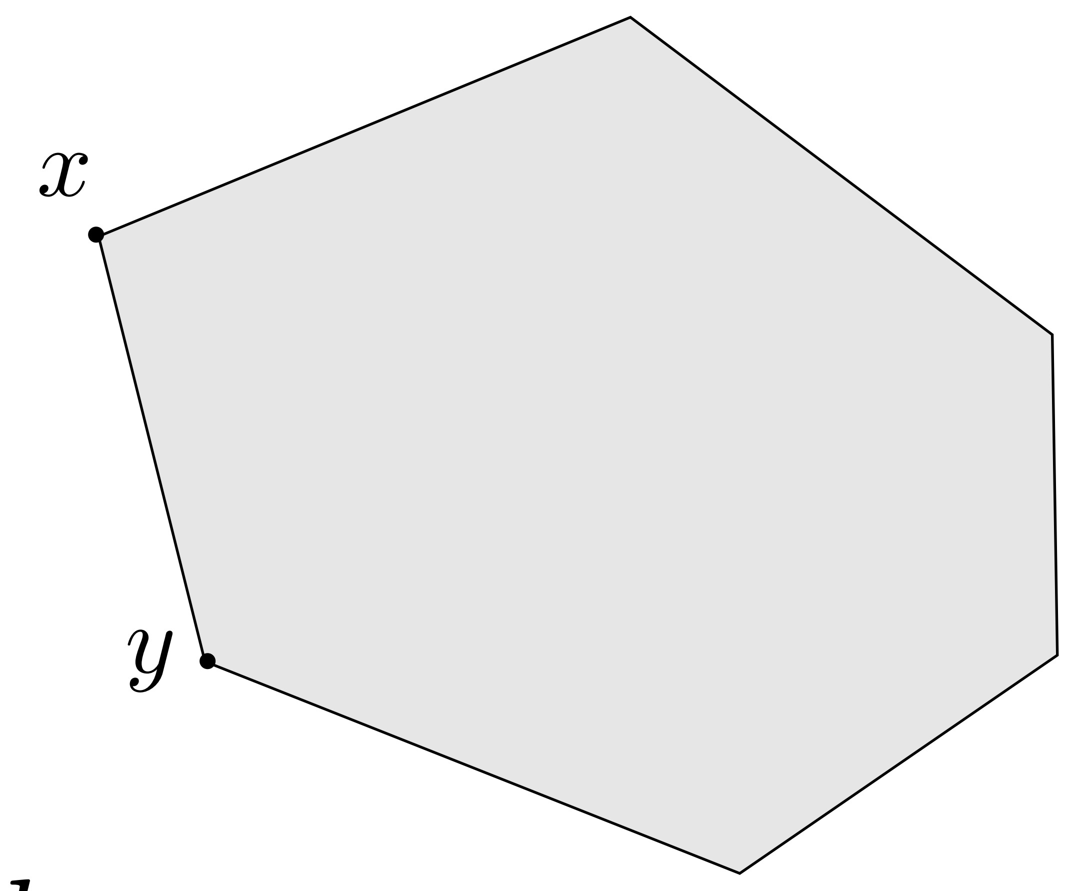
Basis columns

Basic variables

$$A_B = \begin{bmatrix} | & | & & | \\ A_{B(1)} & A_{B(2)} & \ldots & A_{B(m)} \\ | & | & & | \end{bmatrix}, \quad x_B = \begin{bmatrix} x_{B(1)} \\ \vdots \\ x_{B(m)} \end{bmatrix} \longrightarrow \text{Solve } A_B x_B = b$$

If $x_B \geq 0$, then x is a **basic feasible solution**

Neighboring solutions



Two basic solutions are **neighboring** if their basic indices differ by exactly one variable

Example

$$\overset{A}{\begin{bmatrix} 1 & -1 & 0 & 3 & -2 \\ 2 & 0 & -1 & -1 & 0 \\ 0 & 2 & 4 & -1 & 4 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \overset{b}{\begin{bmatrix} -5 \\ -1 \\ 14 \end{bmatrix}}$$

$$B = \{1, 3, 5\} \qquad x_2 = x_4 = 0$$

$$A_B x_B = b \longrightarrow x_B = \begin{bmatrix} x_1 \\ x_3 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2.5 \end{bmatrix}$$

$$\bar{B} = \{1, 3, 4\} \qquad y_2 = y_5 = 0$$

$$A_{\bar{B}} y_{\bar{B}} = b \longrightarrow y_{\bar{B}} = \begin{bmatrix} y_1 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 3.0 \\ -1.7 \end{bmatrix}$$

Feasible directions

Conditions

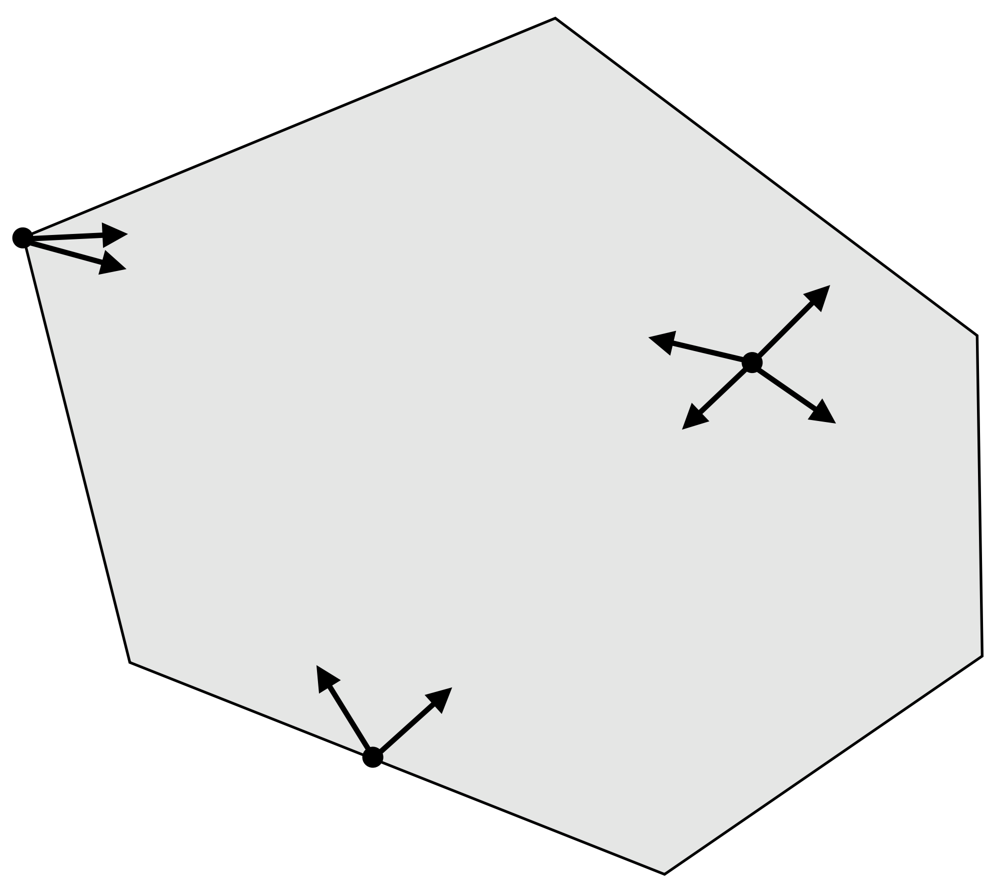
$$P = \{x \mid Ax = b,\ x \geq 0\}$$

Given a basis matrix $A_B = \begin{bmatrix} A_{B(1)} & \dots & A_{B(m)} \end{bmatrix}$

we have basic feasible solution x:

- x_B solves $A_B x_B = b$
- $x_i = 0,\ \forall i \neq B(1), \dots, B(m)$

Let $x \in P$, a vector d is a **feasible direction** at x
if $\exists \theta > 0$ for which $x + \theta d \in P$



Feasible direction d

- $A(x + \theta d) = b \implies Ad = 0$
- $x + \theta d \geq 0$

Feasible directions

Computation

Feasible direction d

- $A(x + \theta d) = b \implies Ad = 0$
- $x + \theta d \geq 0$

Nonbasic indices

- $d_j = 1 \longrightarrow$ **Basic direction**
- $d_k = 0, \ \forall k \notin \{j, B(1), \dots, B(m)\}$

Basic indices

$$Ad = 0 = \sum_{i=1}^{n} A_i d_i = A_B d_B + A_j = 0 \implies d_B = -A_B^{-1} A_j$$

Non-negativity (non-degenerate assumption)

- Non-basic variables: $x_i = 0$. Nonnegative direction $d_i \geq 0$
- Basic variables: $x_B > 0$. Therefore $\exists \theta > 0$ such that $x_B + \theta d_B \geq 0$

Stepsize

What happens if some $\bar{c}_j$ < 0?

We can decrease the cost by bringing x_j into the basis

How far can we go?

$$\theta^\star = \max\{\theta \mid \theta \geq 0 \text{ and } x + \theta d \geq 0\}$$

d is the j-th basic direction

Unbounded

If $d \geq 0$, then $\theta^\star = \infty$. The LP is unbounded.

Bounded

If $d_i < 0$ for some i, then

$$\theta^\star = \min_{\{i | d_i < 0\}} \left(-\frac{x_i}{d_i}\right) = \min_{\{i \in B | d_i < 0\}} \left(-\frac{x_i}{d_i}\right)$$

(Since $d_i \geq 0,\ i \notin B$)

Moving to a new basis

Next feasible solution

$$x + \theta^\star d$$

Let $B(\ell) \in \{B(1), \dots, B(m)\}$ be the index such that $\theta^\star = -\dfrac{x_{B(\ell)}}{d_{B(\ell)}}$. Then,

$$x_{B(\ell)} + \theta^\star d_{B(\ell)} = 0$$

New solution

- $x_{B(\ell)}$ becomes 0 (exits)
- x_j becomes $\theta^\star$ (enters)

New basis

$$A_{\bar{B}} = \begin{bmatrix} A_{B(1)} & \dots & A_{B(\ell-1)} & A_j & A_{B(\ell+1)} & \dots & A_{B(m)} \end{bmatrix}$$

An iteration of the simplex method

Initialization

- a basic feasible solution x
- a basis matrix $A_B = \begin{bmatrix} A_{B(1)} & \dots, A_{B(m)} \end{bmatrix}$

Iteration steps

1. Compute the reduced costs $\bar{c}$
    - Solve $A_B^T p = c_B$
    - $\bar{c} = c - A^T p$
2. If $\bar{c} \geq 0$, x **optimal**. **break**
3. Choose j such that $\bar{c}_j < 0$
4. Compute search direction d with $d_j = 1$ and $A_B d_B = -A_j$
5. If $d_B \geq 0$, the problem is **unbounded** and the optimal value is $-\infty$. **break**
6. Compute step length $\theta^\star = \min_{\{i \in B | d_i < 0\}} \left(-\frac{x_i}{d_i} \right)$
7. Define y such that $y = x + \theta^\star d$
8. Get new basis $\bar{B}$ (i exits and j enters)

Today's agenda
[Chapter 3, LO]

- Find initial feasible solution
- Degeneracy
- Complexity

Find an initial point in simplex method

Initial basic feasible solution

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

How do we get an initial **basic feasible solution** x and a **basis** B ?

Does it **exist**?

Finding an initial basic feasible solution

Auxiliary problem

Minimize violations

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & \boxed{\mathbf{1}^T y} \\ \text{subject to} & Ax + y = b \\ & x \geq 0, y \geq 0 \end{array}$$

Assumption $b \geq 0$ w.l.o.g. (if not multiply constraint by -1)

Trivial basic feasible solution: $x = 0, y = b$

Possible outcomes

- **Feasible problem** (cost = 0): $y^\star = 0$ and $x^\star$ is a basic feasible solution
- **Infeasible problem** (cost > 0): $y^\star > 0$ are the violations

Two-phase simplex method

Phase I

1. Construct **auxiliary problem** such that $b \geq 0$
2. Solve auxiliary problem using simplex method starting from $(x, y) = (0, b)$
3. If the optimal value is greater than 0, **problem infeasible**. **break**.

Phase II

1. Recover original problem (drop variables y and restore original cost)
2. Solve original problem starting from the solution x and its basis B.

Big-M method

Very large constant

$$\begin{aligned} \text{minimize} \quad & c^T x + M \mathbf{1}^T y \\ \text{subject to} \quad & Ax + y = b \\ & x \geq 0, y \geq 0 \end{aligned}$$

Incorporate penalty in the cost

- We can still use $y = b \geq 0$ as initial basic feasible solution
- If the problem is **feasible**, y will not be in the basis.

Remarks

- **Pro:** need to solve only one LP
- **Con:** it is not easy to pick M and it makes the problem badly scaled

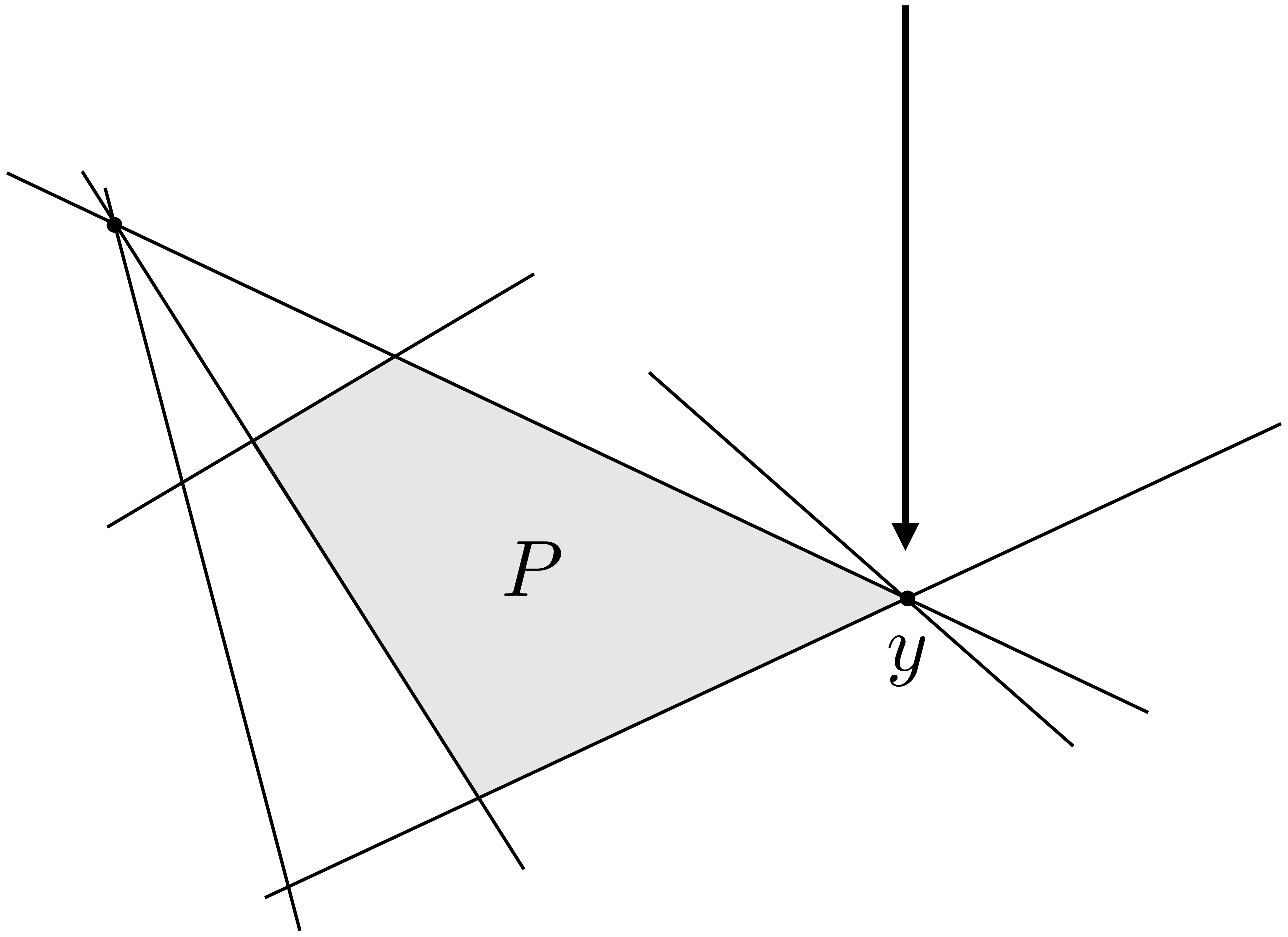
Degeneracy

Degenerate basic feasible solutions

Inequality form polyhedron

A solution y is degenerate if $|\mathcal{I}(\bar{x})| > n$

$$P = \{x \mid Ax \leq b\}$$



Degenerate basic feasible solutions

Standard form polyhedron

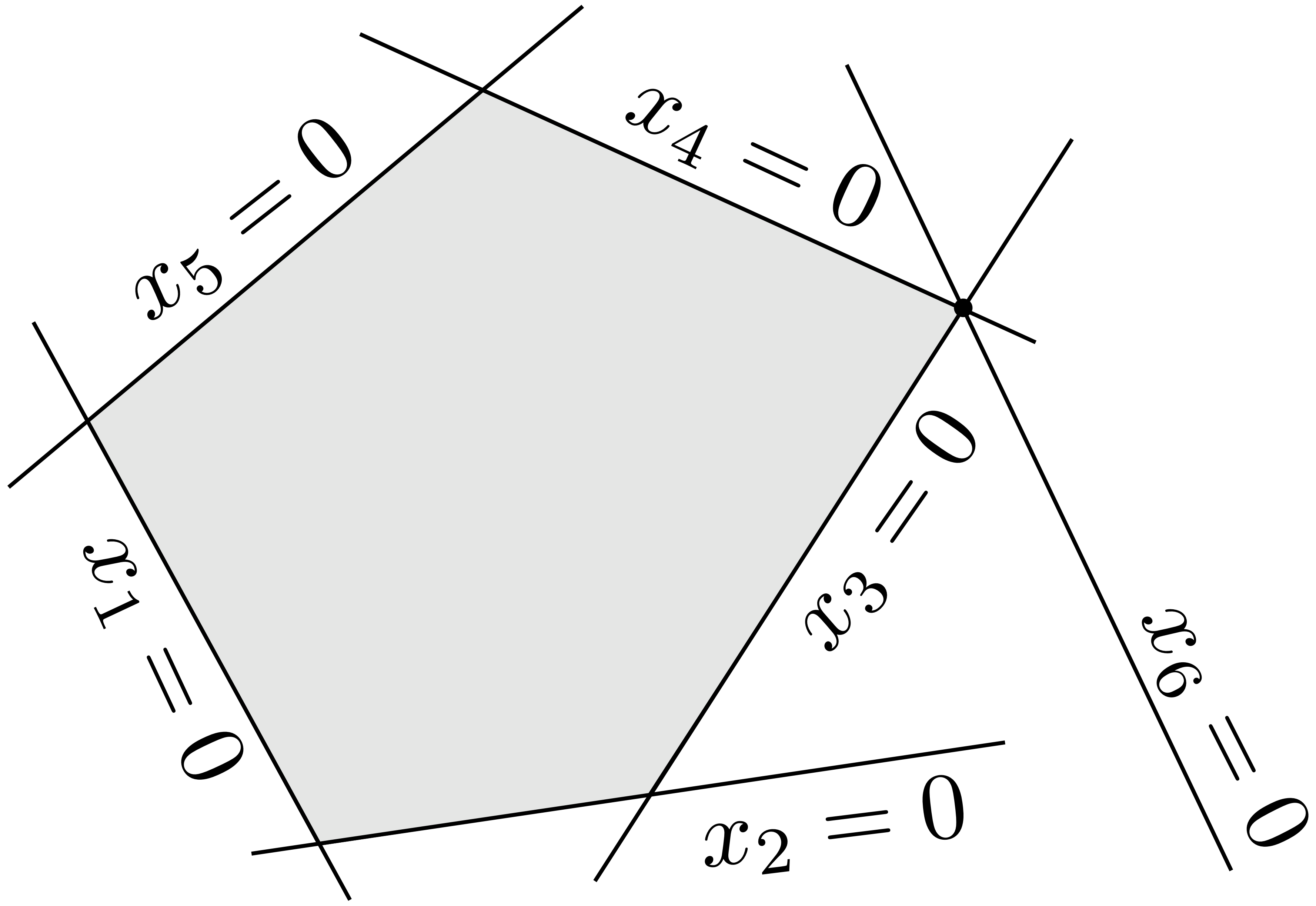
Given a basis matrix $A_B = \begin{bmatrix} A_{B(1)} & \dots & A_{B(m)} \end{bmatrix}$

we have basic feasible solution x:

- $A_B x_B = b$
- $x_i = 0, \ \forall i \neq B(1), \dots, B(m)$

$\longrightarrow$ If some of the $x_B = 0$, then it is a **degenerate solution**

$$P = \{x \mid Ax = b, \ x \geq 0\}$$



Degenerate basic feasible solutions

Example

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 1 \\
-x_1 + x_2 - x_3 &= 1 \\
x_1, x_2, x_3 &\geq 0
\end{aligned}
$$

Degenerate solutions

Basis $B = \{1, 2\}$ $\longrightarrow$ $x = (0, 1, 0)$

Basis $B = \{2, 3\}$ $\longrightarrow$ $y = (0, 1, 0)$

Cycling

Stepsize

6. Compute step length $\theta^\star = \min_{\{i \in B | d_i < 0\}} \left(-\frac{x_i}{d_i} \right)$

$\downarrow$

If $i \in B$, $d_i < 0$ and $x_i = 0$ (degenerate)

$$\theta^\star = 0$$

Therefore $y = x + \theta^\star x = x$ and $B = \bar{B}$

Same solution and cost
Different basis

Finite termination **no longer guaranteed**!

How can we fix it?

Pivoting rules

Pivoting rules

Choose the index entering the basis

Simplex iterations

3. Choose j such that $\bar{c}_j < 0$ $\longrightarrow$ **Which j?**

Possible rules

- **Smallest subscript:** smallest j such that $\bar{c}_j < 0$
- **Most negative:** choose j with the most negative $\bar{c}_j$
- **Largest cost decrement:** choose j with the largest $\theta^\star |\bar{c}_j|$

Pivoting rules

Choose index exiting the basis

Simplex iterations

6. Compute step length $\theta^\star = \min_{\{i \in B \mid d_i < 0\}} \left(-\frac{x_i}{d_i}\right)$ $\longrightarrow$

We can have more than one i for which $x_i = 0$ (**next** solution is **degenerate**)

Which i?

Smallest index rule

Smallest i such that $\theta^\star = -\frac{x_i}{d_i}$

Bland's rule to avoid cycles

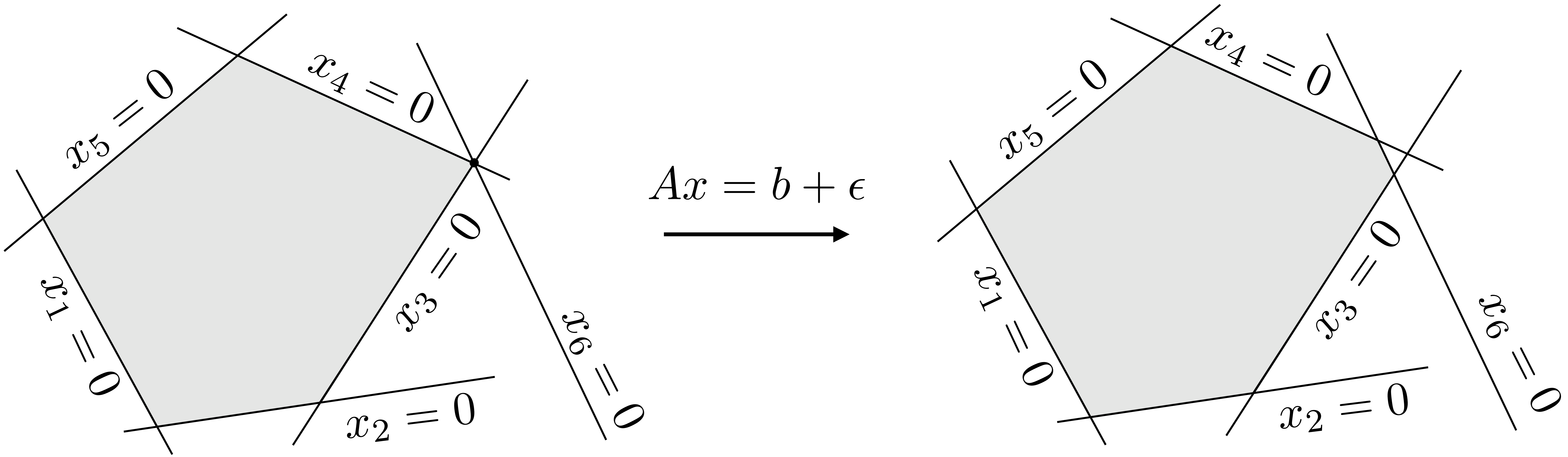
Theorem

If we use the **smallest index rule** for choosing both the j entering the basis and the i leaving the basis, then **no cycling will occur**.

Proof idea [Ch 3, Sec 4, LP][Sec 3.4, LO]

- Assume Bland's rule is applied and there exists a cycle with different bases.
- Obtain contradiction.

Perturbation approach to avoid cycles



Complexity

Complexity

Basic operation: one simplex iteration

Estimate complexity of an algorithm

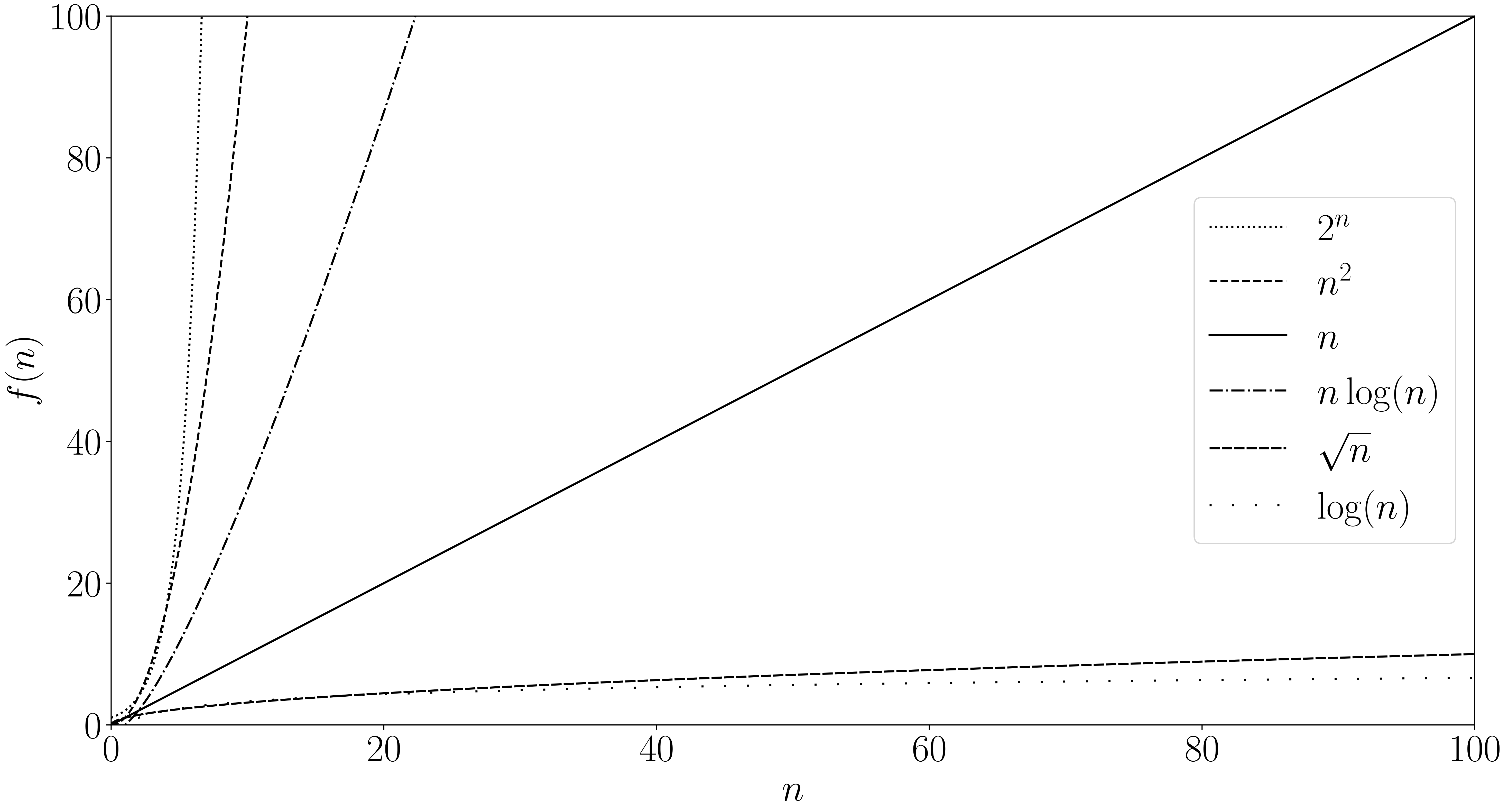
- Write number of basic operations as a **function of problem dimensions**
- Simplify and keep only leading terms

Complexity

Notation

We write $g(x) \sim O(f(x))$ if and only if there exist $c > 0$ and an x_0 such that

$$|g(x)| \leq cf(x), \quad \forall x \geq x_0$$



Polynomial
Practical

Exponential
Impractical!

$\mathcal{P}$ and $\mathcal{NP}$

Complexity class $\mathcal{P}$

There exists a polynomial time algorithms to solve it

Complexity class $\mathcal{NP}$

Given a candidate solution, there exists a polynomial time algorithm to verify it.

Complexity class $\mathcal{NP}$-hard

At least as hard as the hardest problem in $\mathcal{NP}$

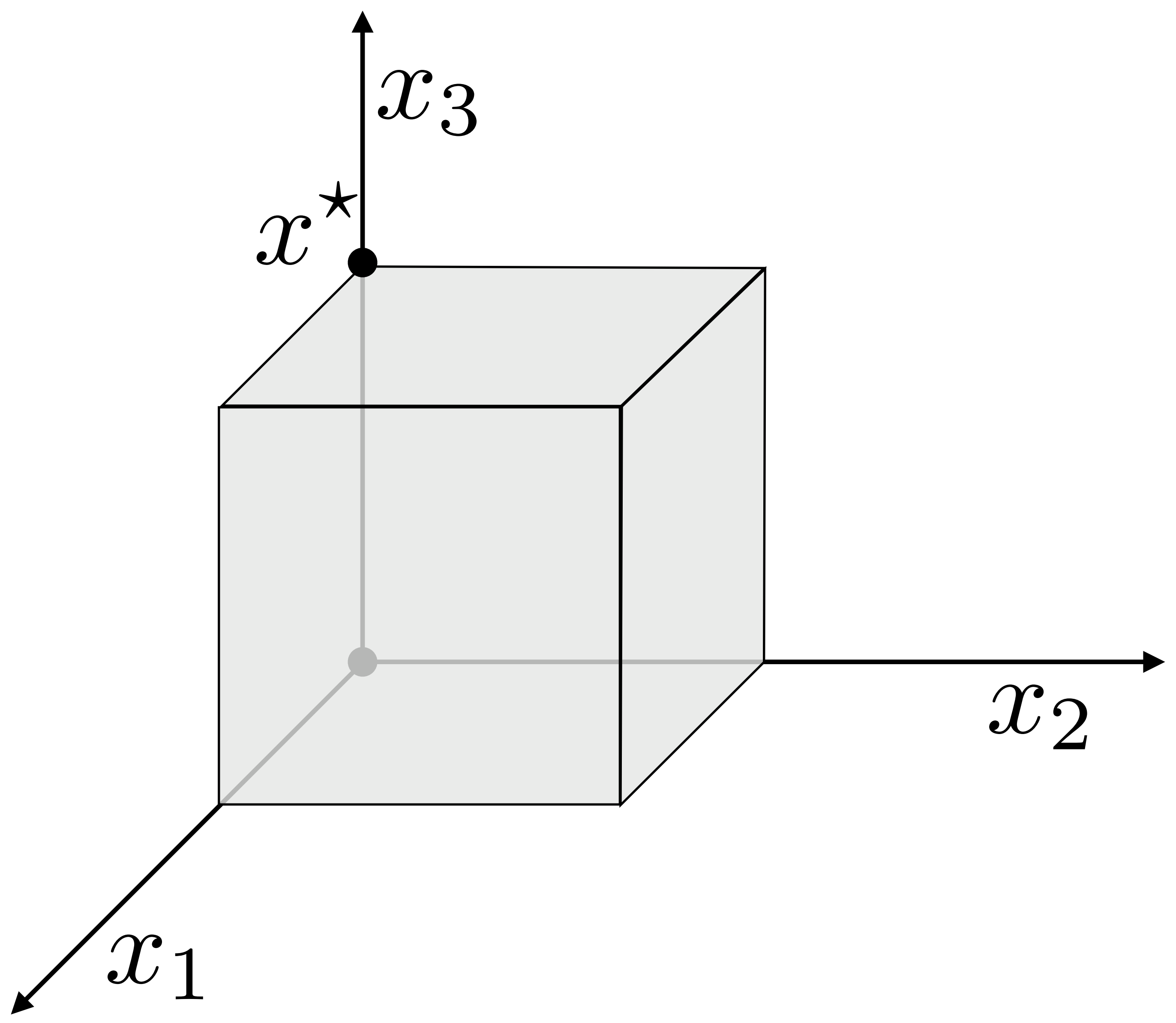
We don't know any **polynomial time algorithm**

Million dollar problem: $\mathcal{P} = \mathcal{NP}$?

- We know that $\mathcal{P} \subset \mathcal{NP}$
- Does it exist a polynomial time algorithm for $\mathcal{NP}$-hard problems?

Complexity of the simplex method

Example of worst-case behavior



Innocent-looking problem

$$\begin{array}{ll} \text{minimize} & -x_n \\ \text{subject to} & 0 \leq x \leq 1 \end{array}$$

2^n **vertices**

$2^n/2$ vertices: cost = 1

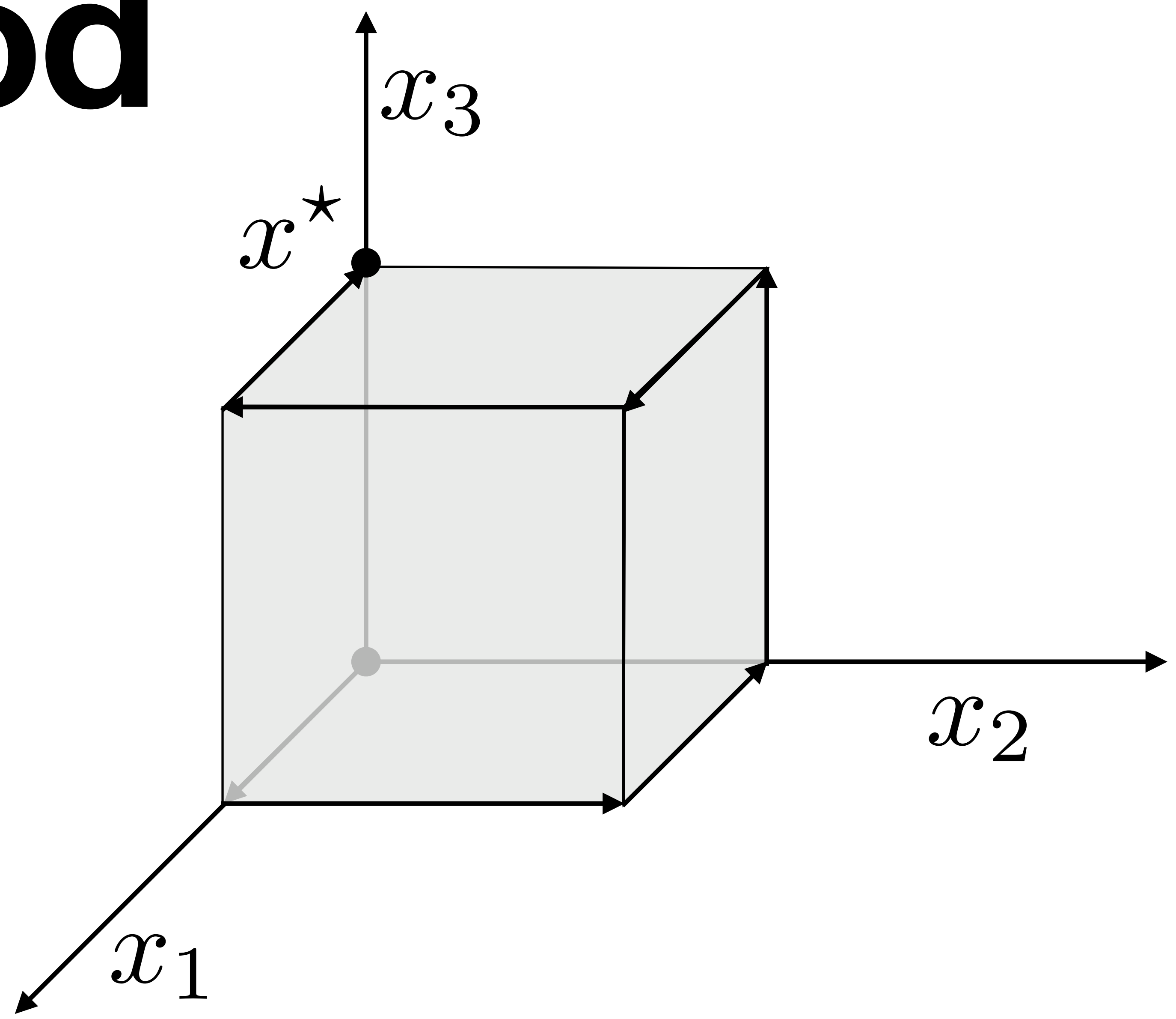
$2^n/2$ vertices: cost = 0

Perturb unit cube

$$\begin{array}{ll} \text{minimize} & -x_n \\ \text{subject to} & \epsilon \leq x_1 \leq 1 \\ & \epsilon x_{i-1} \leq x_i \leq 1 - \epsilon x_{i-1}, \quad i = 2, \dots, n \end{array}$$

Complexity of the simplex method

Example of worst-case behavior



$$
\begin{array}{ll}
\text{minimize} & -x_n \\
\text{subject to} & \epsilon \leq x_1 \leq 1 \\
& \epsilon x_{i-1} \leq x_i \leq 1 - \epsilon x_{i-1}, \quad i = 2, \ldots, n
\end{array}
$$

Theorem

- The vertices can be ordered so that each one is adjacent to and has a **lower cost than the previous one**
- There exists a pivoting rule under which the simplex method terminates after $2^n - 1$ **iterations**

Remark

- A **different pivot rule** would have converged in one iteration.
- We have a bad example for every pivot rule.

Complexity of the simplex method

We do **not know any polynomial version of the simplex method**, no matter which pivoting rule we pick. $\longrightarrow$ Still **open research question!**

Worst-case

There are problem instances where the simplex method will run an **exponential number of iterations** in terms of the dimensions n and m: $O(2^n)$

Good news: average-case

Practical performance is very good. On average, it stops in $O(n)$ iterations.

The simplex method

Today, we learned to:

- **Formulate** auxiliary problem to find starting simplex solutions
- **Apply** pivoting rules to avoid cycling in degenerate linear programs
- **Analyze** complexity of the simplex method

Next lecture

- Numerical linear algebra
- “Realistic" simplex implementation
- Examples