ORF522 – Linear and Nonlinear Optimization
4. The simplex method
Ed Forum

• Notebooks on GitHub: https://github.com/ORF522/companion

• Office hours change:
  Prof. Stellato: Thu 3:30pm-5:30pm
  Scander Mustapha: Mon: 1:30pm-3:30pm

• 10% Participation. The note should **summarize what you learned** in the last lecture, and **highlight the concepts that were most confusing** or that you would like to review. A note will receive full credit if: it is **submitted before the beginning of next lecture**, it is **related to the content** of the lecture, and it is **understandable** and coherent.

• Question: connection between geometry and standard form? Yes, they are equivalent (more in the next slides)
Recap
Equivalence

Theorem

Given a nonempty polyhedron $P = \{x \mid Ax \leq b\}$

Let $x \in P$

$x$ is a vertex $\iff$ $x$ is an extreme point $\iff$ $x$ is a basic feasible solution
Basic feasible solution

\[ P = \{ x \mid a_i^T x \leq b_i, \quad i = 1, \ldots, m \} \]
Basic feasible solution

\[ P = \{ x \mid a_i^T x \leq b_i, \quad i = 1, \ldots, m \} \]

Active constraints at \( \bar{x} \)

\[ \mathcal{I}(\bar{x}) = \{ i \in \{1, \ldots, m\} \mid a_i^T \bar{x} = b_i \} \]

Index of all the constraints satisfied as equality
Basic feasible solution

\[ P = \{ x \mid a_i^T x \leq b_i, \quad i = 1, \ldots, m \} \]

Active constraints at \( \bar{x} \)

\[ \mathcal{I}(\bar{x}) = \{ i \in \{1, \ldots, m\} \mid a_i^T \bar{x} = b_i \} \]

Index of all the constraints satisfied as equality

Basic solution \( \bar{x} \)

- \( \{a_i \mid i \in \mathcal{I}(\bar{x})\} \) has \( n \) linearly independent vectors
Basic feasible solution

$$P = \{ x \mid a_i^T x \leq b_i, \quad i = 1, \ldots, m \}$$

Active constraints at $\bar{x}$

$$\mathcal{I}(\bar{x}) = \{ i \in \{1, \ldots, m\} \mid a_i^T \bar{x} = b_i \}$$

Index of all the constraints satisfied as equality

Basic solution $\bar{x}$

• $\{a_i \mid i \in \mathcal{I}(\bar{x})\}$ has $n$ linearly independent vectors

Basic feasible solution $\bar{x}$

• $\bar{x} \in P$
• $\{a_i \mid i \in \mathcal{I}(\bar{x})\}$ has $n$ linearly independent vectors
Standard form polyhedra

Definition

Standard form LP

minimize \( c^T x \)
subject to \( Ax = b \)
\( x \geq 0 \)

Assumption

\( A \in \mathbb{R}^{m \times n} \) has full row rank \( m \leq n \)

Interpretation

\( P \) lives in \( (n - m) \)-dimensional subspace
Basic solutions
Standard form polyhedra

\[ P = \{ x \mid Ax = b, \ x \geq 0 \} \]

with \( A \in \mathbb{R}^{m \times n} \) has full row rank \( m \leq n \)

\( x \) is a **basic solution** if and only if

- \( Ax = b \)
- There exist indices \( B(1), \ldots, B(m) \) such that
  - columns \( A_{B(1)}, \ldots, A_{B(m)} \) are linearly independent
  - \( x_i = 0 \) for \( i \neq B(1), \ldots, B(m) \)

\( x \) is a **basic feasible solution** if \( x \) is a **basic solution** and \( x \geq 0 \)
From geometry to standard form

minimize \( c^T x \)
subject to \( Ax \leq b \)
From geometry to standard form

minimize \( c^T (x^+ - x^-) \)

subject to \( Ax \leq b \) \[ \rightarrow \]

subject to \[
\begin{bmatrix}
A & -A & I
\end{bmatrix}
\begin{bmatrix}
x^+

x^-
s
\end{bmatrix} = b
\]

\((x^+, x^-, s) \geq 0\)
From geometry to standard form

minimize \[ c^T x \]
subject to \[ Ax \leq b \]
\[ x \in \mathbb{R}^n \]
\[ m \in \mathbb{Z}^+ \]

\[ \tilde{c} \approx (c - c_0) \]

minimize \[ c^T (x^+ - x^-) \]
subject to \[ \begin{bmatrix} A & -A & I \end{bmatrix} \begin{bmatrix} x^+ \\ x^- \\ s \end{bmatrix} = b \]
\[ (x^+, x^-, s) \geq 0 \]

\[ \begin{align*}
\tilde{c}^T \tilde{x} \\
\tilde{A} \tilde{x} = b \\
\tilde{x} \geq 0
\end{align*} \]

Variables: \[ \tilde{n} = 2n + m \]
(Equality) constraints: \[ \tilde{m} = m \implies \text{active} \]
From geometry to standard form

minimize \( c^T (x^+ - x^-) \)
subject to \( Ax \leq b \)

\[
\begin{bmatrix}
A & -A & I
\end{bmatrix}
\begin{bmatrix}
x^+
\end{bmatrix}
= b
\]
\[
(x^+, x^-, s) \geq 0
\]

Variables: \( \tilde{n} = 2n + m \)
(Equality) constraints: \( \tilde{m} = m \rightarrow \text{active} \)

For a basic solution

We need \( \tilde{n} - \tilde{m} = 2n \)
active inequalities \( \Rightarrow \tilde{x}_i = 0 \) (non basic)
From geometry to standard form

minimize \( c^T (x^+ - x^-) \)
subject to \( Ax \leq b \)

\[
\begin{bmatrix}
A & -A & I
\end{bmatrix}
\begin{bmatrix}
x^+
-x^-
s
\end{bmatrix}
= b
\]
\((x^+, x^-, s) \geq 0\)

Variables: \( \tilde{n} = 2n + m \)
(Equality) constraints: \( \tilde{m} = m \implies \text{active} \)

For a basic solution \( \implies \) We need \( \tilde{n} - \tilde{m} = 2n \)
active inequalities \( \Rightarrow \tilde{x}_i = 0 \) (non basic)

Which corresponds to \( m \) inequalities inactive \( \Rightarrow \tilde{x}_i > 0 \) (basic)
From geometry to standard form

minimize \( c^T(x^+ - x^-) \)
subject to \( Ax \leq b \)

minimize \( \tilde{c}^T \tilde{x} \)
subject to \( \tilde{A} \tilde{x} = b \)
\( \tilde{x} \geq 0 \)

Variables: \( \tilde{n} = 2n + m \)
(Equality) constraints: \( \tilde{m} = m \implies \text{active} \)

For a basic solution

We need \( \tilde{n} - \tilde{m} = 2n \)
active inequalities \( \Rightarrow \tilde{x}_i = 0 \) (non basic)

Which corresponds to \( m \) inequalities inactive \( \Rightarrow \tilde{x}_i > 0 \) (basic)
Constructing basic solution

1. Choose any $m$ independent columns of $A$: $A_{B(1)}, \ldots, A_{B(m)}$
2. Let $x_i = 0$ for all $i \neq B(1), \ldots, B(m)$
3. Solve $Ax = b$ for the remaining $x_{B(1)}, \ldots, x_{B(m)}$
Constructing basic solution

1. Choose any $m$ independent columns of $A$: $A_{B(1)}, \ldots, A_{B(m)}$
2. Let $x_i = 0$ for all $i \neq B(1), \ldots, B(m)$
3. Solve $Ax = b$ for the remaining $x_{B(1)}, \ldots, x_{B(m)}$

\[
\begin{align*}
\text{Basis} & \quad \text{Basis columns} & \quad \text{Basic variables} \\
A_B & = \begin{bmatrix} A_{B(1)} & A_{B(2)} & \cdots & A_{B(m)} \end{bmatrix}, & \quad x_B = \begin{bmatrix} x_{B(1)} \\ \vdots \\ x_{B(m)} \end{bmatrix} \\
\end{align*}
\]

$\rightarrow$ Solve $A_Bx_B = b$
Constructing basic solution

1. Choose any $m$ independent columns of $A$: $A_B(1), \ldots, A_B(m)$
2. Let $x_i = 0$ for all $i \neq B(1), \ldots, B(m)$
3. Solve $Ax = b$ for the remaining $x_{B(1)}, \ldots, x_{B(m)}$

Basis matrix

$$A_B = \begin{bmatrix} A_{B(1)} & A_{B(2)} & \ldots & A_{B(m)} \end{bmatrix}$$

Basis columns

Basic variables

$$x_B = \begin{bmatrix} x_{B(1)} \\ \vdots \\ x_{B(m)} \end{bmatrix}$$

Solve $A_B x_B = b$

If $x_B \geq 0$, then $x$ is a basic feasible solution
Optimality of extreme points

minimize \( c^T x \)
subject to \( Ax \leq b \)

- \( P \) has at least one extreme point
- There exists an optimal solution \( x^* \)

Then, there exists an optimal solution which is an extreme point of \( P \)

We only need to search between extreme points
Conceptual algorithm

- Start at corner
- Visit neighboring corner that improves the objective
Today’s agenda
Readings: [Chapter 3, LO]

Simplex method
- Iterate between neighboring basic solutions
- Optimality conditions
- Simplex iterations
The simplex method

Top 10 algorithms of the 20th century

1946: Metropolis algorithm
1947: Simplex method
1950: Krylov subspace method
1951: The decompositional approach to matrix computations
1957: The Fortran optimizing compiler
1959: QR algorithm
1962: Quicksort
1965: Fast Fourier transform
1977: Integer relation detection
1987: Fast multipole method
The simplex method
Top 10 algorithms of the 20th century

1946: Metropolis algorithm
1947: Simplex method
1950: Krylov subspace method
1951: The decompositional approach to matrix computations
1957: The Fortran optimizing compiler
1959: QR algorithm
1962: Quicksort
1965: Fast Fourier transform
1977: Integer relation detection
1987: Fast multipole method

[SIAM News (2000)]
Neighboring basic solutions
Neighboring solutions

Two basic solutions are **neighboring** if their basic indices differ by exactly one variable.
Neighboring solutions

Two basic solutions are neighboring if their basic indices differ by exactly one variable

Example

\[
\begin{bmatrix}
1 & -1 & 0 & 3 & -2 \\
2 & 0 & -1 & -1 & 0 \\
0 & 2 & 4 & -1 & 4 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
\end{bmatrix}
= 
\begin{bmatrix}
-5 \\
-1 \\
14 \\
\end{bmatrix}
\]
Neighboring solutions

Two basic solutions are \textbf{neighboring} if their basic indices differ by exactly one variable

\textbf{Example}

\[
\begin{align*}
A & = \begin{bmatrix}
1 & -1 & 0 & 3 & -2 \\
2 & 0 & -1 & -1 & 0 \\
0 & 2 & 4 & -1 & 4
\end{bmatrix} \\
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix} & = \begin{bmatrix}
b_1 \\
b_2 \\
b_3 \\
b_4 \\
b_5
\end{bmatrix}
\end{align*}
\]

\[B = \{1, 3, 5\} \quad x_2 = x_4 = 0\]

\[
A_B x_B = b \quad \rightarrow \quad x_B = \begin{bmatrix}
x_1 \\
x_3 \\
x_5
\end{bmatrix} = \begin{bmatrix}
0 \\
1 \\
2.5
\end{bmatrix}
\]
Neighboring solutions

Two basic solutions are neighboring if their basic indices differ by exactly one variable.

Example

\[
\begin{bmatrix}
1 & -1 & 0 & 3 & -2 \\
2 & 0 & -1 & -1 & 0 \\
0 & 2 & 4 & -1 & 4 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
\end{bmatrix}
= 
\begin{bmatrix}
b_1 \\
b_2 \\
b_3 \\
\end{bmatrix}
\]

\(B = \{1, 3, 5\}\) \quad \(x_2 = x_4 = 0\)

\(A_Bx_B = b\) \quad \(x_B = \begin{bmatrix} x_1 \\ x_3 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2.5 \end{bmatrix}\)

\(\bar{B} = \{1, 3, 4\}\) \quad \(y_2 = y_5 = 0\)

\(A_By_B = b\) \quad \(y_B = \begin{bmatrix} y_1 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 3.0 \\ -1.7 \end{bmatrix}\)
Feasible directions

Conditions

\[ P = \{ x \mid Ax = b, \ x \geq 0 \} \]

Given a basis matrix \( A_B = \begin{bmatrix} A_B(1) & \cdots & A_B(m) \end{bmatrix} \)
we have basic feasible solution \( x \):

\begin{itemize}
  \item \( x_B \) solves \( A_B x_B = b \)
  \item \( x_i = 0, \ \forall i \neq B(1), \ldots, B(m) \)
\end{itemize}
Feasible directions

Conditions

\[ P = \{ x \mid Ax = b, \ x \geq 0 \} \]

Given a basis matrix \( A_B = \begin{bmatrix} A_{B(1)} & \ldots & A_{B(m)} \end{bmatrix} \)

we have basic feasible solution \( x \):

- \( x_B \) solves \( A_B x_B = b \)
- \( x_i = 0, \ \forall i \neq B(1), \ldots, B(m) \)

Let \( x \in P \), a vector \( d \) is a **feasible direction** at \( x \)

if \( \exists \theta > 0 \) for which \( x + \theta d \in P \)

**Feasible direction** \( d \)

- \( A(x + \theta d) = b \implies Ad = 0 \)
- \( x + \theta d \geq 0 \)
Feasible directions

Computation

Nonbasic indices $(\forall j \neq 0)$

- $d_j = 1 \longrightarrow \text{Basic direction}$
- $d_k = 0, \forall k \notin \{j, B(1), \ldots, B(m)\}$

Feasible direction $d$

- $A(x + \theta d) = b \implies Ad = 0$
- $x + \theta d \geq 0$
Feasible directions

Computation

Nonbasic indices
• \( d_j = 1 \) → Basic direction
• \( d_k = 0 \), \( \forall k \notin \{j, B(1), \ldots, B(m)\} \)

Basic indices

\[
Ad = 0 \Rightarrow \sum_{i=1}^{n} A_i d_i = A_B d_B + A_j = 0 \Rightarrow d_B = -A_B^{-1} A_j
\]
Feasible directions

Computation

Nonbasic indices
• $d_j = 1$ → Basic direction
• $d_k = 0, \forall k \notin \{j, B(1), \ldots, B(m)\}$

Basic indices

\[ Ad = 0 = \sum_{i=1}^{n} A_i d_i = A_g d_B + A_j = 0 \implies d_B = -A_g^{-1} A_j \]

Non-negativity (non-degenerate assumption)
• Non-basic variables: $x_i = 0$. Nonnegative direction $d_i \geq 0$
• Basic variables: $x_B > 0$. Therefore $\exists \theta > 0$ such that $x_B + \theta d_B \geq 0$

Feasible direction $d$
• $A(x + \theta d) = b \implies Ad = 0$
• $x + \theta d \geq 0$
Feasible directions

Example

\[ P = \{ x \mid x_1 + x_2 + x_3 = 2, \quad x \geq 0 \} \]

\[ x = (2, 0, 0) \quad B = \{1\} \]
Feasible directions

Example

\[ P = \{ x \mid x_1 + x_2 + x_3 = 2, \quad x \geq 0 \} \]

\[ x = (2, 0, 0) \quad B = \{1\} \]

**Now**

**Basic index** \( j = 3 \) \[ \rightarrow d = (-1, 0, 1) \]

\[ A_B d_B = -A_j \quad \Rightarrow \quad d_B = -1 \]
How does the cost change?

Cost improvement

\[ c^T (x + \theta d) - c^T x = \theta c^T d \]
How does the cost change?

Cost improvement

\[ c^T(x + \theta d) - c^T x = \theta c^T d \]
How does the cost change?

Cost improvement

\[ c^T (x + \theta d) - c^T x = \theta c^T d \]

New cost

Old cost
How does the cost change?

Cost improvement

\[ c^T(x + \theta d) - c^T x = \theta c^T d \]

New cost \quad Old cost

We call \( \bar{c}_j \) the **reduced cost** of (introducing) variable \( x_j \) in the basis

\[
\bar{c}_j = c^T d = \sum_{i=1}^{n} c_i \bar{d}_i = c_j + c_B^T d_B = c_j - c_B^T A_B^{-1} A_j
\]
Reduced costs

Interpretation
Change in objective/marginal cost of adding $x_j$ to the basis

$$\bar{c}_j = c_j - c_B^T A_B^{-1} A_j$$

- $\bar{c}_j > 0$: adding $x_j$ will increase the objective (bad)
- $\bar{c}_j < 0$: adding $x_j$ will decrease the objective (good)
Reduced costs

Interpretation
Change in objective/marginal cost of adding $x_j$ to the basis

$$\bar{c}_j = c_j - c_B^T A_B^{-1} A_j$$

Cost per-unit increase of variable $x_j$

- $\bar{c}_j > 0$: adding $x_j$ will increase the objective (bad)
- $\bar{c}_j < 0$: adding $x_j$ will decrease the objective (good)
Reduced costs

Interpretation
Change in objective/marginal cost of adding \( x_j \) to the basis

\[
\bar{c}_j = c_j - c_B^T A_B^{-1} A_j
\]

Cost per-unit increase of variable \( x_j \)

Cost to change other variables compensating for \( x_j \) to enforce \( Ax = b \)

- \( \bar{c}_j > 0 \): adding \( x_j \) will increase the objective (bad)
- \( \bar{c}_j < 0 \): adding \( x_j \) will decrease the objective (good)
Reduced costs

Interpretation
Change in objective/marginal cost of adding $x_j$ to the basis

$$\bar{c}_j = c_j - c_B^T A_B^{-1} A_j$$

Cost per-unit increase of variable $x_j$

Cost to change other variables compensating for $x_j$ to enforce $Ax = b$

- $\bar{c}_j > 0$: adding $x_j$ will increase the objective (bad)
- $\bar{c}_j < 0$: adding $x_j$ will decrease the objective (good)

Reduced costs for basic variables is 0

$$\bar{c}_B(i) = c_B(i) - c_B^T A_B^{-1} A_B(i) = c_B(i) - c_B^T (A_B^{-1} A_B) e_i$$

$$= c_B(i) - c_B^T e_i = c_B(i) - c_B(i) = 0$$
Vector of reduced costs

Reduced costs
\[ \bar{c}_j = c_j - c_B^T A_B^{-1} A_j \]

Full vector in one shot?
\[ \bar{c} = (\bar{c}_1, \ldots, \bar{c}_n) \]
Vector of reduced costs

Reduced costs

\[ \bar{c}_j = c_j - c_B^T A_B^{-1} A_j \]

Isolate basis \( B \)-related components \( \bar{p} \) (they are the same across \( j \))

\[ \bar{c}_j = c_j - A_j^T (A_B^{-1})^T c_B = c_j - A_j^T \bar{p} \]

Full vector in one shot?

\[ \bar{c} = (\bar{c}_1, \ldots, \bar{c}_n) \]
Vector of reduced costs

Reduced costs
\[ \bar{c}_j = c_j - c_B^T A_B^{-1} A_j \]

Isolate basis \( B \)-related components \( p \)
(they are the same across \( j \))
\[ \bar{c}_j = c_j - A_j^T (A_B^{-1})^T c_B = c_j - A_j^T p \]

Full vector in one shot?
\[ \bar{c} = (\bar{c}_1, \ldots, \bar{c}_n) \]

Obtain \( p \) by solving linear system
\[ p = (A_B^{-1})^T c_B \quad \Rightarrow \quad A_B^T p = c_B \]

Note: \( (M^{-1})^T = (M^T)^{-1} \)
for any square invertible \( M \)
Vector of reduced costs

Reduced costs
\[ \bar{c}_j = c_j - c_B^T A_B^{-1} A_j \]

Isolate basis \( B \)-related components \( p \)
(they are the same across \( j \))
\[ \bar{c}_j = c_j - A_j^T (A_B^{-1})^T c_B = c_j - A_j^T p \]

Full vector in one shot?
\[ \bar{c} = (\bar{c}_1, \ldots, \bar{c}_n) \]

Obtain \( p \) by solving linear system
\[ p = (A_B^{-1})^T c_B \quad \Rightarrow \quad A_B^T p = c_B \]

Note: \( (M^{-1})^T = (M^T)^{-1} \)
for any square invertible \( M \)

Computing reduced cost vector
1. Solve \( A_B^T p = c_B \)
2. \( \bar{c} = c - A^T p \)
Optimality conditions
Optimality conditions

Theorem

Let $x$ be a basic feasible solution associated with basis matrix $A_B$
Let $\bar{c}$ be the vector of reduced costs.

If $\bar{c} \geq 0$, then $x$ is optimal
Optimality conditions

Theorem

Let $x$ be a basic feasible solution associated with basis matrix $B$
Let $\bar{c}$ be the vector of reduced costs.

If $\bar{c} \geq 0$, then $x$ is optimal

Remark

This is a stopping criterion for the simplex algorithm.
If the neighboring solutions do not improve the cost, we are done (because of convexity).
Optimality conditions

Proof

For a basic feasible solution $x$ with basis $B$ the reduced costs are $\bar{c} \geq 0$. 
Optimality conditions

Proof

For a basic feasible solution \( x \) with basis \( B \) the reduced costs are \( \bar{c} \geq 0 \).

Consider any feasible solution \( y \) and define \( d = y - x \).
Optimality conditions

Proof

For a basic feasible solution $x$ with basis $B$ the reduced costs are $\bar{c} \geq 0$. Consider any feasible solution $y$ and define $d = y - x$

Since $x$ and $y$ are feasible, then $Ax = Ay = b$ and $Ad = 0$

$$Ad = A_B d_B + \sum_{i \in N} A_i d_i = 0 \quad \Rightarrow \quad d_B = -\sum_{i \in N} A_B^{-1} A_i d_i$$

$N$ are the nonbasic indices
Optimality conditions

Proof

For a basic feasible solution $x$ with basis $B$ the reduced costs are $\bar{c} \geq 0$. Consider any feasible solution $y$ and define $d = y - x$.

Since $x$ and $y$ are feasible, then $Ax = Ay = b$ and $Ad = 0$

$$Ad = A_B d_B + \sum_{i \in N} A_i d_i = 0 \quad \Rightarrow \quad d_B = -\sum_{i \in N} A_B^{-1} A_i d_i$$

The change in objective is

$$c^T d = c_B^T d_B + \sum_{i \in N} c_i d_i = \sum_{i \in N} (c_i - c_B^T A_B^{-1} A_i) d_i = \sum_{i \in N} \bar{c}_i d_i$$
Optimality conditions

Proof

For a basic feasible solution $x$ with basis $B$ the reduced costs are $\bar{c} \geq 0$.
Consider any feasible solution $y$ and define $d = y - x$

Since $x$ and $y$ are feasible, then $Ax = Ay = b$ and $Ad = 0$

$$Ad = A_B d_B + \sum_{i \in N} A_i d_i = 0 \quad \Rightarrow \quad d_B = - \sum_{i \in N} A_B^{-1} A_i d_i$$

The change in objective is

$$c^T d = c_B^T d_B + \sum_{i \in N} c_i d_i = \sum_{i \in N} (c_i - c_B^T A_B^{-1} A_i) d_i = \sum_{i \in N} \bar{c}_i d_i$$

Since $y \geq 0$ and $x_i = 0$, $i \in N$, then $d_i = y_i - x_i \geq 0$, $i \in N$

$$c^T d = c^T (y - x) \geq 0 \quad \Rightarrow \quad c^T y \geq c^T x.$$
Simplex iterations
Stepsizes

What happens if some $\tilde{c}_j < 0$?
We can decrease the cost by bringing $x_j$ into the basis
Stepsize

What happens if some $c_j < 0$?
We can decrease the cost by bringing $x_j$ into the basis

How far can we go?

$$\theta^* = \max\{\theta \mid \theta \geq 0 \text{ and } x + \theta d \geq 0\}$$

$d$ is the $j$-th basic direction
Stepsize

What happens if some $\bar{c}_j < 0$?
We can decrease the cost by bringing $x_j$ into the basis

How far can we go?

$$\theta^* = \max \{ \theta \mid \theta \geq 0 \text{ and } x + \theta d \geq 0\}$$

$d$ is the $j$-th basic direction

Unbounded
If $d \geq 0$, then $\theta^* = \infty$. The LP is unbounded.
Stepsizes

What happens if some $\bar{c}_j < 0$?
We can decrease the cost by bringing $x_j$ into the basis

How far can we go?

$$\theta^* = \max\{\theta \mid \theta \geq 0 \text{ and } x + \theta d \geq 0\}$$

$d$ is the $j$-th basic direction

Unbounded
If $d \geq 0$, then $\theta^* = \infty$. The LP is unbounded.

Bounded
If $d_i < 0$ for some $i$, then

$$\theta^* = \min_{\{i \mid d_i < 0\}} \left( -\frac{x_i}{d_i} \right) = \min_{\{i \in B \mid d_i < 0\}} \left( -\frac{x_i}{d_i} \right)$$

(Since $d_i \geq 0$, $i \notin B$)
Moving to a new basis

Next feasible solution

\[ x + \theta^* d \]
Moving to a new basis

Next feasible solution

\[ x + \theta^* d \]

Let \( B(\ell) \in \{B(1), \ldots, B(m)\} \) be the index such that \( \theta^* = -\frac{x_{B(\ell)}}{d_{B(\ell)}} \). Then,

\[ x_{B(\ell)} + \theta^* d_{B(\ell)} = 0 \]
Moving to a new basis

**Next feasible solution**

\[ x + \theta^* d \]

Let \( B(\ell) \in \{ B(1), \ldots, B(m) \} \) be the index such that \( \theta^* = -\frac{x_{B(\ell)}}{d_{B(\ell)}} \). Then,

\[ x_{B(\ell)} + \theta^* d_{B(\ell)} = 0 \]

**New solution**

- \( x_{B(\ell)} \) becomes 0 (exits)
- \( x_j \) becomes \( \theta^* \) (enters)

\[ d_J = 1 \]
Moving to a new basis

Next feasible solution

\[ x + \theta^* d \]

Let \( B(\ell) \in \{ B(1), \ldots, B(m) \} \) be the index such that \( \theta^* = -\frac{x_{B(\ell)}}{d_{B(\ell)}} \). Then,

\[ x_{B(\ell)} + \theta^* d_{B(\ell)} = 0 \]

New solution

- \( x_{B(\ell)} \) becomes 0 (exits)
- \( x_j \) becomes \( \theta^* \) (enters)

New basis

\[
A_{\bar{B}} = \begin{bmatrix}
A_{B(1)} & \cdots & A_{B(\ell-1)} & [A_j] & A_{B(\ell+1)} & \cdots & A_{B(m)}
\end{bmatrix}
\]
An iteration of the simplex method

First part

We start with

- a basic feasible solution \( x \)
- a basis matrix \( A_B = \begin{bmatrix} A_B(1) & \cdots & A_B(m) \end{bmatrix} \)

1. Compute the reduced costs \( \bar{c} \)
   - Solve \( A_B^T p = c_B \)
   - \( \bar{c} = c - A_B^T p \)

2. If \( \bar{c} \geq 0 \), \( x \) optimal. break

3. Choose \( j \) such that \( \bar{c}_j < 0 \)
An iteration of the simplex method
Second part

4. Compute search direction \( d \) with \( d_j = 1 \) and \( A_B d_B = -A_j \)

5. If \( d_B \geq 0 \), the problem is **unbounded** and the optimal value is \(-\infty\). **break**

6. Compute step length \( \theta^* = \min_{\{i \in B | d_i < 0\}} \left( -\frac{x_i}{d_i} \right) \)

7. Define \( y \) such that \( y = x + \theta^* d \)

8. Get new basis \( \tilde{B} \) (\( i \) exits and \( j \) enters)
Example

\[ P = \{ x \mid x_1 + x_2 + x_3 = 2, \quad x \geq 0 \} \]

\[ x = (2, 0, 0) \quad B = \{1\} \]

**Basic index** \( j = 3 \quad d = (-1, 0, 1) \)

\[ d_j = 1 \]

\[ A_B d_B = -A_j \quad \Rightarrow \quad d_B = -1 \]
Example

\[ P = \{ x \mid x_1 + x_2 + x_3 = 2, \quad x \geq 0 \} \]

\[ x = (2, 0, 0) \quad B = \{1\} \]

**Basic index** \( j = 3 \quad d = (-1, 0, 1) \)

\[ d_j = 1 \]

\[ A_B d_B = -A_j \quad \Rightarrow \quad d_B = -1 \]

**Stepsise** \( \theta^* = -\frac{x_1}{d_1} = 2 \)
Example

\[ P = \{ x \mid x_1 + x_2 + x_3 = 2, \quad x \geq 0 \} \]

\[ x = (2, 0, 0) \quad B = \{1\} \]

**Basic index** \( j = 3 \) \( \Rightarrow \) \( d = (-1, 0, 1) \)

\[ d_j = 1 \quad A_B d_B = -A_j \quad \Rightarrow \quad d_B = -1 \]

**Stepsizes** \( \theta^* = -\frac{x_1}{d_1} = 2 \)

**New solution** \( y = x + \theta^* d = (0, 0, 2) \quad \bar{B} = \{3\} \)
Finite convergence

Assume that

- \( P = \{x \mid Ax = b, x \geq 0\} \) not empty
- Every basic feasible solution non degenerate
Finite convergence

Assume that

\( P = \{ x \mid Ax = b, x \geq 0 \} \) not empty
\( \) Every basic feasible solution non degenerate

Then

\( \) The simplex method terminates after a finite number of iterations
\( \) At termination we either have one of the following

\( \) an optimal basis \( B \)
\( \) a direction \( d \) such that \( Ad = 0, \ d \geq 0, \ c^T d < 0 \) and the optimal cost is \( -\infty \)
Finite convergence
Proof sketch

At each iteration the algorithm improves
• by a positive amount $\theta^*$
• along the direction $d$ such that $c^T d < 0$
Finite convergence
Proof sketch

At each iteration the algorithm improves
• by a positive amount $\theta^*$
• along the direction $d$ such that $c^T d < 0$

Therefore
• The cost strictly decreases
• No basic feasible solution can be visited twice
Finite convergence
Proof sketch

At each iteration the algorithm improves
• by a **positive** amount $\theta^*$
• along the **direction** $d$ such that $c^T d < 0$

Therefore
• The cost strictly decreases
• No basic feasible solution can be visited twice

Since there is a **finite number of basic feasible solutions**
The algorithm **must eventually terminate**
The simplex method

Today, we learned to:

• **Iterate** between basic feasible solutions

• **Verify** optimality and unboundedness conditions

• **Apply** a single iteration of the simplex method

• **Prove** finite convergence of the simplex method in the non-degenerate case
Next lecture

- Finding initial basic feasible solution
- Degeneracy
- Complexity