

ORF522 – Linear and Nonlinear Optimization

3. Geometry and polyhedra

Ed forum

Questions and notes

- Ed Forum: put questions/comments for each lecture under “Notes - Lecture XX” instead of General “Questions”.
- SVM setting: why from $a^T v_i - b > 0 (< 0)$ to $s_i(a^T v_i - b_i) \geq 1$? (numerical issues + ambiguity)
- $\ell_1 - \ell_0$ equivalence? Not always and difficult to verify! (Compressed sensing literature).
- Why is ℓ_0 penalty not linear? It can be modeled with mixed-integer constraints. We will see it in the branch-and-bound lecture.

Today's agenda

Readings [Chapter 2, Bertsimas and Tsitsiklis]

- Polyhedra and linear algebra
- Corners: extreme points, vertices, basic feasible solutions
- Constructing basic solutions
- Existence and optimality of extreme points

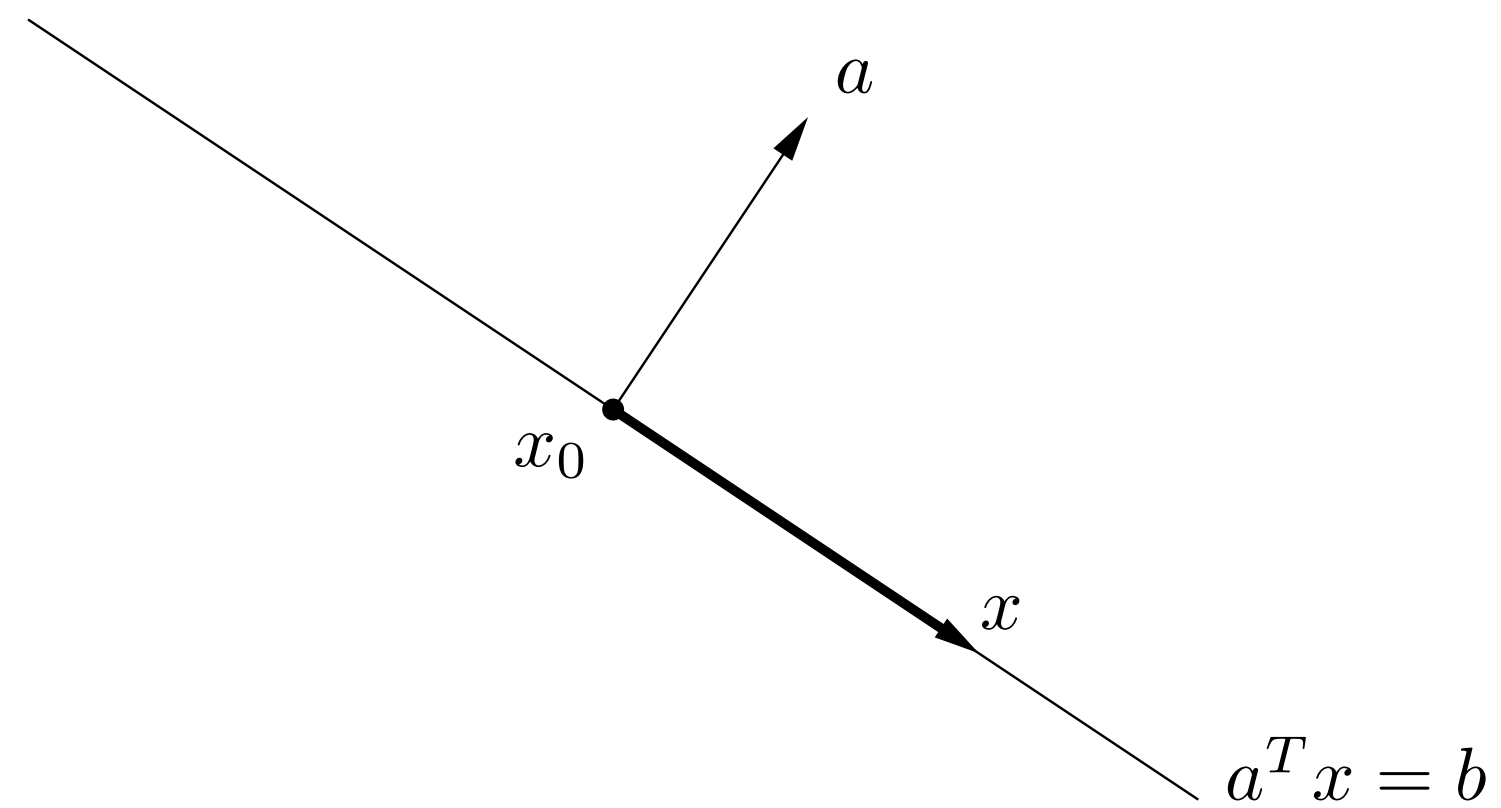
Polyhedra and linear algebra

Hyperplanes and halfspaces

Definitions

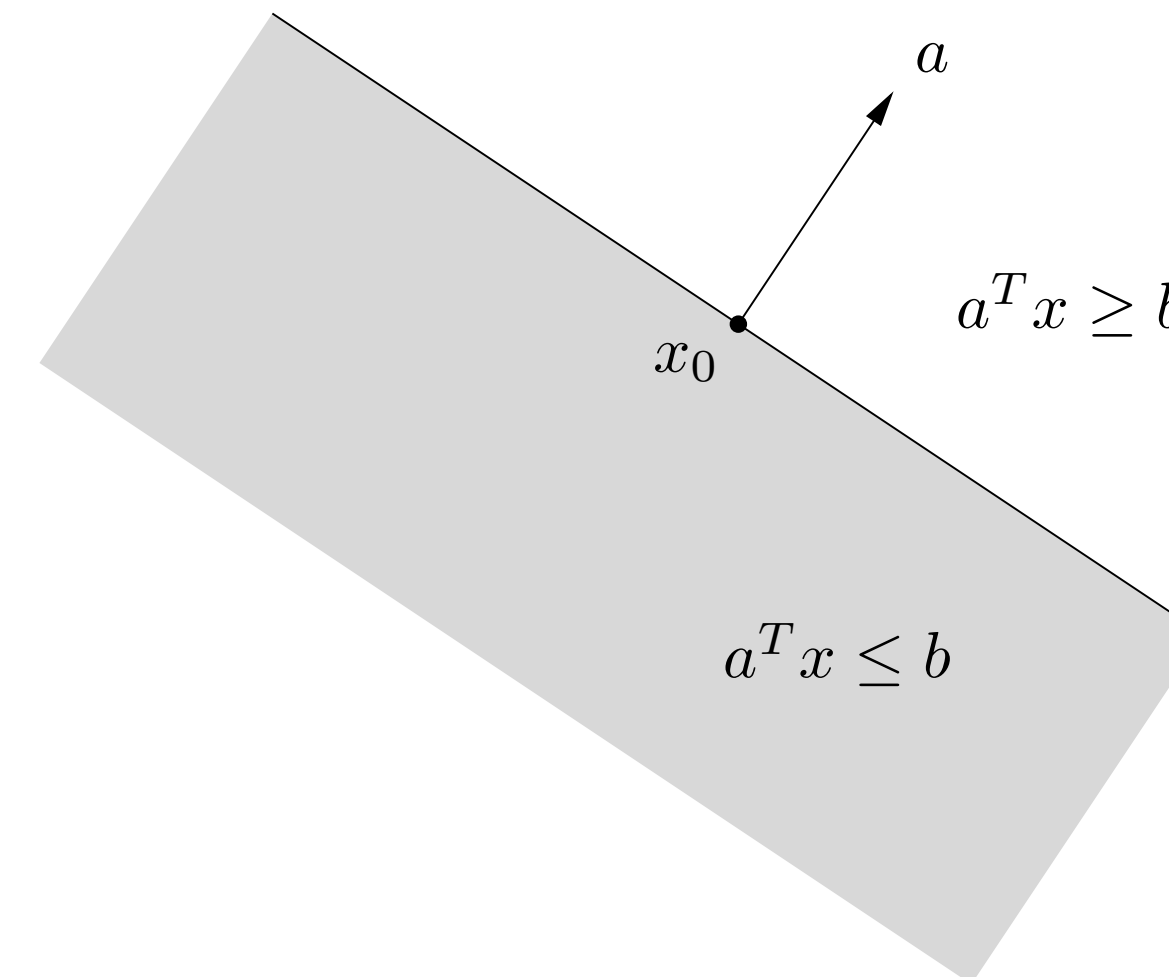
Hyperplane

$$\{x \mid a^T x = b\}$$



Halfspace

$$\{x \mid a^T x \leq b\}$$

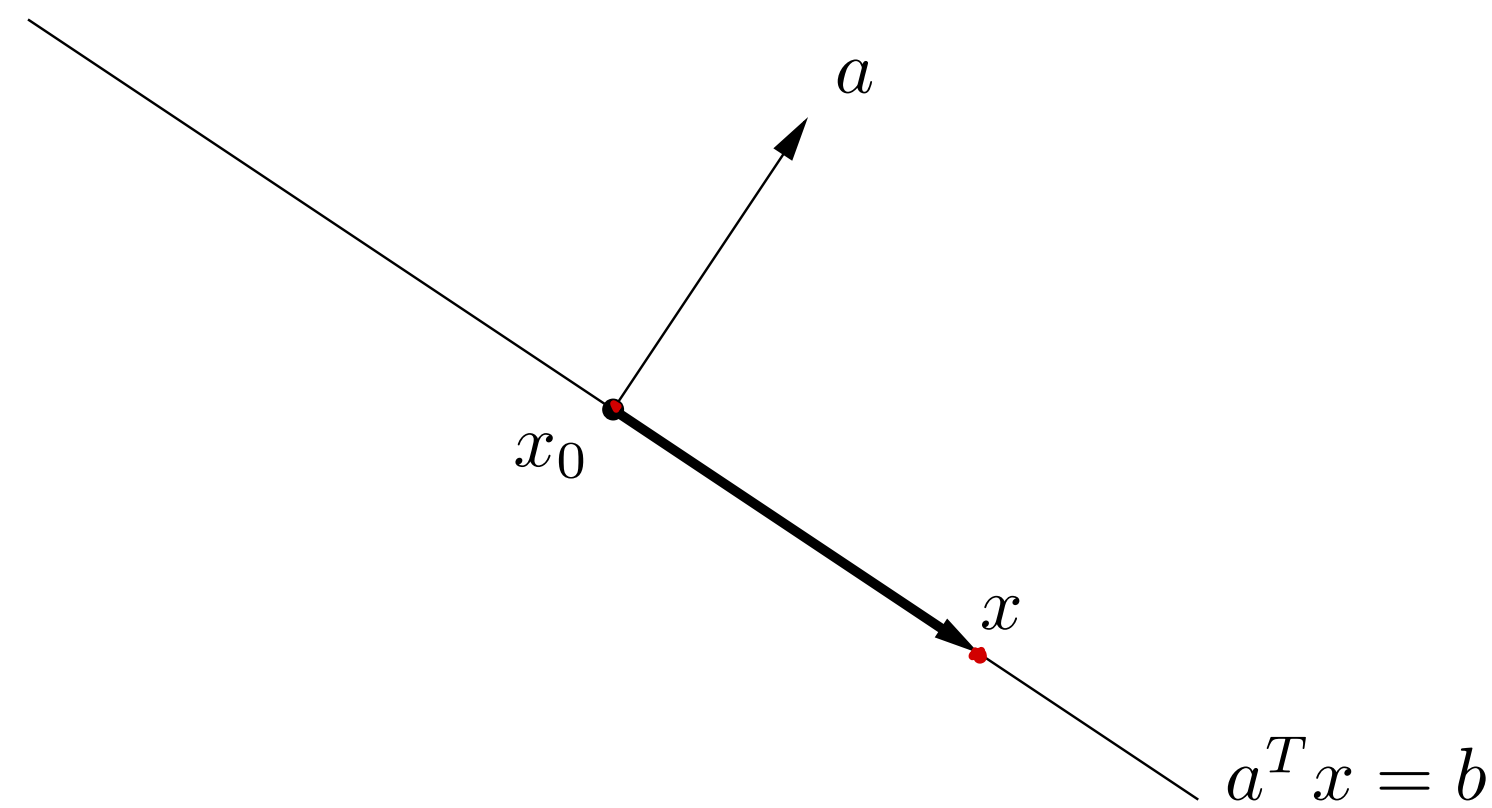


Hyperplanes and halfspaces

Definitions

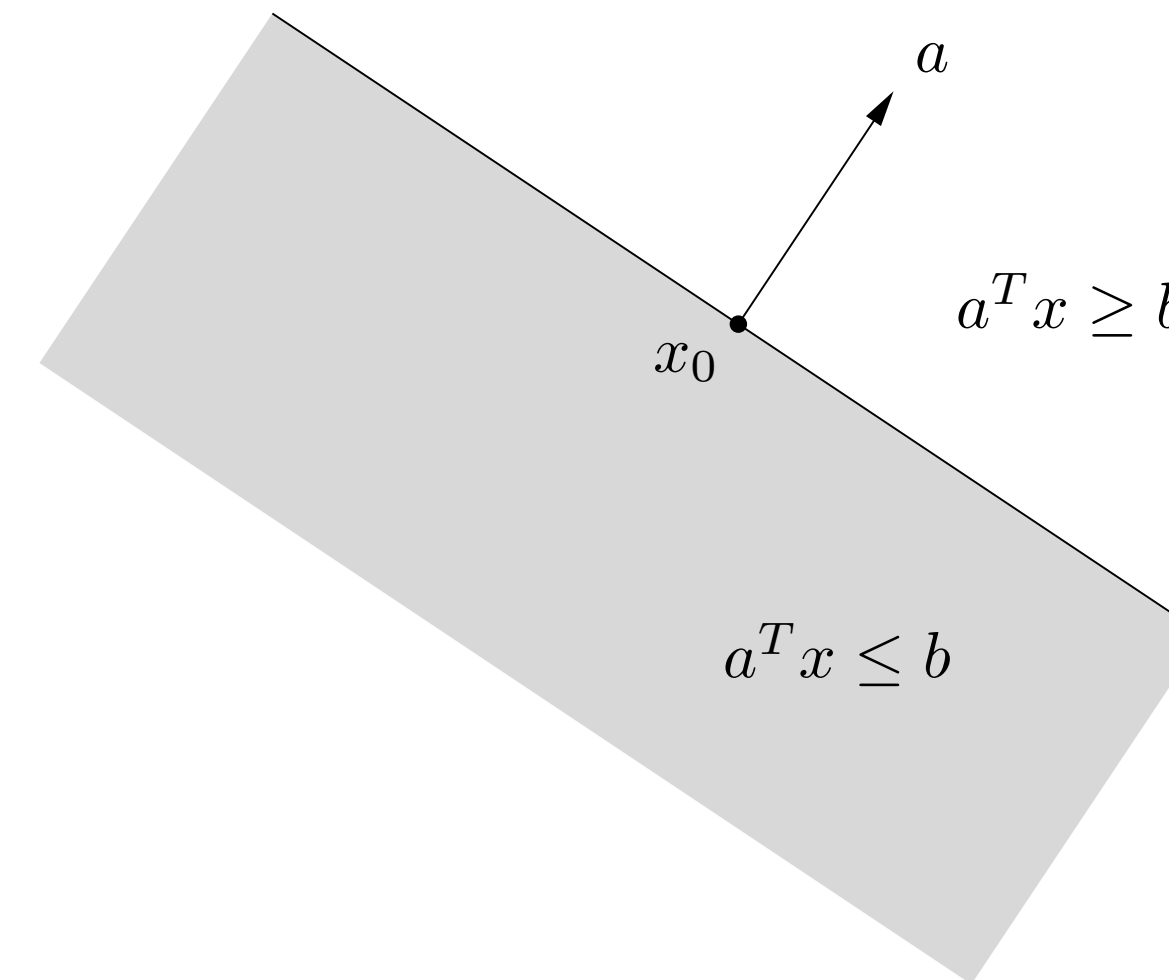
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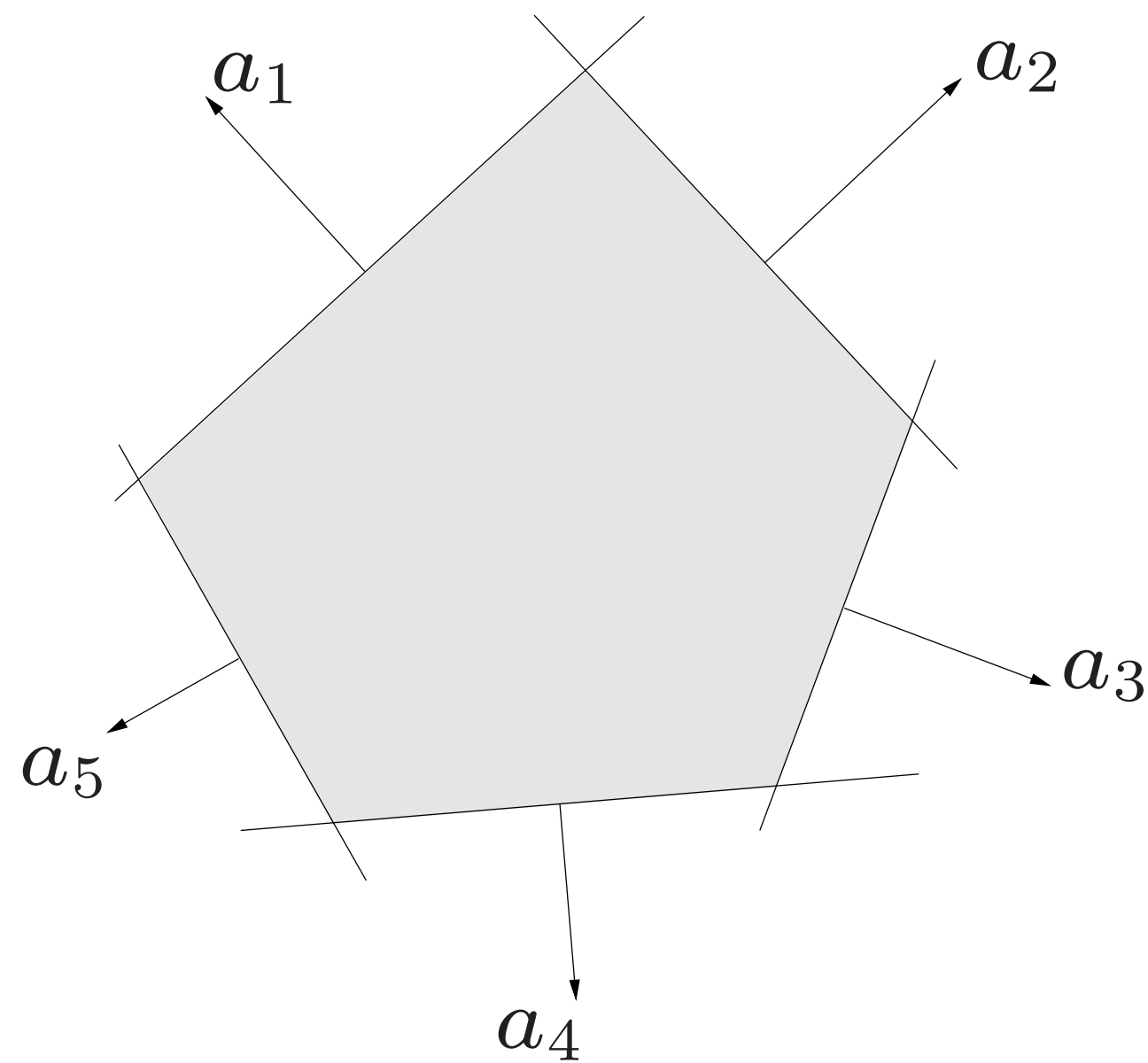
- x_0 is a specific point in the hyperplane
- For any x in the hyperplane defined by $a^T x = b$, $x - x_0 \perp a$
- The halfspace determined by $a^T x \leq b$ extends in the direction of $-a$

Polyhedron

Definition

$$a_i^T x = b_i \Leftrightarrow \begin{cases} a_i^T x > b_i \\ a_i^T x < b_i \end{cases}$$

$$P = \{x \mid a_i^T x \leq b_i, \quad i = 1, \dots, m\} = \{x \mid Ax \leq b\}$$



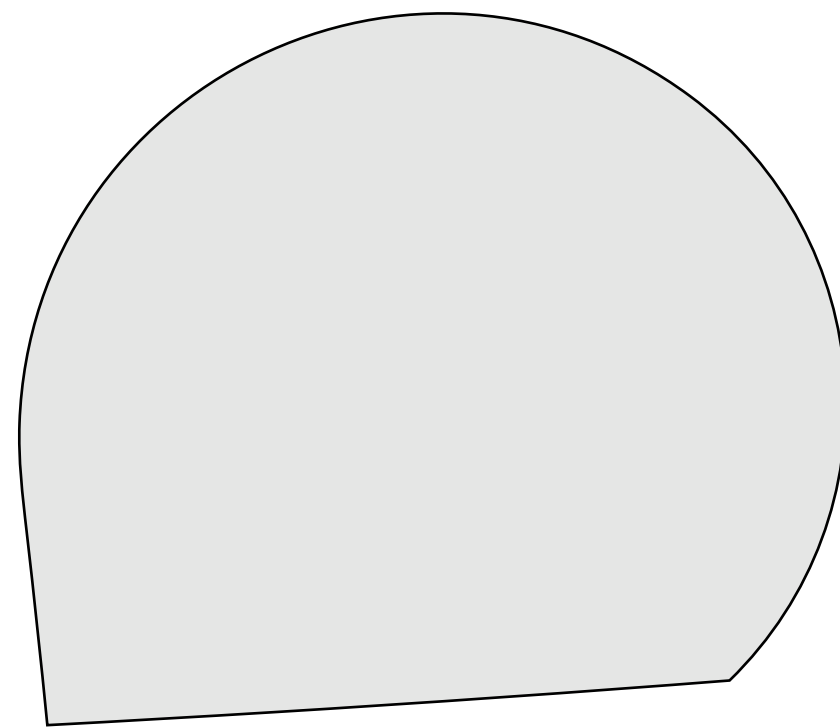
- Intersection of finite number of halfspaces
- Can include equalities

Convex set

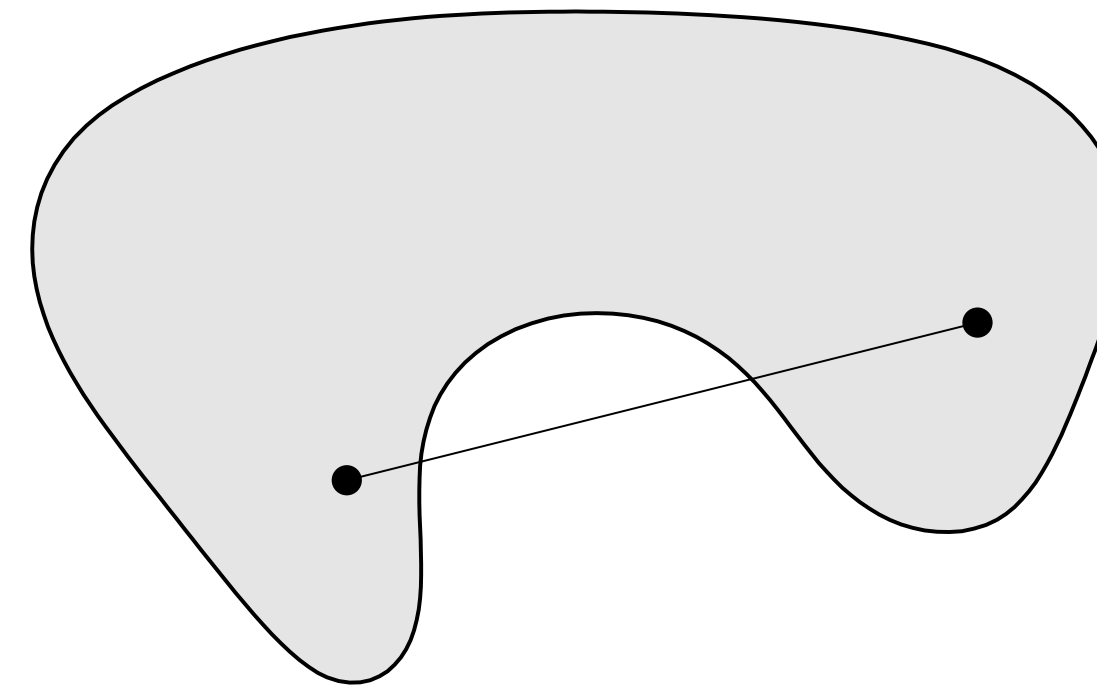
Definition

For any $x, y \in C$ and any $\alpha \in [0, 1]$

$$\alpha x + (1 - \alpha)y \in C$$



Convex



Not convex

Examples

- \mathbb{R}^n
- Hyperplanes
- Halfspaces
- Polyhedra

Convex combinations

Convex combination

$\alpha_1 x_1 + \cdots + \alpha_k x_k$ for any x_1, \dots, x_k and $\alpha_1, \dots, \alpha_k$ such that $\alpha_i \geq 0$, $\sum_{i=1}^k \alpha_i = 1$

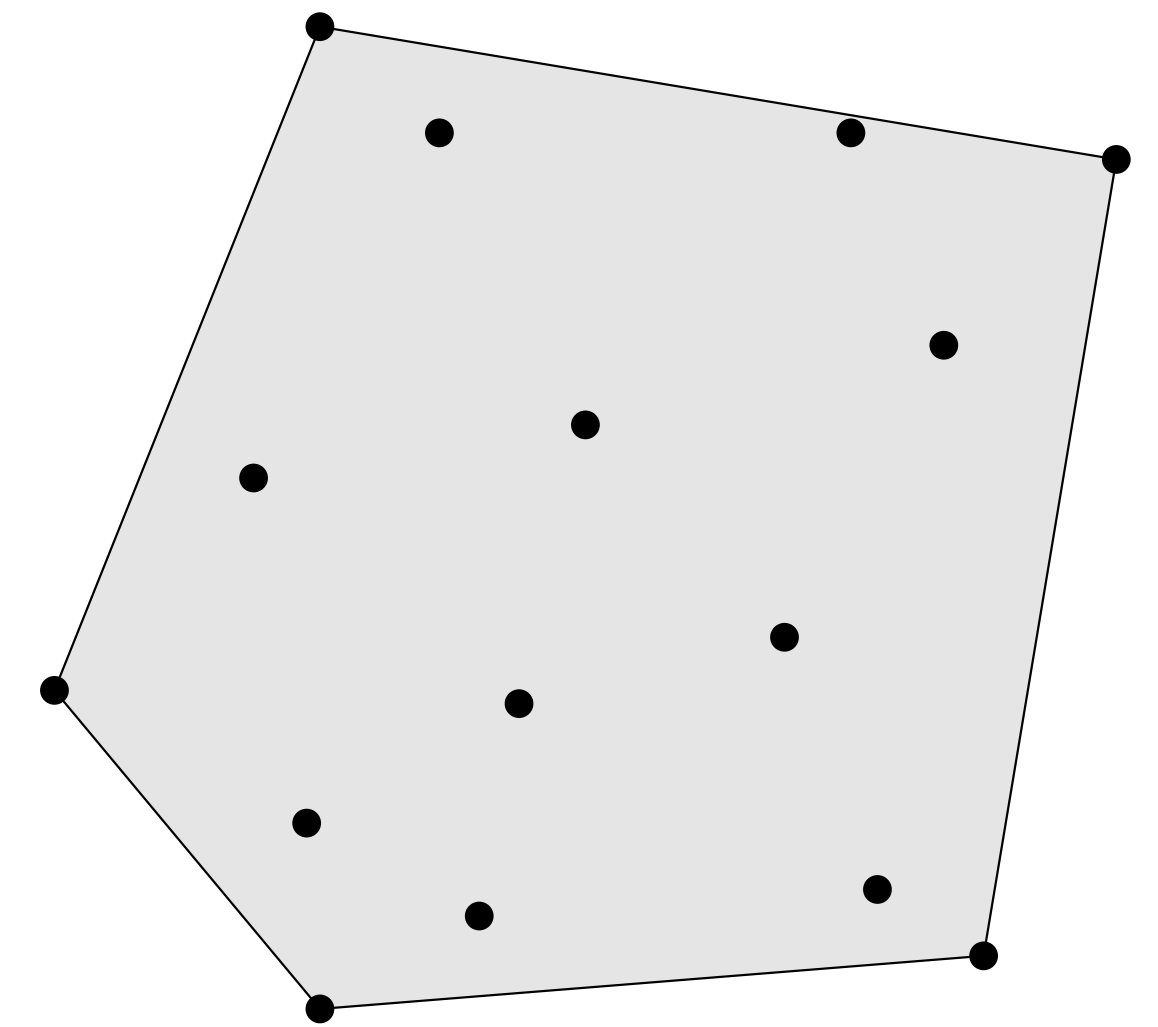
Convex combinations

Convex combination

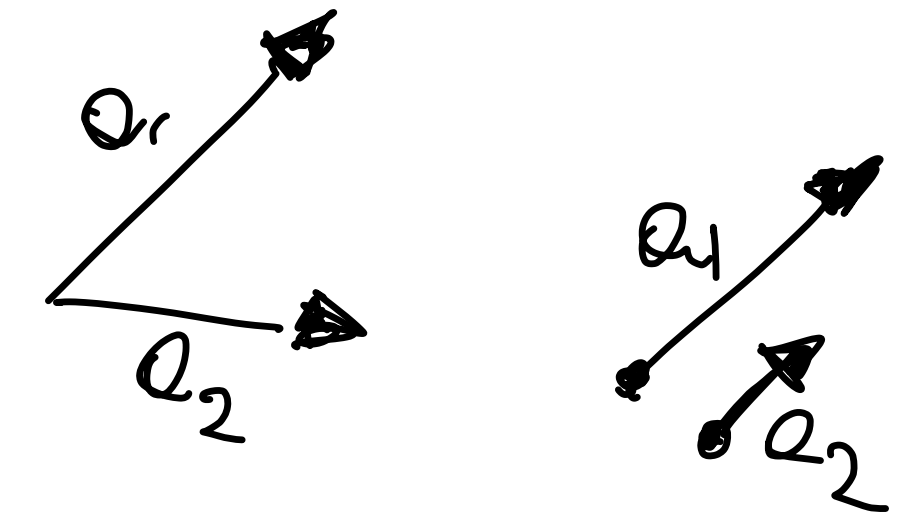
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Convex hull

$$\text{conv } C = \left\{ \sum_{i=1}^k \alpha_i x_i \mid x_i \in C, \alpha_i \geq 0, \mathbf{1}^T \alpha = 1 \right\}$$



Linear independence



a nonempty set of vectors $\{v_1, \dots, v_k\}$ is **linearly independent** if

$$\alpha_1 v_1 + \dots + \alpha_k v_k = 0$$

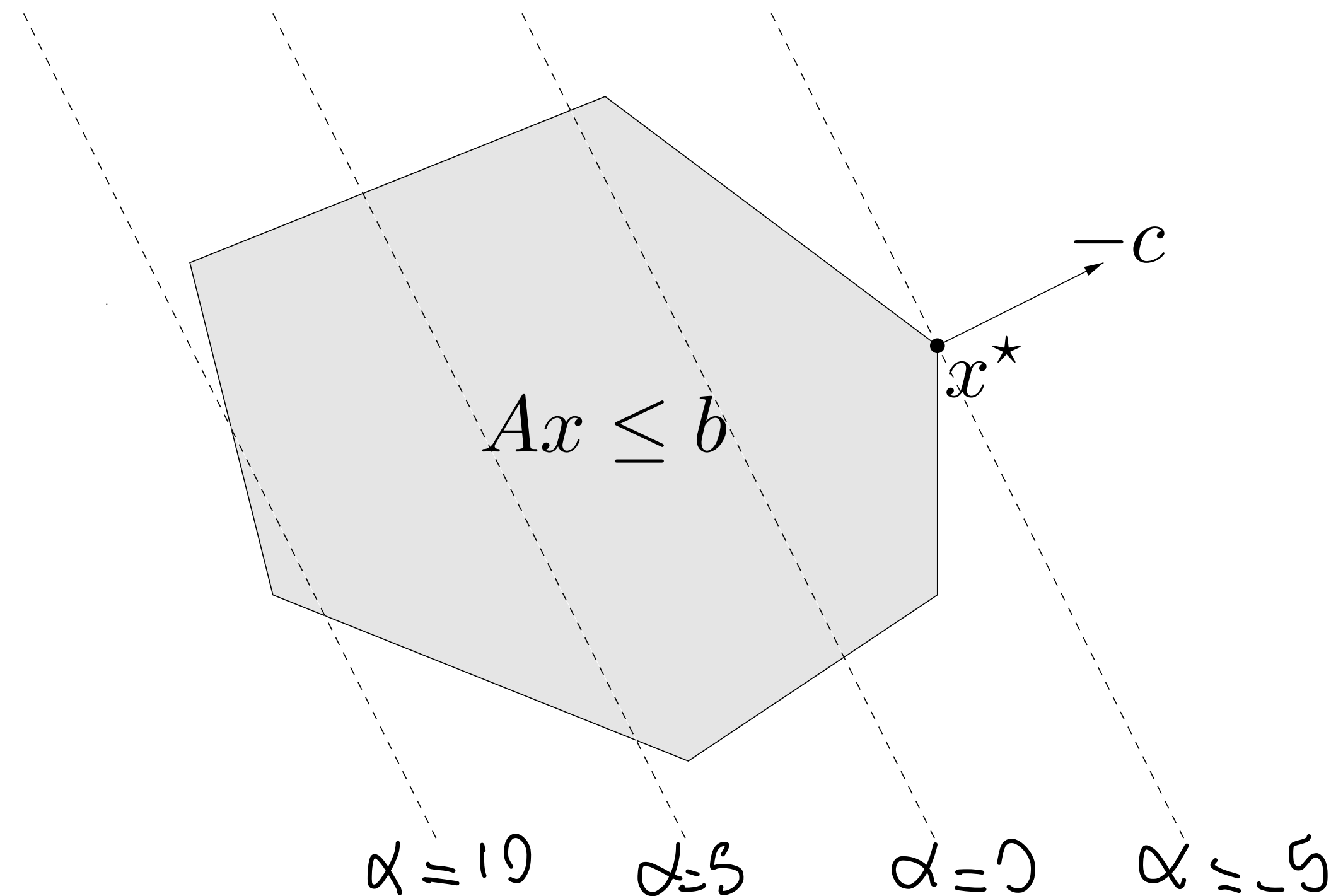
holds only for $\alpha_1 = \dots = \alpha_k = 0$

Properties

- The coefficients α_k in a linear combination $x = \alpha_1 v_1 + \dots + \alpha_k v_k$ are unique
- None of the vectors v_i is a linear combination of the other vectors

Geometrical interpretation of linear optimization

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

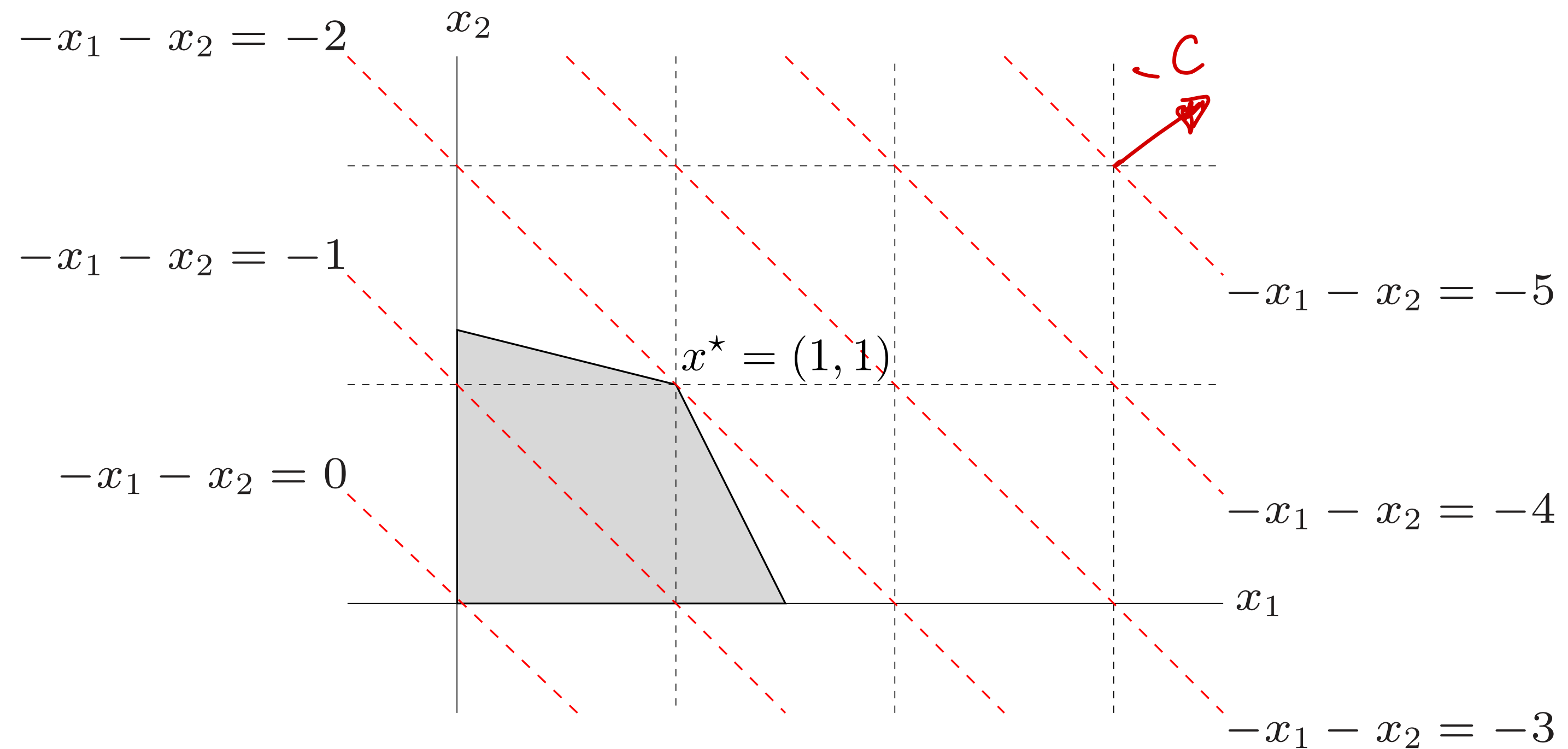


Dashed lines (hyperplanes) are level sets $c^T x = \alpha$ for different α

Example of linear optimization



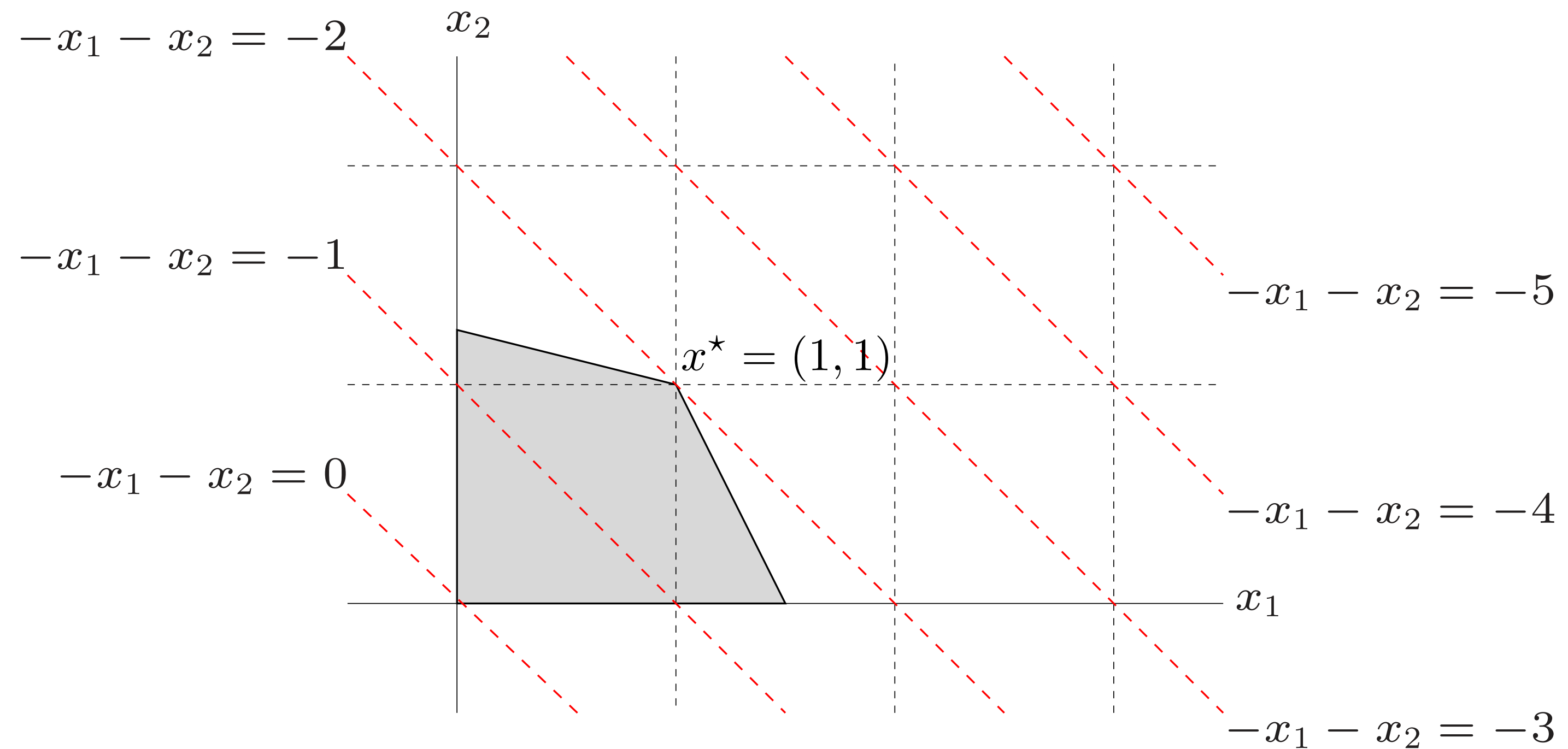
minimize $-x_1 - x_2$
subject to $2x_1 + x_2 \leq 3$
 $x_1 + 4x_2 \leq 5$
 $x_1 \geq 0, x_2 \geq 0$



Optimal solutions tend to be at a “**corner**” of the feasible set

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How do we formalize it?

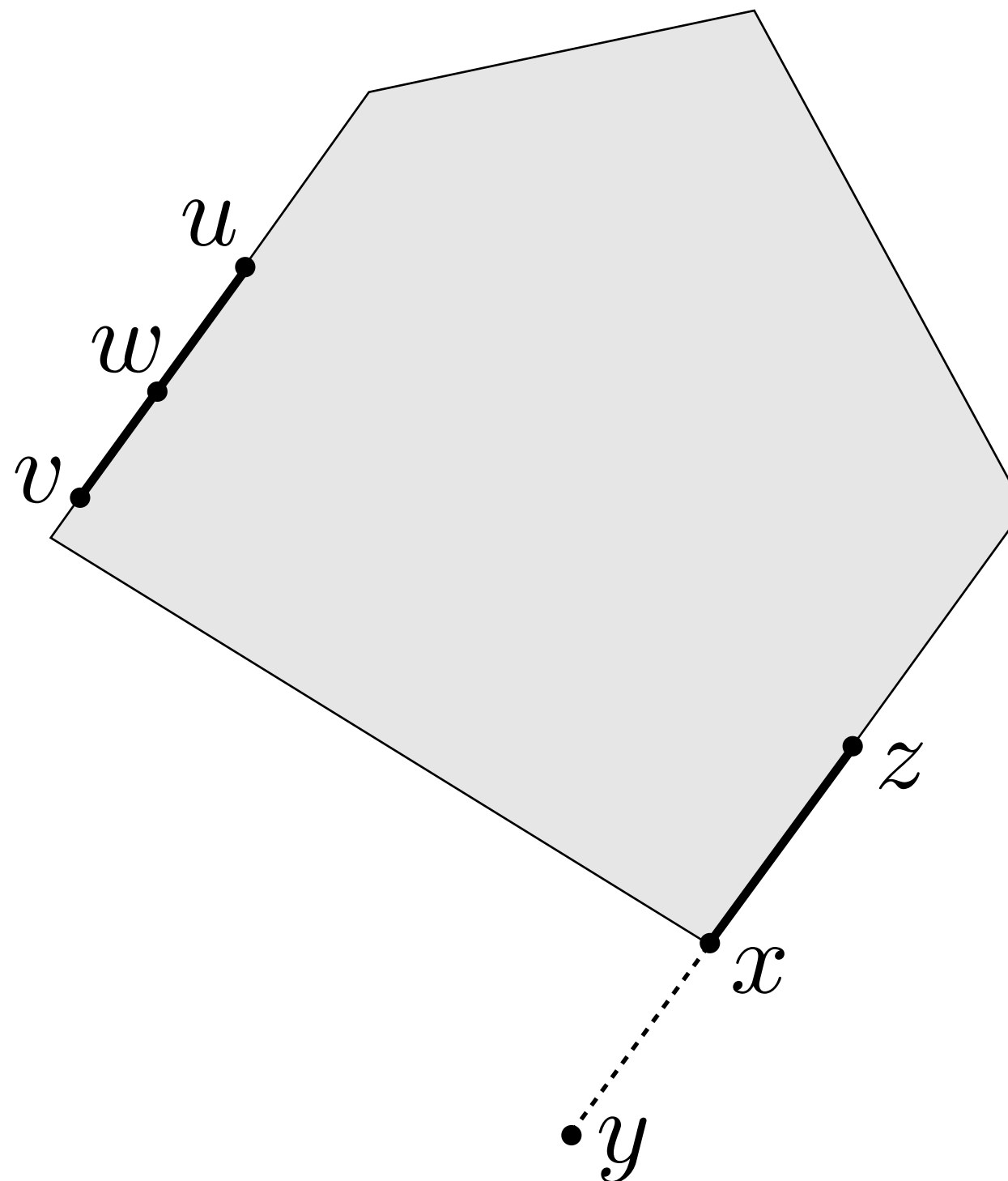
Corners of linear optimization

Extreme points

Definition

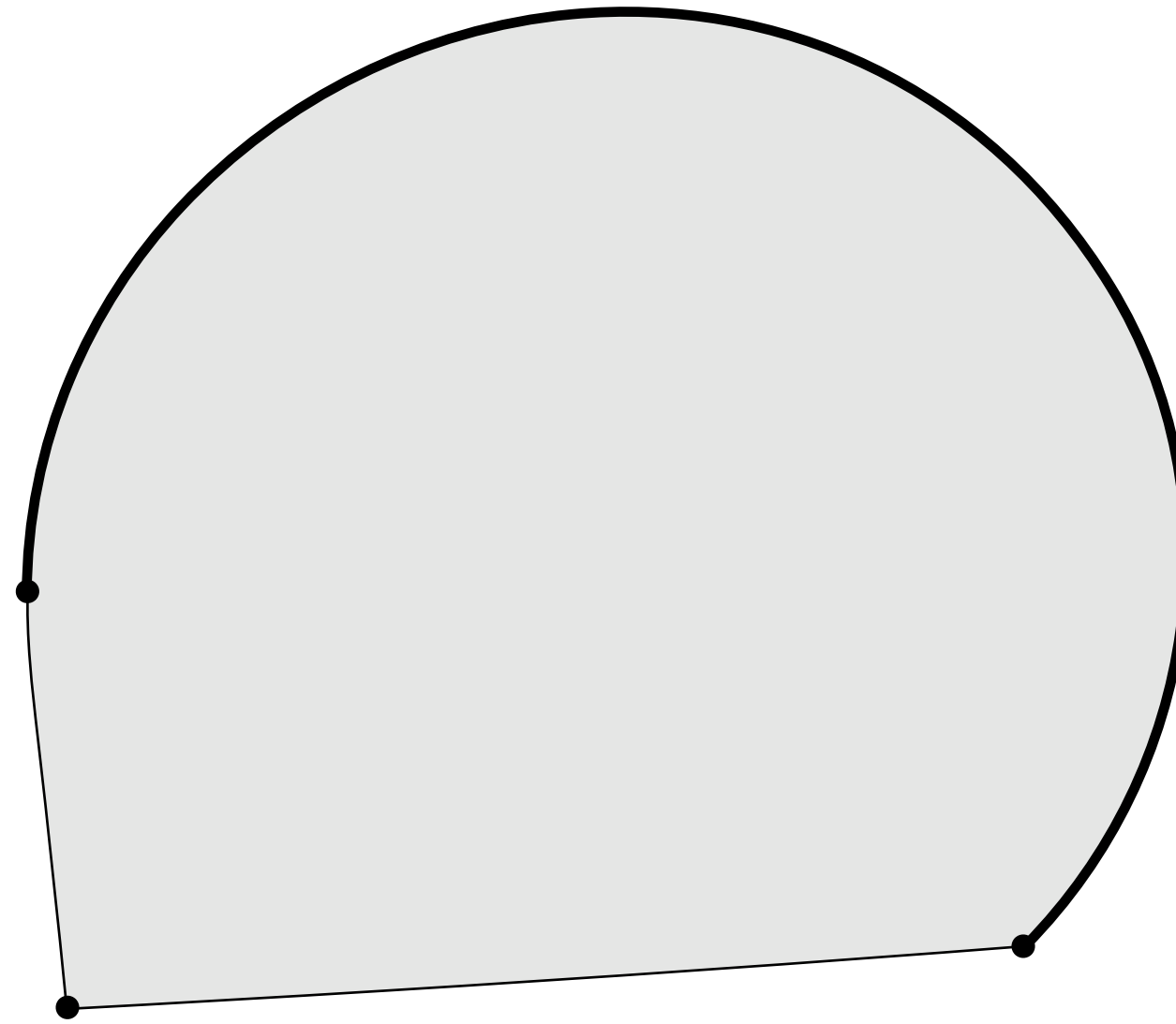
$x \in P$ is said to be an **extreme point** of P if

$\nexists y, z \in P$ ($y \neq x, z \neq x$) and $\alpha \in [0, 1]$ such that $x = \alpha y + (1 - \alpha)z$



Extreme points

Convex sets



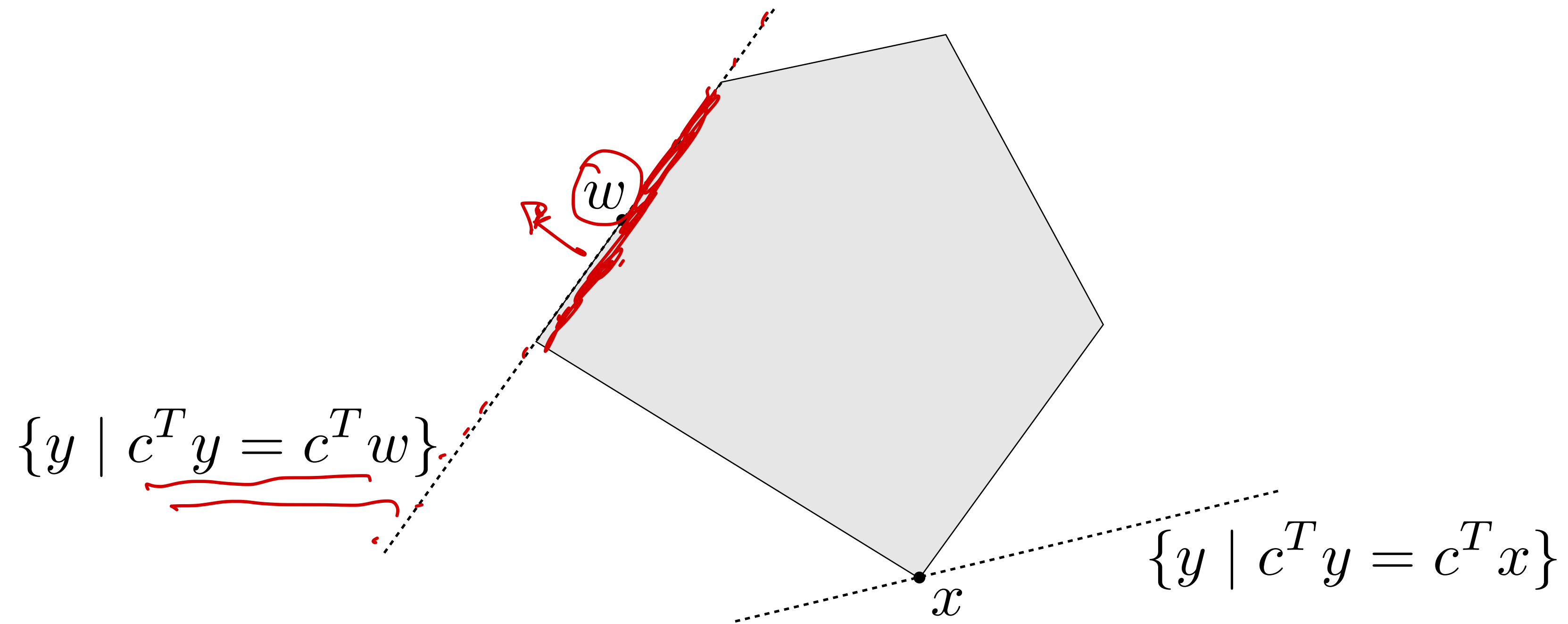
- Convex sets can have an infinite number of extreme points
- Polyhedra are convex sets with a finite number of extreme points

Vertices

Definition

$x \in P$ is a **vertex** if $\exists c$ such that x is the unique optimum of

$$\begin{aligned} &\text{minimize} && c^T y \\ &\text{subject to} && y \in P \end{aligned}$$



Basic feasible solution

$$P = \{x \mid a_i^T x \leq b_i, \quad i = 1, \dots, m\}$$

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Active constraints at \bar{x}

$$\mathcal{I}(\bar{x}) = \{i \in \{1, \dots, m\} \mid a_i^T \bar{x} = b_i\}$$

Index of all the constraints
satisfied as **equality**

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Basic solution \bar{x}

- $\{a_i \mid i \in \mathcal{I}(\bar{x})\}$ has n linearly independent vectors

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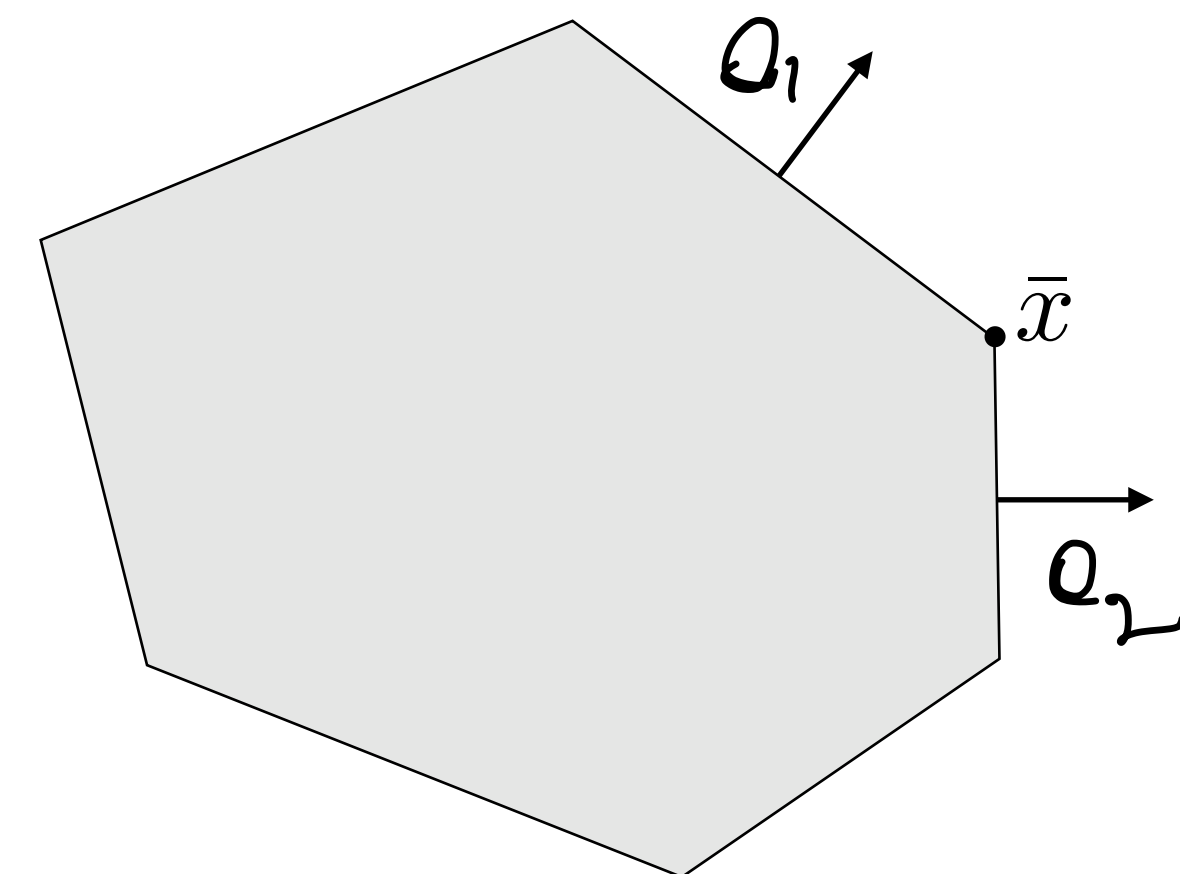
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Basic solution \bar{x}

- $\{a_i \mid i \in \mathcal{I}(\bar{x})\}$ has n linearly independent vectors

Basic feasible solution \bar{x}

- $\bar{x} \in P$
- $\{a_i \mid i \in \mathcal{I}(\bar{x})\}$ has n linearly independent vectors



Degenerate basic feasible solutions

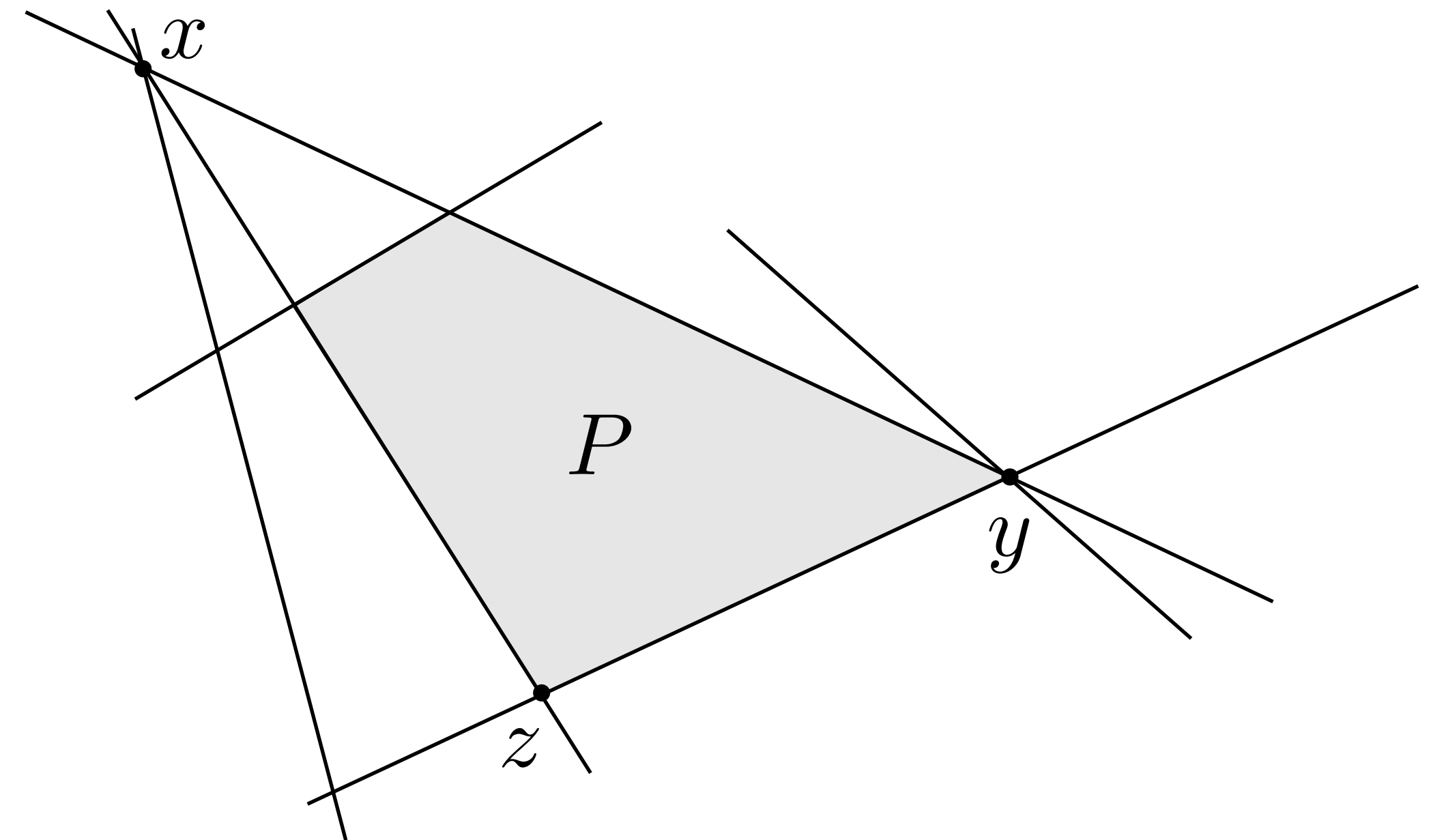
A solution \bar{x} is degenerate if $|\mathcal{I}(\bar{x})| > n$

Degenerate basic feasible solutions

A solution \bar{x} is degenerate if $|\mathcal{I}(\bar{x})| > n$

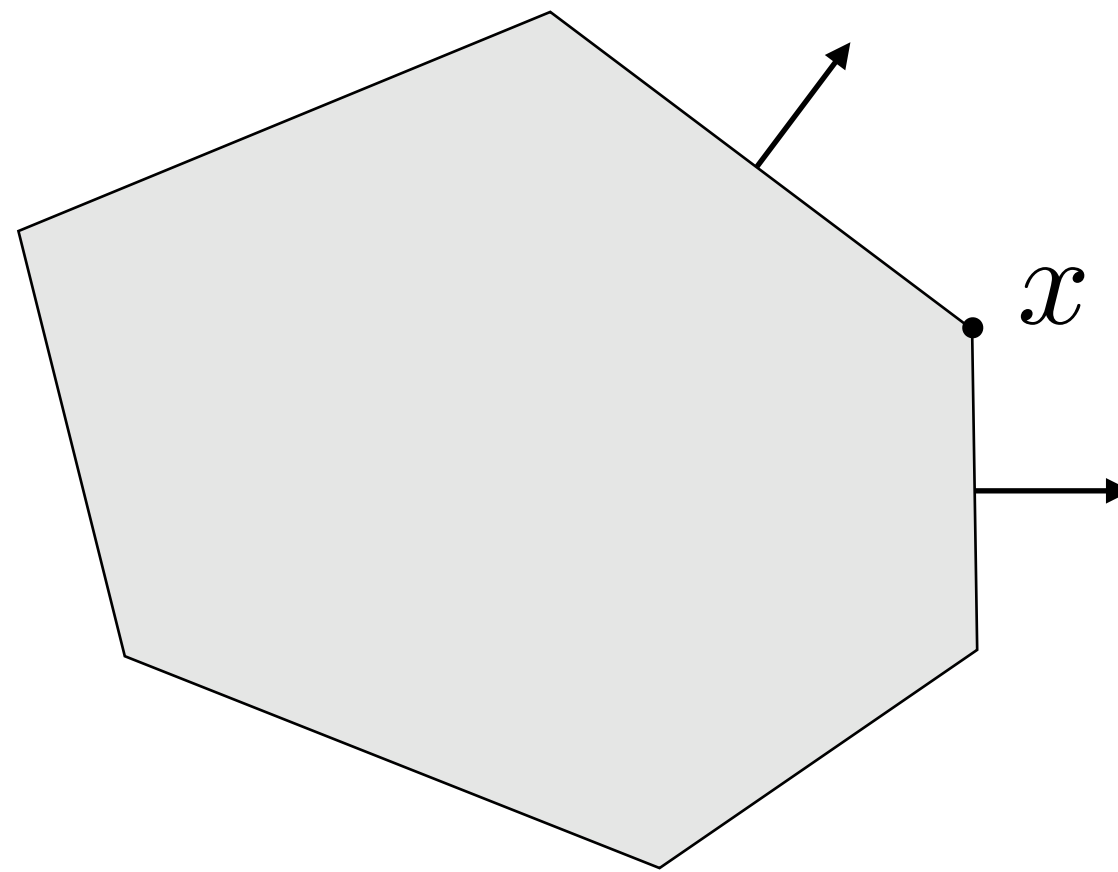
True or False?

	Basic	Feasible	Degenerate
x	✓	✗	✓
y	✓	✓	✓
z	✓	✓	✗



Equivalence Theorem

Given a nonempty polyhedron $P = \{x \mid Ax \leq b\}$



Let $x \in P$

x is a **vertex** $\iff x$ is an **extreme point** $\iff x$ is a **basic feasible solution**

Equivalent theorem proof

Vertex \rightarrow Extreme point

If x is a vertex, $\exists c$ such that $c^T x < c^T y, \quad \forall y \in P, y \neq x$

Let's assume x is not an extreme point:

$\exists y, z \neq x$ such that $x = \lambda y + (1 - \lambda)z$

Since x is a vertex, $c^T x < c^T y$ and $c^T x < c^T z$

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Therefore, $c^T x = \lambda c^T y + (1 - \lambda)c^T z > \lambda c^T x + (1 - \lambda)c^T x = c^T x$

\implies **contradiction**



Equivalent theorem proof

Extreme point \rightarrow Basic feasible solution

(proof by contraposition)

Suppose $x \in P$ is not basic feasible solution

Equivalent theorem proof

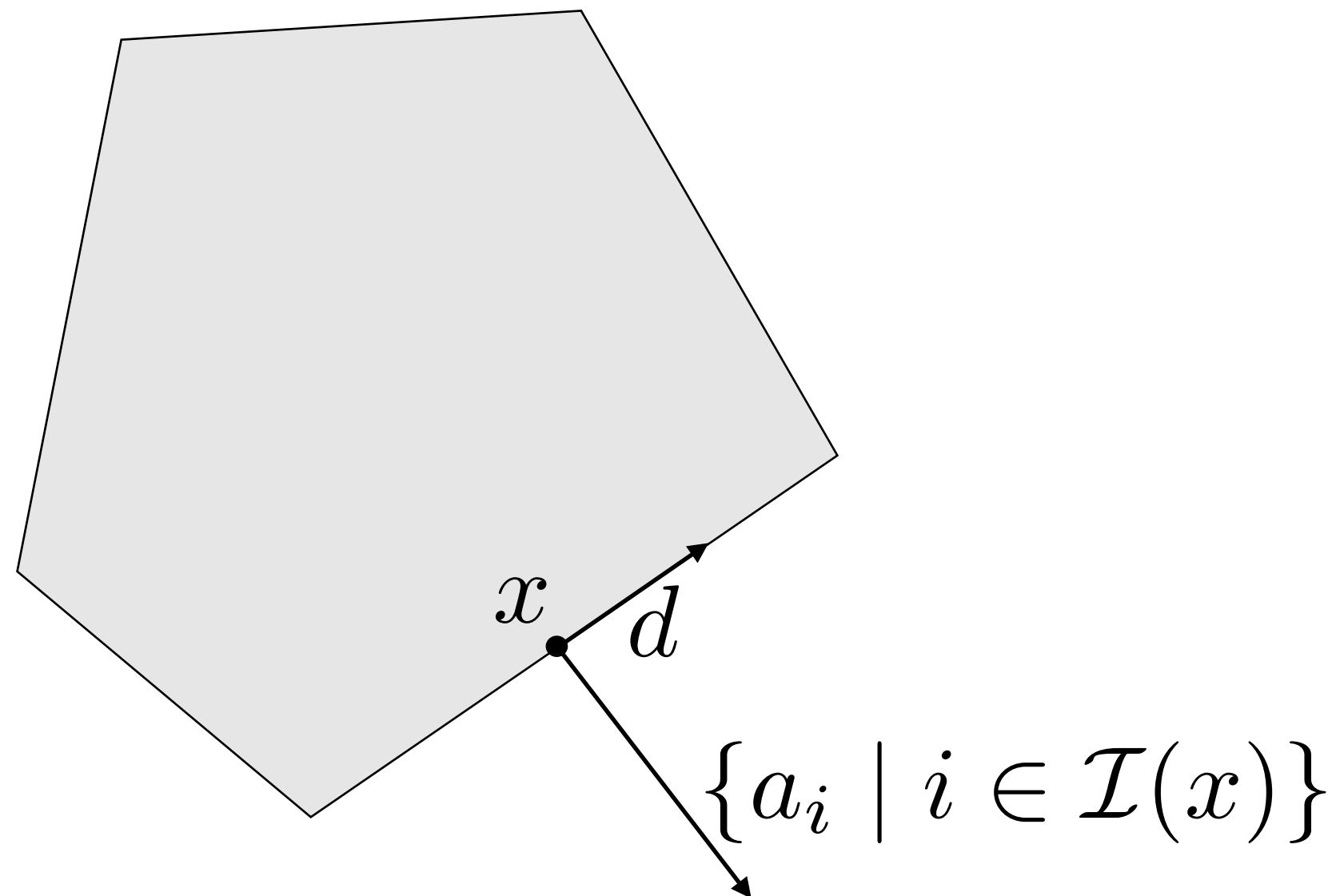
Extreme point \rightarrow Basic feasible solution

(proof by contraposition)

Suppose $x \in P$ is **not basic feasible solution**

$\{a_i \mid i \in \mathcal{I}(x)\}$ does not span \mathbf{R}^n

$\exists d \in \mathbf{R}^n$ perpendicular to all of them: $a_i^T d = 0, \quad \forall i \in \mathcal{I}(x)$



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Let $\epsilon > 0$ and define $y = x + \epsilon d$ and $z = x - \epsilon d$

For $i \in \mathcal{I}(x)$ we have $a_i^T y = b_i$ and $a_i^T z = b_i$

For $i \notin \mathcal{I}(x)$ we have $a_i^T x < b_i \Rightarrow a_i^T (x + \epsilon d) < b_i$ and $a_i^T (x - \epsilon d) < b_i$

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Hence, $y, z \in P$ and $x = \lambda y + (1 - \lambda)z$ with $\lambda = 0.5$.

$\implies x$ is **not an extreme point**

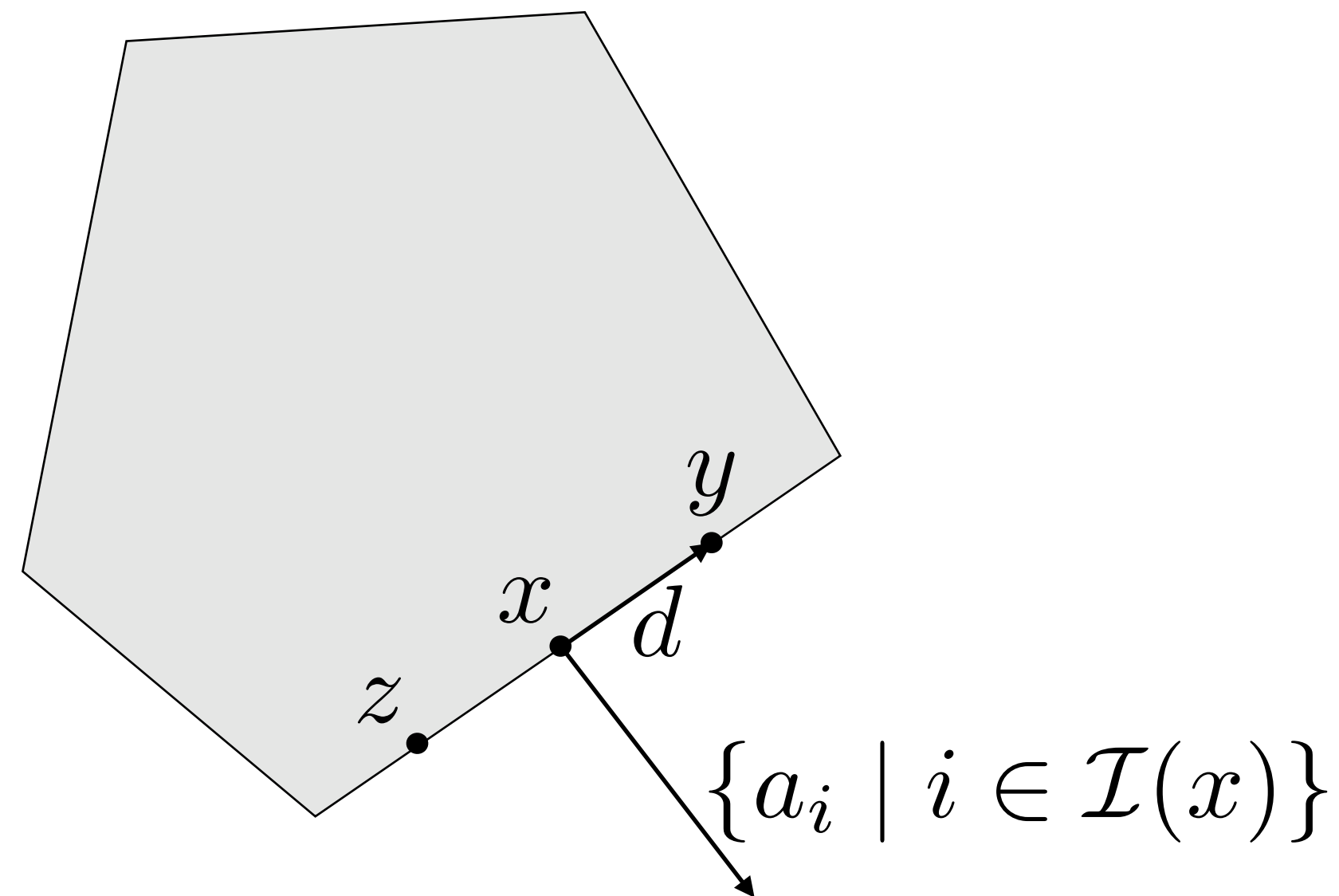


Equivalent theorem proof

Extreme point \rightarrow Basic feasible solution

(proof by contraposition)

Suppose $x \in P$ is not basic feasible solution



Hence, $y, z \in P$ and $x = \lambda y + (1 - \lambda)z$ with $\lambda = 0.5$.

$\implies x$ is not an extreme point



Equivalent theorem proof

Basic feasible solution \rightarrow Vertex

Left as exercise

Hint

Define $c = \sum_{i \in \mathcal{I}(x)} a_i$

Constructing basic solutions

Standard form polyhedra

Definition

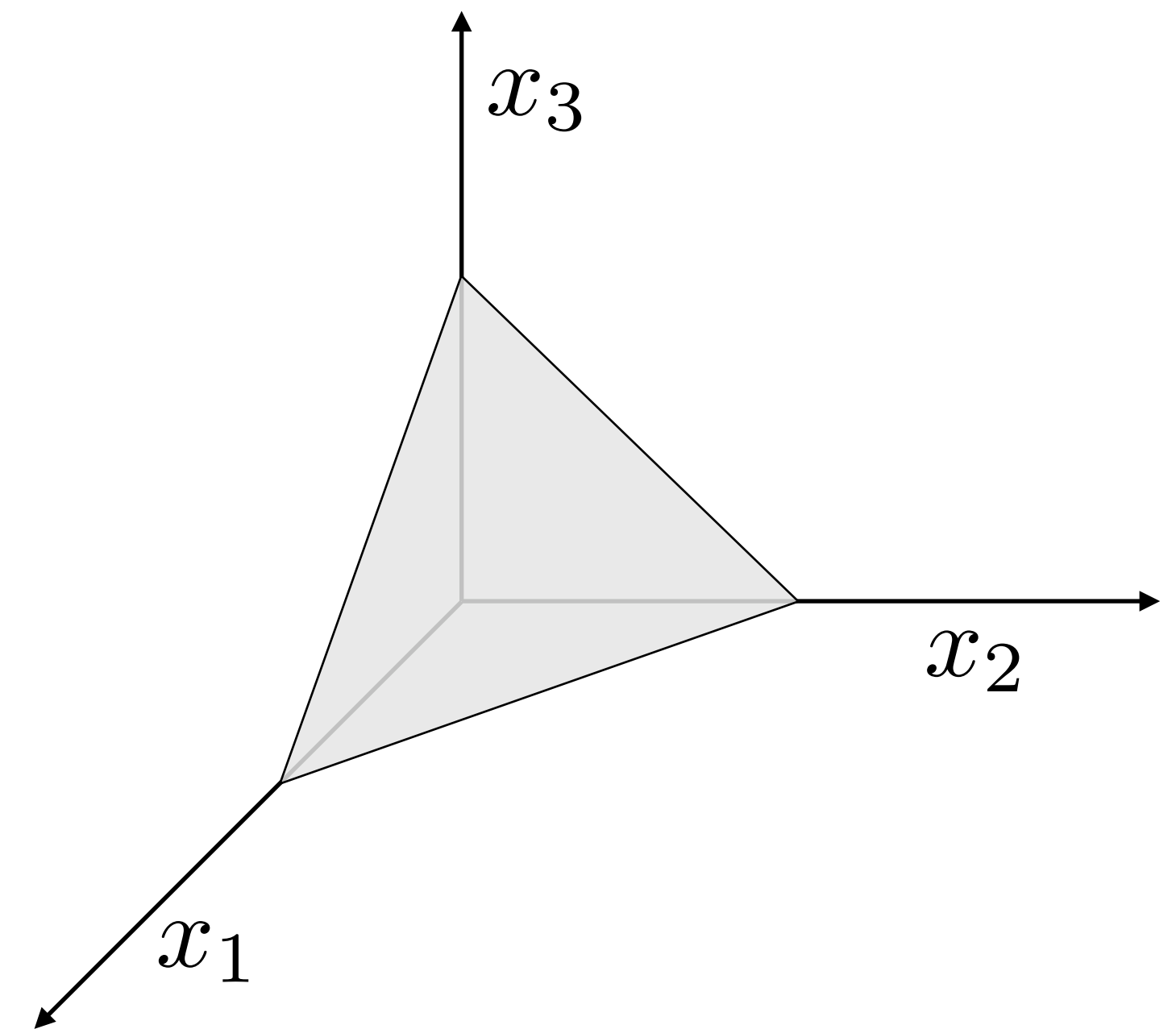
Standard form LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

$$\mathbb{R}^3 \quad b \in \mathbb{R}^1$$

Standard form polyhedron

$$P = \{x \mid Ax = b, x \geq 0\}$$



Standard form polyhedra

Definition

Standard form LP

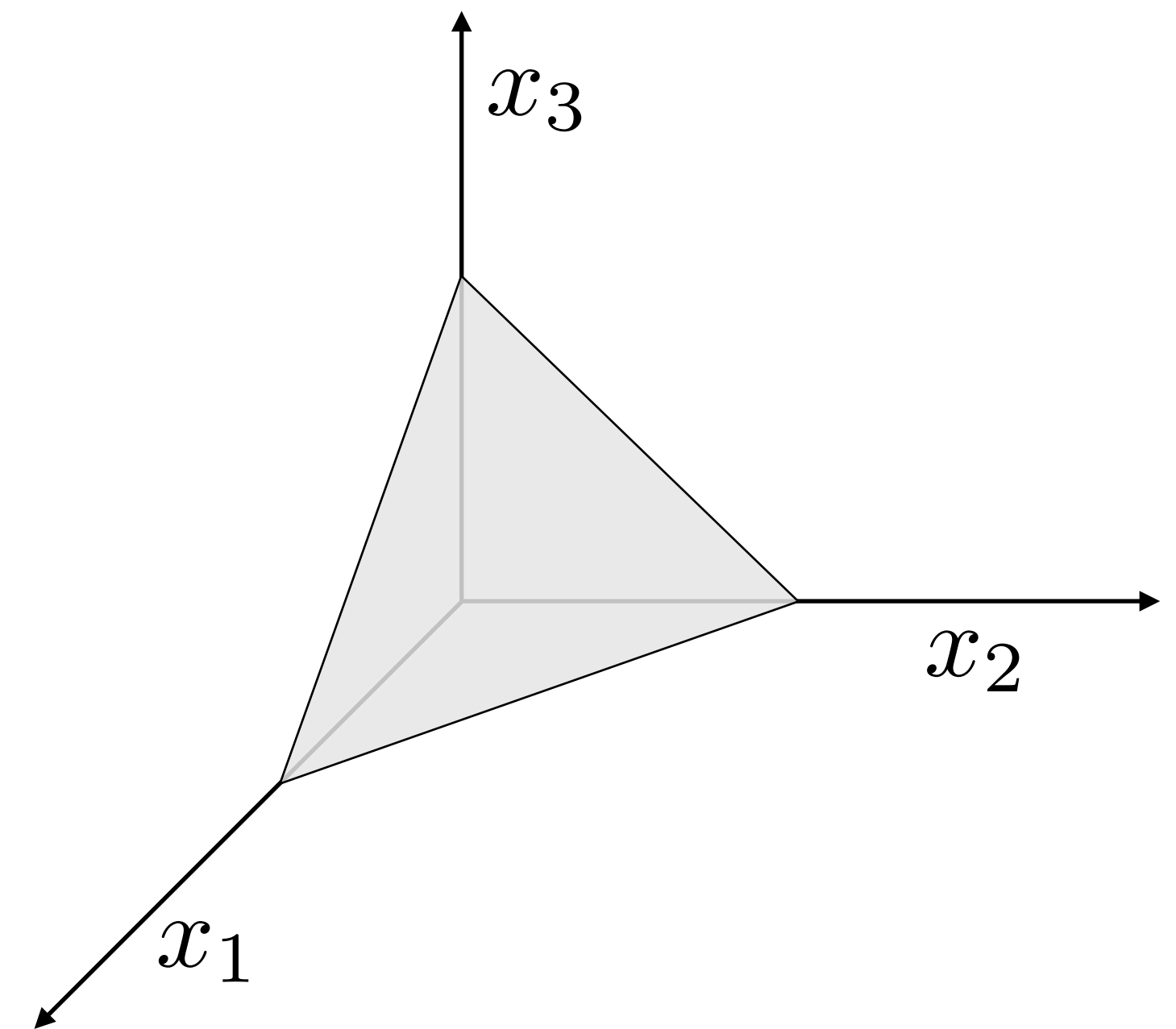
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Assumption

$A \in \mathbf{R}^{m \times n}$ has full row rank $m \leq n$

Standard form polyhedron

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Standard form polyhedra

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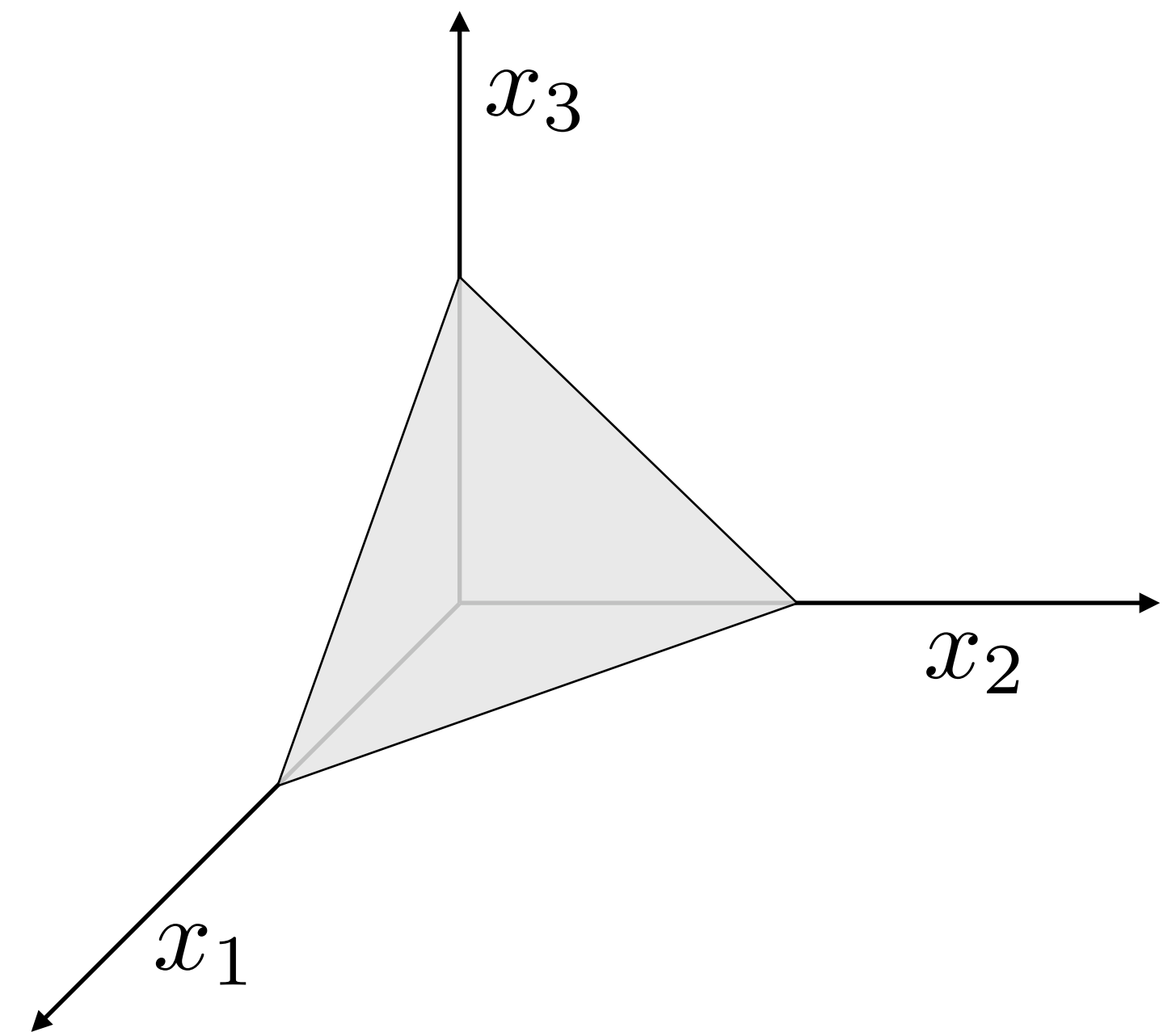
$A \in \mathbf{R}^{m \times n}$ has full row rank $m \leq n$

Interpretation

P lives in $(n - m)$ -dimensional subspace
} (

Standard form polyhedron

$$P = \{x \mid Ax = b, x \geq 0\}$$



Basic solutions

Standard form polyhedra

$$P = \{x \mid Ax = b, x \geq 0\} \quad \text{with} \quad A \in \mathbf{R}^{m \times n} \text{ has full row rank } m \leq n$$

Basic solutions

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x is a **basic solution** if and only if

- $Ax = b$
- There exist indices $B(1), \dots, B(m)$ such that
 - columns $A_{B(1)}, \dots, A_{B(m)}$ are linearly independent
 - $x_i = 0$ for $i \neq B(1), \dots, B(m)$

Basic solutions

Standard form polyhedra

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x is a **basic feasible solution** if x is a ~~basic~~ **basic solution** and $x \geq 0$

Constructing basic solution

1. Choose any m independent columns of A : $A_{B(1)}, \dots, A_{B(m)}$
2. Let $x_i = 0$ for all $i \neq B(1), \dots, B(m)$
3. Solve $Ax = b$ for the remaining $x_{B(1)}, \dots, x_{B(m)}$

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Basis
matrix

Basis columns

Basic variables

$$A_B = \begin{bmatrix} | & | & & | \\ A_{B(1)} & A_{B(2)} & \dots & A_{B(m)} \\ | & | & & | \end{bmatrix}, \quad x_B = \begin{bmatrix} x_{B(1)} \\ \vdots \\ x_{B(m)} \end{bmatrix} \longrightarrow \text{Solve } A_B x_B = b$$

$$x_i \geq 0$$

Constructing basic solution

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If $x_B \geq 0$, then x is a **basic feasible solution**

Finding a basic solution

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 2 & -1 & -3 & 0 & 0 \\ 0 & 2 & 8 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 6 \end{bmatrix}$$

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The matrix is partitioned into three basic columns, $A_{B(1)}$, $A_{B(2)}$, and $A_{B(3)}$, which are circled in red. Red arrows point from the labels $A_{B(1)}$, $A_{B(2)}$, and $A_{B(3)}$ to their respective columns. The variable x_2 is also circled in red.

Finding a basic solution

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$A_{B(1)}$ $A_{B(2)}$ $A_{B(3)}$

Solve

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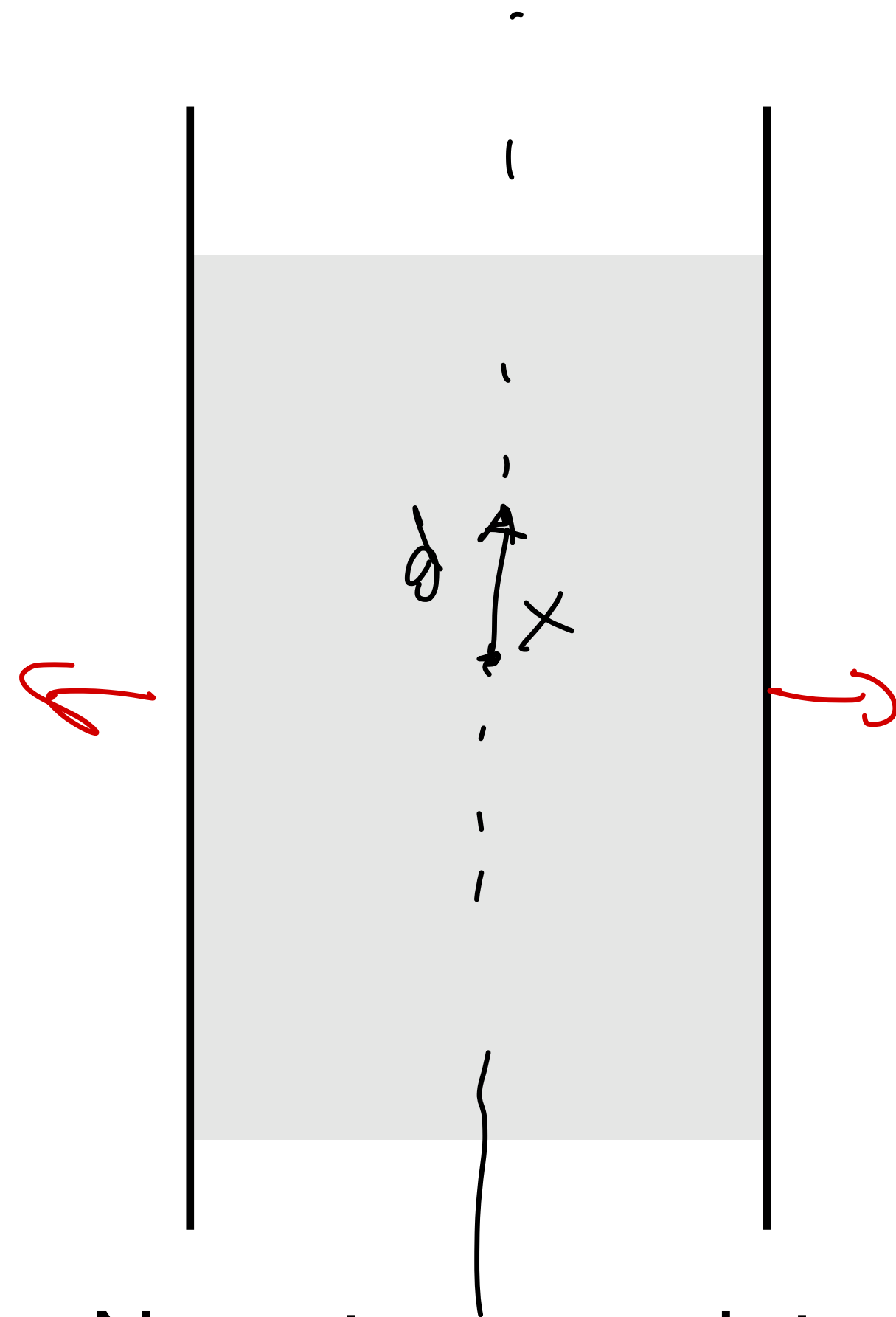
$$x_B = \begin{bmatrix} x_2 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \geq 0$$

$x_1, x_3 = 0$

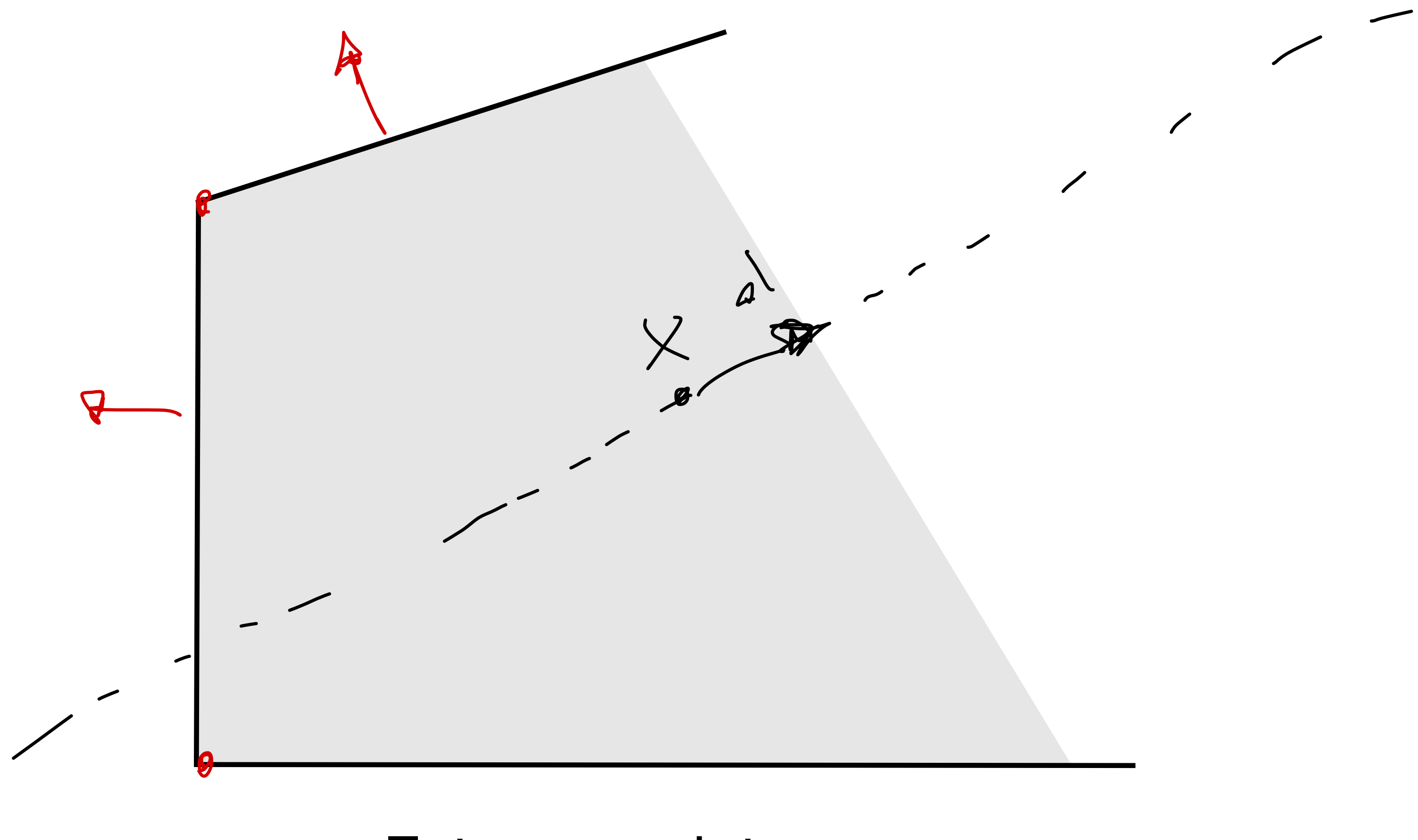
Existence and optimality of extreme points

Existence of extreme points

Example



No extreme points



Extreme points

Existence of extreme points

Characterization

A polyhedron P **contains a line** if

$\exists x \in P$ and a nonzero vector d such that $x + \lambda d \in P, \forall \lambda \in \mathbf{R}$.

Existence of extreme points

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- P does not contain a line
- P has at least one extreme point
- n of the a_i vectors are linearly independent

Existence of extreme points

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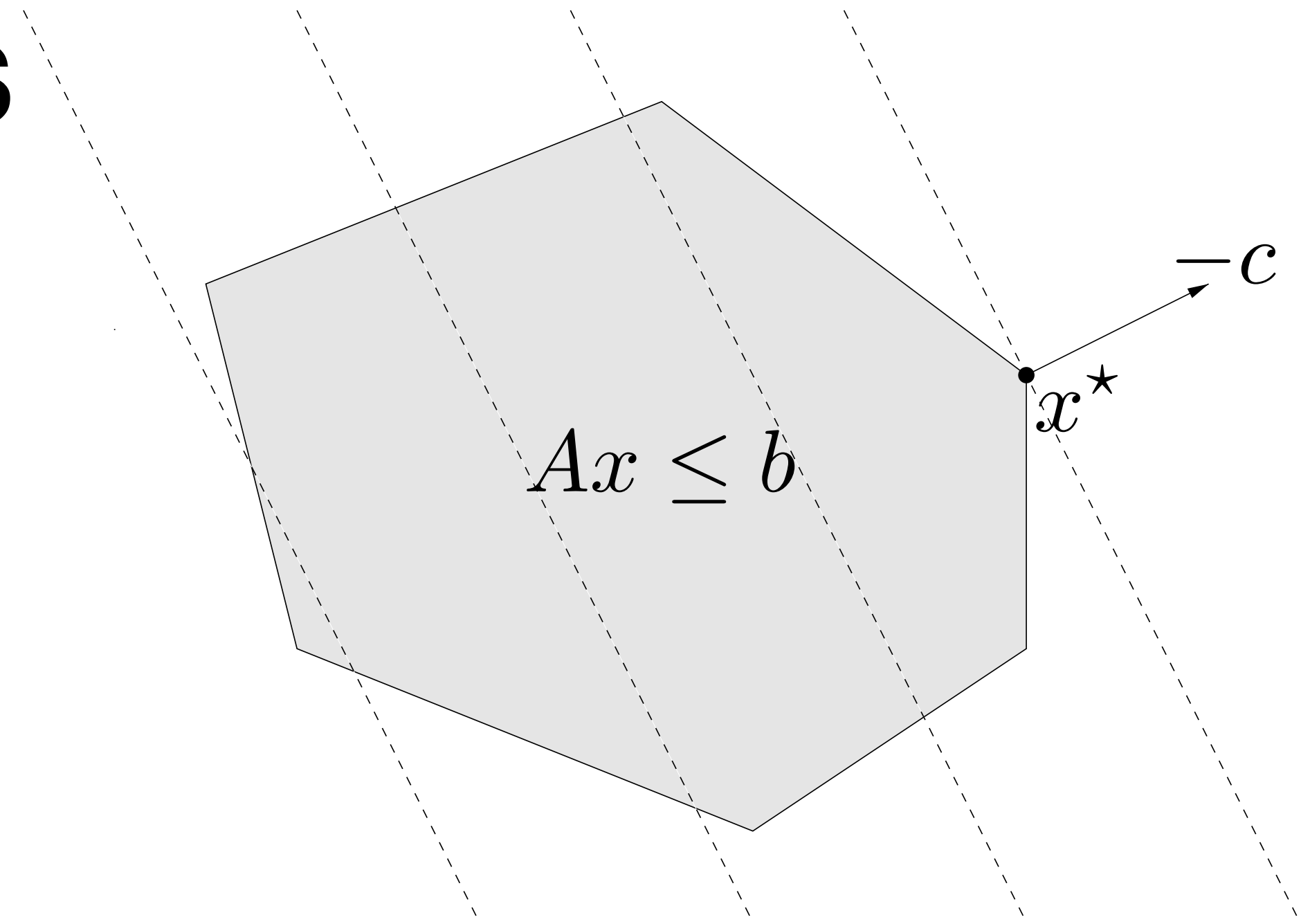
✓ G.P.S

Corollary

Every nonempty **bounded polyhedron** has
at least one basic feasible solution

Optimality of extreme points

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

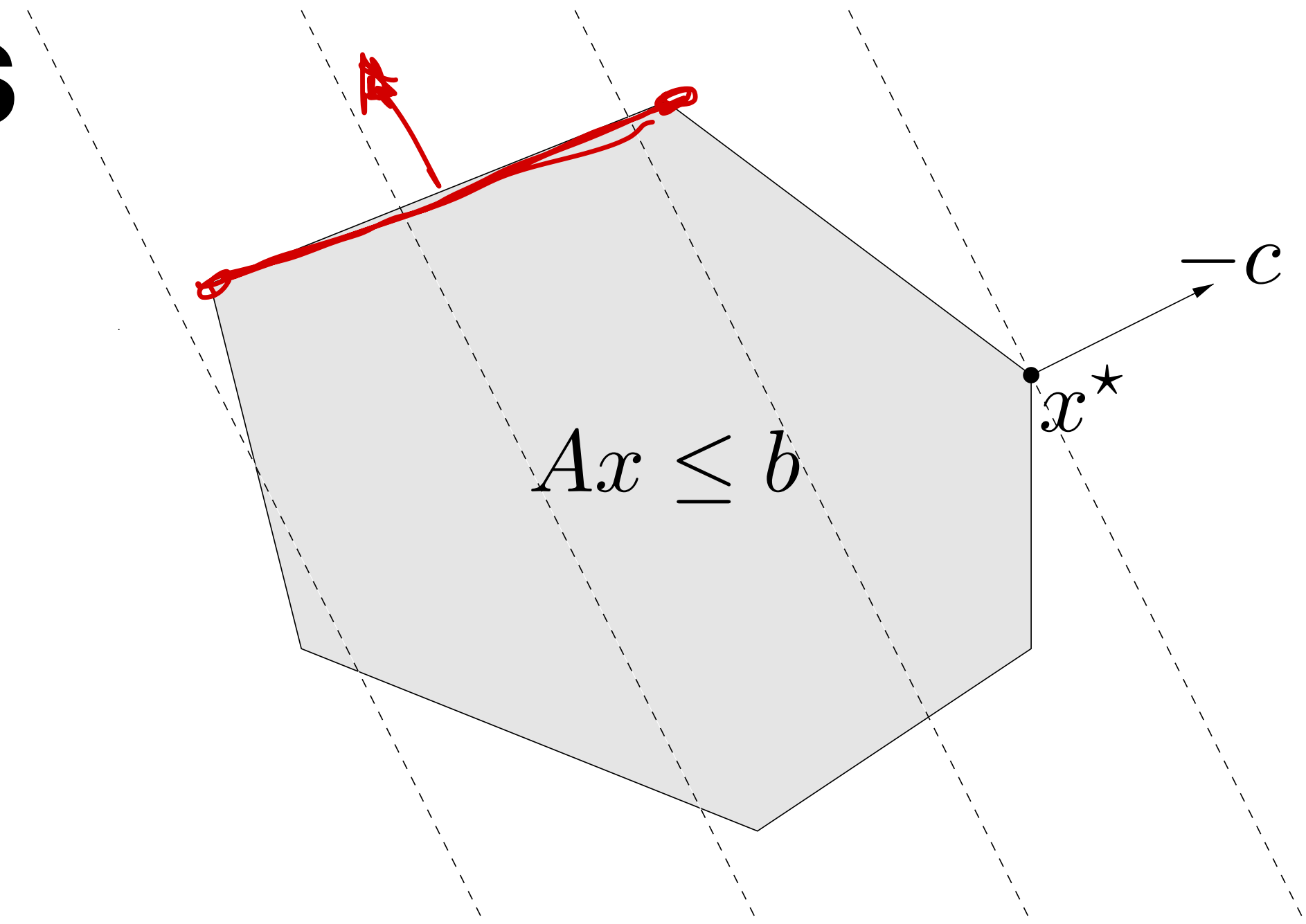


Optimality of extreme points

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

- If
- P has at least one extreme point
 - There exists an optimal solution x^*

Then, there exists an optimal solution which is an **extreme point** of P

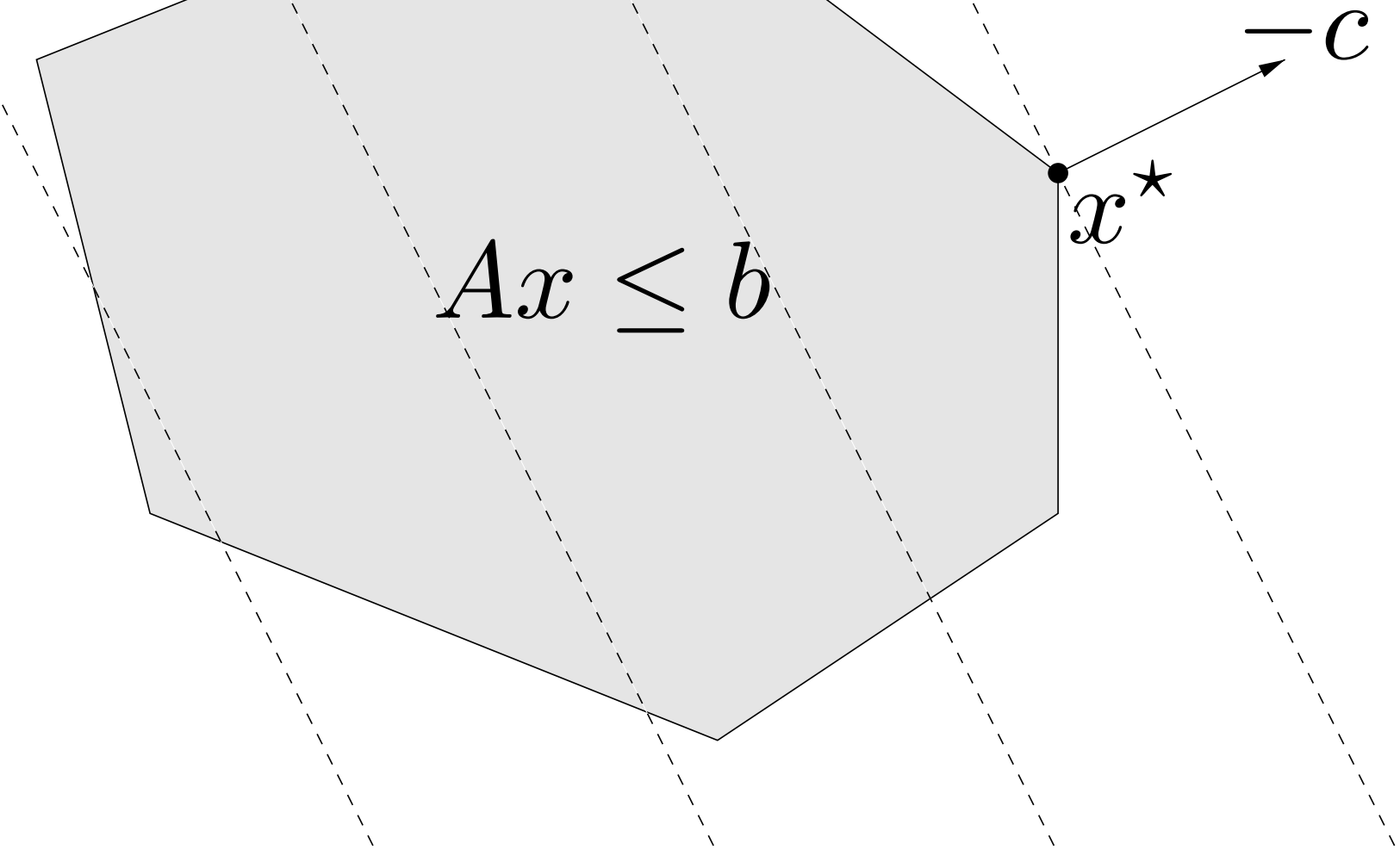


Optimality of extreme points

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

- If
- P has at least one extreme point
 - There exists an optimal solution x^*

Then, there exists an optimal solution which is an **extreme point** of P



We only need to search between **extreme points**

How to search among basic feasible solutions?

How to search among basic feasible solutions?

Idea

List all the basic feasible solutions, compare objective values and pick the best one.

How to search among basic feasible solutions?

Idea

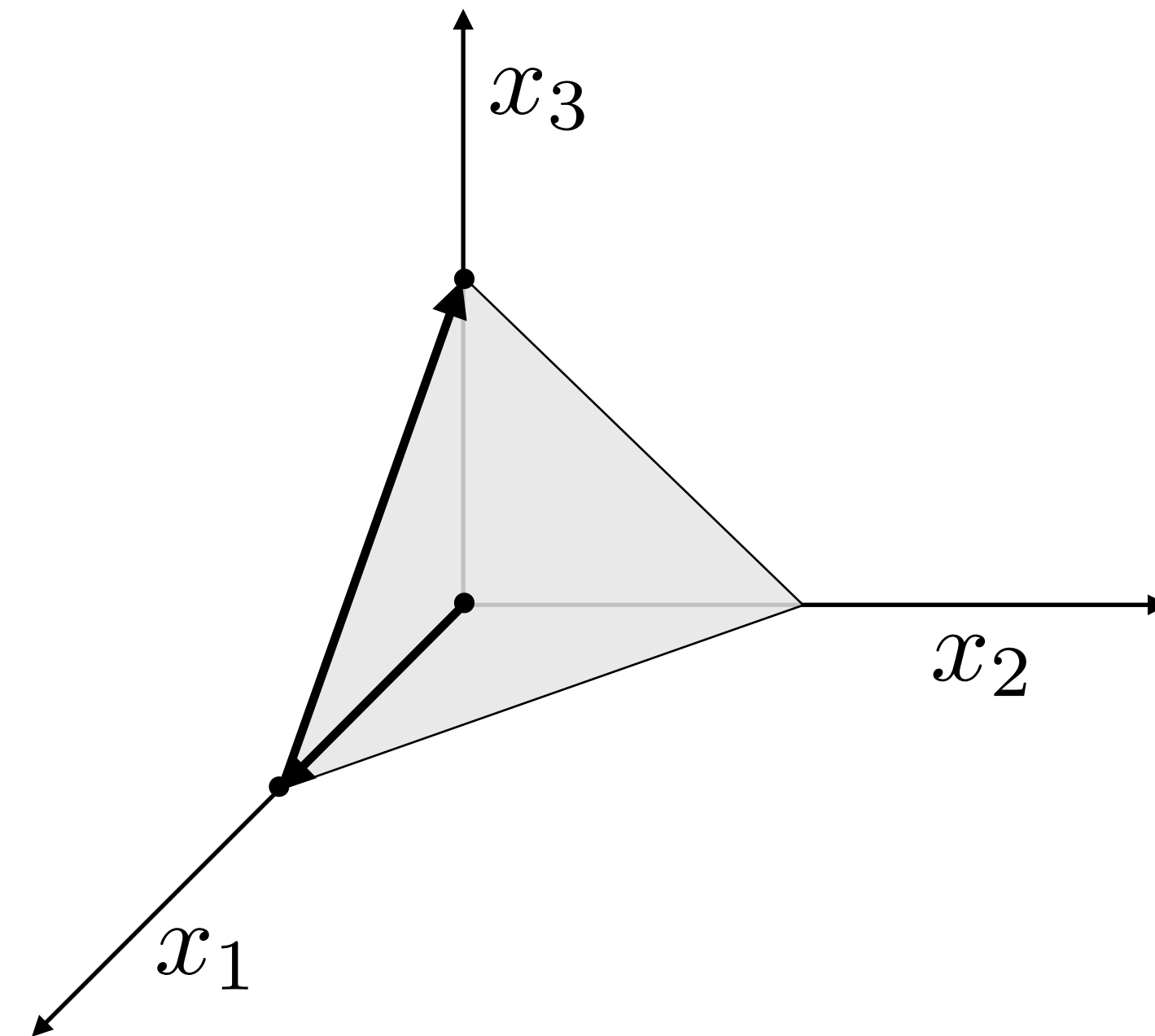
List all the basic feasible solutions, compare objective values and pick the best one.

Intractable!

If $n = 1000$ and $m = 100$, we have 10^{143} combinations!

Conceptual algorithm

- Start at corner
- Visit neighboring corner that improves the objective



Geometry of linear optimization

Today, we learned to:

- **Apply geometric and algebraic properties** of polyhedra to characterize the “corners” of the feasible region.
- **Construct basic feasible solutions** by solving a linear system.
- **Recognize existence and optimality** of extreme points.

Next lecture

The simplex method

- Iterations
- Convergence
- Complexity