ORF522 – Linear and Nonlinear Optimization

7. Linear optimization duality
Ed forum

• How computationally expensive is it for computers to "detect" special patterns such as sparsity or orthogonality? Or do people input these features and the algorithm just assumes it?

• Do solvers usually check to see what B is before deciding which method to use, or are they usually coded to use some default method? Does one need to create custom code when when the structure is known and the solver can be sped up a bit for some particular problems?

• I recall the final solution having 4 nonzero x values, while the basis consists of 3 elements - is this still a basic feasible solution, since there is a nonbasic variable that has a nonzero value? (It was a typo!)

• If we want to factor A into A1 A2 ... Ak, the matrices it can factor into, and the order they appear, is probably not unique. How does the computer typically do it? Perhaps, a related question is, how does the computer take inverses of large matrices?

• Are there situations, such as in physics, electrical engineering, robotics, etc where one knows for such that B will be tridiagonal and/or positive definite so that one uses a custom simplex method solver to speed up the process by not doing, say, and LU factorization but some faster method that applies to the given situation?
Recap
Linear optimization formulations

**Standard form LP**
- minimize $c^T x$
- subject to $Ax = b$
- $x \geq 0$

**Inequality form LP**
- minimize $c^T x$
- subject to $Ax \leq b$
Today’s agenda
Readings: [Chapter 4, Bertsimas, Tsitsiklis][Chapter 5, Vanderbei]

• Obtaining lower bounds
• The dual problem
• Weak and strong duality
Obtaining lower bounds
Obtaining lower bounds

A simple example

\[
\begin{align*}
\text{minimize} & \quad x_1 + 3x_2 \\
\text{subject to} & \quad x_1 + 3x_2 \geq 2
\end{align*}
\]

What is a lower bound on the optimal cost?
Obtaining lower bounds

A simple example

minimize \( x_1 + 3x_2 \)
subject to \( x_1 + 3x_2 \geq 2 \)

What is a lower bound on the optimal cost?

A lower bound is 2 because \( x_1 + 3x_2 \geq 2 \)
Obtaining lower bounds

Another example

minimize \( x_1 + 3x_2 \)
subject to \( x_1 + x_2 \geq 2 \)
\( x_2 \geq 1 \)

What is a lower bound on the optimal cost?
Obtaining lower bounds

Another example

minimize $x_1 + 3x_2$
subject to $x_1 + x_2 \geq 2$
$x_2 \geq 1$

What is a lower bound on the optimal cost?

Let’s sum the constraints

$1 \cdot (x_1 + x_2 \geq 2)$
$+ 2 \cdot (x_2 \geq 1)$
$= x_1 + 3x_2 \geq 4$
Obtaining lower bounds

Another example

minimize \( x_1 + 3x_2 \)
subject to \( x_1 + x_2 \geq 2 \)
\( x_2 \geq 1 \)

What is a lower bound on the optimal cost?

Let’s sum the constraints
\[
1 \cdot (x_1 + x_2 \geq 2) + 2 \cdot (x_2 \geq 1)
= x_1 + 3x_2 \geq 4
\]

A lower bound is 4
Obtaining lower bounds
A more interesting example

minimize \( x_1 + 3x_2 \)
subject to
\( x_1 + x_2 \geq 2 \)
\( x_2 \geq 1 \)
\( x_1 - x_2 \geq 3 \)

How can we obtain a lower bound?
Obtaining lower bounds
A more interesting example

minimize \( x_1 + 3x_2 \)
subject to \( x_1 + x_2 \geq 2 \)
\( x_2 \geq 1 \)
\( x_1 - x_2 \geq 3 \)

How can we obtain a lower bound?

Add constraints
\( y_1 \cdot (x_1 + x_2 \geq 2) \)
\[ + y_2 \cdot (x_2 \geq 1) \]
\[ + y_3 \cdot (x_1 - x_2 \geq 3) \]
\[ = x_1 + 3x_2 \geq 2y_1 + y_2 + 3y_3 \]
Obtaining lower bounds

A more interesting example

minimize \[ x_1 + 3x_2 \]

subject to

\[ x_1 + x_2 \geq 2 \]
\[ x_2 \geq 1 \]
\[ x_1 - x_2 \geq 3 \]

How can we obtain a lower bound?

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\[ y_1 \cdot (x_1 + x_2 \geq 2) \]
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\[ = x_1 + 3x_2 \geq 2y_1 + y_2 + 3y_3 \]

Bound
Obtaining lower bounds

A more interesting example

minimize \( x_1 + 3x_2 \)
subject to \( x_1 + x_2 \geq 2 \)
\( x_2 \geq 1 \)
\( x_1 - x_2 \geq 3 \)

How can we obtain a lower bound?

Add constraints
\[
\begin{align*}
  y_1 \cdot (x_1 + x_2 \geq 2) \\
  + y_2 \cdot (x_2 \geq 1) \\
  + y_3 \cdot (x_1 - x_2 \geq 3)
\end{align*}
\]
\[
= x_1 + 3x_2 \geq 2y_1 + y_2 + 3y_3
\]

Match the cost
\[
\begin{align*}
y_1 + y_3 &= 1 \\
y_1 + y_2 - y_3 &= 3 \\
y_1, y_2, y_3 &\geq 0
\end{align*}
\]
**Obtaining lower bounds**

**A more interesting example**

minimize \( x_1 + 3x_2 \)

subject to

\[
\begin{align*}
    & x_1 + x_2 \geq 2 \\
    & x_2 \geq 1 \\
    & x_1 - x_2 \geq 3
\end{align*}
\]

How can we obtain a lower bound?

**Add constraints**

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\begin{align*}
    y_1 \cdot (x_1 + x_2 & \geq 2) \\
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    + y_3 \cdot (x_1 - x_2 & \geq 3)
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**Match the cost**

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**Many options**

\[
y = (1, 2, 0) \Rightarrow \text{Bound 4}
\]
Obtaining lower bounds
A more interesting example

minimize \( x_1 + 3x_2 \)
subject to \( x_1 + x_2 \geq 2 \)
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How can we obtain a lower bound?

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Match the cost
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\begin{align*}
  y_1 + y_3 &= 1 \\
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\]

Many options
\[
y = (1, 2, 0) \Rightarrow \text{Bound 4} \\
y = (0, 4, 1) \Rightarrow \text{Bound 7}
\]
Obtaining lower bounds
A more interesting example

minimize \( x_1 + 3x_2 \)
subject to \( x_1 + x_2 \geq 2 \)
\( x_2 \geq 1 \)
\( x_1 - x_2 \geq 3 \)

How can we obtain a lower bound?

Add constraints
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\begin{align*}
y_1 \cdot (x_1 + x_2 \geq 2) \\
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\[= x_1 + 3x_2 \geq 2y_1 + y_2 + 3y_3\]

Bound

Match the cost
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\begin{align*}
y_1 + y_3 &= 1 \\
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Many options
\[
y = (1, 2, 0) \Rightarrow \text{Bound 4}
\]
\[
y = (0, 4, 1) \Rightarrow \text{Bound 7}
\]

How can we get the best one?
Obtaining lower bounds
A more interesting example — Best lower bound

We can obtain the **best lower bound** by solving the following problem

\[
\begin{align*}
\text{maximize} & \quad 2y_1 + y_2 + 3y_3 \\
\text{subject to} & \quad y_1 + y_3 = 1 \\
& \quad y_1 + y_2 - y_3 = 3 \\
& \quad y_1, y_2, y_3 \geq 0
\end{align*}
\]
Obtaining lower bounds
A more interesting example — Best lower bound

We can obtain the **best lower bound** by solving the following problem

maximize \[ 2y_1 + y_2 + 3y_3 \]
subject to \[ y_1 + y_3 = 1 \]
\[ y_1 + y_2 - y_3 = 3 \]
\[ y_1, y_2, y_3 \geq 0 \]

This linear optimization problem is called the **dual problem**
The dual problem
Lagrange multipliers

Consider the LP in standard form

minimize \[ c^T x \]
subject to \[ Ax = b \]
\[ x \geq 0 \]
Lagrange multipliers

Consider the LP in standard form

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\begin{align*}
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\end{align*}
\]

Relax the constraint

\[
\begin{align*}
g(y) = & \quad \text{minimize}_x \quad c^T x + y^T (Ax - b) \\
& \quad \text{subject to} \quad x \geq 0
\end{align*}
\]
Lagrange multipliers

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\end{align*}

Lower bound

\[ g(y) \leq c^T x^* + y^T (Ax^* - b) = c^T x^* \]
Lagrange multipliers

Consider the LP in standard form

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\begin{align*}
\text{minimize} & \quad c^T x \\
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Lower bound

\[
g(y) \leq c^T x^* + y^T (Ax^* - b) = c^T x^*
\]

Best lower bound

\[
\max_y \quad g(y)
\]
The dual

**Dual function**

\[ g(y) = \minimize_{x \geq 0} \left( c^T x + y^T (Ax - b) \right) \]

\[ -b^T y + \minimize_{x \geq 0} \left( c + A^T y \right)^T x \]
The dual

**Dual function**

\[
g(y) = \min_{x \geq 0} \left( c^T x + y^T (Ax - b) \right) - b^T y + \min_{x \geq 0} \left( c + A^T y \right)^T x
\]

\[
g(y) = \begin{cases} 
-b^T y & \text{if } c + A^T y \geq 0 \\
-\infty & \text{otherwise}
\end{cases}
\]
The dual

**Dual function**

\[ g(y) = \minimize_{x \geq 0} \left( c^T x + y^T (Ax - b) \right) \]

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\[ g(y) = \begin{cases} 
- b^T y & \text{if } c + A^T y \geq 0 \\
-\infty & \text{otherwise} 
\end{cases} \]

**Dual problem** (find the best bound)

\[ \maximize_{y} g(y) = \maximize \quad - b^T y \]

subject to \( A^T y + c \geq 0 \)
Primal and dual problems

**Primal problem**
- minimize $c^T x$
- subject to $Ax = b$
  - $x \geq 0$

**Dual variable** $x \in \mathbb{R}^n$

**Dual problem**
- maximize $-b^T y$
- subject to $A^T y + c \geq 0$

**Dual variable** $y \in \mathbb{R}^m$
Primal and dual problems

**Primal problem**

minimize \( c^T x \)
subject to \( Ax = b \)
\( x \geq 0 \)

Primal variable \( x \in \mathbb{R}^n \)

**Dual problem**

maximize \(-b^T y\)
subject to \( A^T y + c \geq 0\)

Dual variable \( y \in \mathbb{R}^m \)

The dual problem carries **useful information** for the primal problem.
Primal and dual problems

Primal problem

minimize \( c^T x \)
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\( x \geq 0 \)

Primal variable \( x \in \mathbb{R}^n \)

Dual problem

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Dual variable \( y \in \mathbb{R}^m \)

The dual problem carries **useful information** for the primal problem

Duality is useful also to **solve** optimization problems
Dual of inequality form LP

What if you find an LP with inequalities?

minimize \( c^T x \)

subject to \( Ax \leq b \)
Dual of inequality form LP

What if you find an LP with inequalities?

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax \leq b
\end{align*}
\]

1. We could first transform it to standard form
Dual of inequality form LP

What if you find an LP with inequalities?

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax \leq b
\end{align*}
\]

1. We could first transform it to standard form
2. We can compute the dual function (same procedure as before)

Relax the constraint

\[
g(y) = \minimize_x c^T x + y^T (Ax - b)
\]
Dual of inequality form LP

What if you find an LP with inequalities?

minimize $c^T x$

subject to $Ax \leq b$

1. We could first transform it to standard form
2. We can compute the dual function (same procedure as before)

Relax the constraint

$$g(y) = \minimize_x c^T x + y^T (Ax - b)$$

Lower bound

$$g(y) \leq c^T x^* + y^T (Ax^* - b) \leq c^T x^*$$

we must have $y \geq 0$
Dual of LP with inequalities

Derivation

Dual function

\[ g(y) = \minimize_x \left( c^T x + y^T (Ax - b) \right) \]

\[ - b^T y + \minimize_x \left( c + A^T y \right)^T x \]
Dual of LP with inequalities

Derivation

**Dual function**

\[ g(y) = \min_x \left( c^T x + y^T (Ax - b) \right) \]

\[ -b^T y + \min_x (c + A^T y)^T x \]

\[ g(y) = \begin{cases} 
-b^T y & \text{if } c + A^T y = 0 \quad \text{(and } y \geq 0) \\
-\infty & \text{otherwise} 
\end{cases} \]
Dual of LP with inequalities

Derivation

**Dual function**

\[
g(y) = \min_x \left( c^T x + y^T (Ax - b) \right) - b^T y + \min_x \left( c + A^T y \right)^T x
\]

\[
g(y) = \begin{cases} 
-b^T y & \text{if } c + A^T y = 0 \quad (\text{and } y \geq 0) \\
-\infty & \text{otherwise}
\end{cases}
\]

**Dual problem** (find the best bound)

\[
\begin{align*}
\text{maximize} & \quad g(y) = -b^T y \\
\text{subject to} & \quad A^T y + c = 0 \\
& \quad y \geq 0
\end{align*}
\]
**General forms**

<table>
<thead>
<tr>
<th>Primal</th>
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<th>Dual</th>
</tr>
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<tbody>
<tr>
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</tr>
<tr>
<td></td>
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### General forms

**Primal**

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### Inequality form LP

**Primal**

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### LP with inequalities and equalities

**Primal**

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<td></td>
<td>$C x = d$</td>
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Example from before

minimize \[ x_1 + 3x_2 \]
subject to \[ x_1 + x_2 \geq 2 \]
\[ x_2 \geq 1 \]
\[ x_1 - x_2 \geq 3 \]
Example from before

minimize \( x_1 + 3x_2 \)
subject to
\[
\begin{align*}
  x_1 + x_2 & \geq 2 \\
  x_2 & \geq 1 \\
  x_1 - x_2 & \geq 3
\end{align*}
\]

Inequality form LP

minimize \( c^T x \)
subject to
\[
\begin{align*}
  A x & \leq b
\end{align*}
\]

\[
\begin{align*}
  c &= (1, 3) \\
  A &= \begin{bmatrix} -1 & -1 \\ 0 & -1 \\ -1 & 1 \end{bmatrix} \\
  b &= (-2, -1, -3)
\end{align*}
\]
Example from before

minimize \quad x_1 + 3x_2 \\
subject to \quad x_1 + x_2 \geq 2 \\
\quad x_2 \geq 1 \\
\quad x_1 - x_2 \geq 3 \\

Dual
maximize \quad -b^T y \\
subject to \quad A^T y + c = 0 \\
\quad y \geq 0 \\

Inequality form LP
minimize \quad c^T x \\
subject to \quad Ax \leq b \\

\begin{align*}
c &= (1, 3) \\
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minimize \( x_1 + 3x_2 \)
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Inequality form LP

minimize \( c^T x \)
subject to \( Ax \leq b \)

\( c = (1, 3) \)
\( A = \begin{bmatrix} -1 & -1 \\ 0 & -1 \\ -1 & 1 \end{bmatrix} \)
\( b = (-2, -1, -3) \)

Dual

maximize \( -b^T y \)
subject to \( A^T y + c = 0 \)
\( y \geq 0 \)

maximize \( 2y_1 + y_2 + 3y_3 \)
subject to \( -y_1 - y_3 = -1 \)
\( -y_1 - y_2 + y_3 = -3 \)
\( y_1, y_2, y_3 \geq 0 \)

\( A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ -1 & -1 \end{bmatrix} \)
\( C = \begin{bmatrix} A \\ -A \end{bmatrix} \)

\( A^T x \leq 0 \quad \Rightarrow \quad Cx \leq 0 \)
To memorize

Ways to get the dual
• Derive dual function directly
• Transform the problem in inequality form LP and dualize

Sanity-checks and signs convention
• Consider constraints as $g(x) \leq 0$ or $g(x) = 0$
• Each dual variable is associated to a primal constraint
• $y$ free for primal equalities and $y \geq 0$ for primal inequalities
Dual of the dual

**Theorem**

If we transform the primal into its dual and then transform the dual to its dual, we obtain a problem equivalent to the original problem. In other words, the **dual of the dual is the primal**.
Dual of the dual

Theorem
If we transform the primal into its dual and then transform the dual to its dual, we obtain a problem equivalent to the original problem. In other words, the dual of the dual is the primal.

Exercise
Derive dual and dualize again

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Dual of the dual

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If we transform the primal into its dual and then transform the dual to its dual, we obtain a problem equivalent to the original problem. In other words, the dual of the dual is the primal.

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Theorem
If we transform a linear optimization problem to another form (inequality form, standard form, inequality and equality form), the dual of the two problems will be equivalent.
Weak and strong duality
Optimal objective values

**Primal**

minimize \( c^T x \)

subject to \( Ax \leq b \)

\( p^* \) is the primal optimal value

Primal infeasible: \( p^* = +\infty \)

Primal unbounded: \( p^* = -\infty \)

**Dual**

maximize \( -b^T y \)

subject to \( A^T y + c = 0 \)

\( y \geq 0 \)

\( d^* \) is the dual optimal value

Dual infeasible: \( d^* = -\infty \)

Dual unbounded: \( d^* = +\infty \)
Weak duality

Theorem
If \(x, y\) satisfy:

- \(x\) is a feasible solution to the primal problem
- \(y\) is a feasible solution to the dual problem

\[-b^T y \leq c^T x\]
Weak duality

Theorem
If \( x, y \) satisfy:

- \( x \) is a feasible solution to the primal problem
- \( y \) is a feasible solution to the dual problem

\[
-b^T y \leq c^T x
\]

Proof
We know that \( Ax \leq b, A^T y + c = 0 \) and \( y \geq 0 \). Therefore,

\[
0 \leq y^T (b - Ax) = b^T y - y^T Ax = c^T x + b^T y
\]

\( \square \)
Weak duality

Theorem
If $x, y$ satisfy:

- $x$ is a feasible solution to the primal problem
- $y$ is a feasible solution to the dual problem

\[-b^T y \leq c^T x\]

Proof
We know that $Ax \leq b$, $A^T y + c = 0$ and $y \geq 0$. Therefore,

\[0 \leq y^T (b - Ax) = b^T y - y^T Ax = c^T x + b^T y\]

Remark
- Any dual feasible $y$ gives a **lower bound** on the primal optimal value
- Any primal feasible $x$ gives an **upper bound** on the dual optimal value
- $c^T x + b^T y$ is the **duality gap**
Weak duality

Corollaries

Unboundedness vs feasibility

- Primal unbounded ($p^* = -\infty$) $\Rightarrow$ dual infeasible ($d^* = -\infty$)
- Dual unbounded ($d^* = +\infty$) $\Rightarrow$ primal infeasible ($p^* = +\infty$)
Weak duality
Corollaries

Unboundedness vs feasibility
- Primal unbounded \((p^* = -\infty)\) ⇒ dual infeasible \((d^* = -\infty)\)
- Dual unbounded \((d^* = +\infty)\) ⇒ primal infeasible \((p^* = +\infty)\)

Optimality condition
If \(x, y\) satisfy:
- \(x\) is a feasible solution to the primal problem
- \(y\) is a feasible solution to the dual problem
- The duality gap is zero, i.e., \(c^T x + b^T y = 0\)

Then \(x\) and \(y\) are optimal solutions to the primal and dual problem respectively
Strong duality

**Theorem**
If a linear optimization problem has an optimal solution, so does its dual, and the optimal value of primal and dual are equal

\[ d^* = p^* \]
Strong duality

Constructive proof

Given a primal optimal solution $x^*$ we will construct a dual optimal solution $y^*$
Strong duality
Constructive proof

Given a primal optimal solution $x^*$ we will construct a dual optimal solution $y^*$

Apply simplex to problem in **standard form**

- minimize $c^T x$
- subject to $Ax = b$
- $x \geq 0$
Strong duality

Constructive proof

Given a primal optimal solution $x^*$ we will construct a dual optimal solution $y^*$

Apply simplex to problem in **standard form**

minimize \[ c^T x \]

subject to \[ Ax = b \]

\[ x \geq 0 \]

- optimal basis $B$
- optimal solution $x^*$ with $Bx^*_B = b$
- reduced costs $\bar{c} = c - A^T B^{-T} c_B \geq 0$

\[ \bar{c}_J = c_J - c_B^T B^{-1} A_J \]
Strong duality
Constructive proof
Given a primal optimal solution $x^*$ we will construct a dual optimal solution $y^*$

Apply simplex to problem in **standard form**

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\]

- optimal basis $B$
- optimal solution $x^*$ with $B x_B^* = b$
- reduced costs $\bar{c} = c - A^T B^{-T} c_B \geq 0$

Define $y^*$ such that $y^* = -B^{-T} c_B$. Therefore, $A^T y^* + c \geq 0$ ($y^* \text{ dual feasible}$).
Strong duality

Constructive proof

Given a primal optimal solution $x^*$ we will construct a dual optimal solution $y^*$

Apply simplex to problem in **standard form**

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\begin{align*}
\text{minimize} & \quad c^T x \\
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- optimal basis $B$
- optimal solution $x^*$ with $B x^*_B = b$
- reduced costs $\bar{c} = c - A^T B^{-T} c_B \geq 0$

Define $y^*$ such that $y^* = -B^{-T} c_B$. Therefore, $A^T y^* + c \geq 0$ ($y^*$ dual feasible).

\[
-b^T y^* = -b^T (-B^{-T} c_B) = c_B^T (B^{-1} b) = c_B^T x^*_B = c^T x^*
\]
Strong duality
Constructive proof

Given a primal optimal solution \( x^* \) we will construct a dual optimal solution \( y^* \).

Apply simplex to problem in **standard form**

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\]

- optimal basis \( B \)
- optimal solution \( x^* \) with \( Bx^*_B = b \)
- reduced costs \( \bar{c} = c - A^T B^{-T} c_B \geq 0 \)

Define \( y^* \) such that \( y^* = -B^{-T} c_B \). Therefore, \( A^T y^* + c \geq 0 \) (\( y^* \) dual feasible).

\[
-b^T y^* = -b^T (-B^{-T} c_B) = c_B^T (B^{-1} b) = c_B^T x^*_B = c^T x^*
\]

By weak duality theorem corollary, \( y^* \) is an optimal solution of the dual. Therefore, \( d^* = p^* \).
Exception to strong duality

Primal

minimize \( x \)
subject to \( 0 \cdot x \leq -1 \)

Optimal value is \( p^* = +\infty \)

Dual

maximize \( y \)
subject to \( 0 \cdot y + 1 = 0 \)
\( y \geq 0 \)

Optimal value is \( d^* = -\infty \)
Exception to strong duality

Primal

minimize \( x \)
subject to \( 0 \cdot x \leq -1 \)

Optimal value is \( p^* = +\infty \)

Dual

maximize \( y \)
subject to \( 0 \cdot y + 1 = 0 \)
\( y \geq 0 \)

Optimal value is \( d^* = -\infty \)

Both primal and dual infeasible
## Relationship between primal and dual

<table>
<thead>
<tr>
<th></th>
<th>$p^* = +\infty$</th>
<th>$p^*$ finite</th>
<th>$p^* = -\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d^* = +\infty$</td>
<td>primal inf.</td>
<td></td>
<td></td>
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<tr>
<td></td>
<td>dual unb.</td>
<td></td>
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</tr>
<tr>
<td>$d^*$ finite</td>
<td></td>
<td>optimal values equal</td>
<td></td>
</tr>
<tr>
<td>$d^* = -\infty$</td>
<td>exception</td>
<td></td>
<td>primal unbounded dual inf</td>
</tr>
</tbody>
</table>

- Upper-right excluded by **weak duality**
- (1, 1) and (3, 3) proven by **weak duality**
- (3, 1) and (2, 2) proven by **strong duality**
Example
Production problem

maximize \( x_1 + 2x_2 \)

subject to \( x_1 \leq 100 \)
\( 2x_2 \leq 200 \)
\( x_1 + x_2 \leq 150 \)
\( x_1, x_2 \geq 0 \)
Production problem

maximize \[ x_1 + 2x_2 \] ← Profits

subject to
\[ x_1 \leq 100 \]
\[ 2x_2 \leq 200 \]
\[ x_1 + x_2 \leq 150 \]
\[ x_1, x_2 \geq 0 \]
Production problem

maximize \( x_1 + 2x_2 \) \hspace{1cm} \text{Profits}

subject to

\begin{align*}
  x_1 & \leq 100 \\
  2x_2 & \leq 200 \\
  x_1 + x_2 & \leq 150 \\
  x_1, x_2 & \geq 0
\end{align*} \hspace{1cm} \text{Resources}
Production problem

maximize \( x_1 + 2x_2 \) \hspace{1cm} \text{Profits}
subject to
\[
\begin{align*}
  x_1 & \leq 100 \\
  2x_2 & \leq 200 \\
  x_1 + x_2 & \leq 150 \\
  x_1, x_2 & \geq 0
\end{align*}
\]

\[
\begin{align*}
  c &= (-1, -2) \\
  A &= \begin{bmatrix}
    1 & 0 \\
    0 & 2 \\
    1 & 1 \\
    -1 & 0 \\
    0 & -1
  \end{bmatrix} \\
  b &= (100, 200, 150, 0, 0)
\end{align*}
\]

Dualize

1. Transform in inequality form

minimize \( c^T x \) \hspace{1cm} \text{Profits}
subject to
\[
Ax \leq b
\]
Production problem

maximize \( x_1 + 2x_2 \) \(--\) Profits
subject to
\[
\begin{align*}
  x_1 & \leq 100 \\
  2x_2 & \leq 200 \\
  x_1 + x_2 & \leq 150 \\
  x_1, x_2 & \geq 0
\end{align*}
\]

\[ c = (-1, -2) \]
\[ A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \]
\[ b = (100, 200, 150, 0, 0) \]

Dualize

1. Transform in inequality form

2. Derive dual

\[
\begin{align*}
  \text{minimize} & \quad c^T x \\
  \text{subject to} & \quad Ax \leq b \\
  \text{maximize} & \quad -b^T y \\
  \text{subject to} & \quad A^T y + c = 0 \\
  & \quad y \geq 0
\end{align*}
\]
Production problem

The dual

minimize \[ 100y_1 + 200y_2 + 150y_3 \]
subject to 
\[ y_1 + y_3 \geq 1 \]
\[ 2y_2 + y_3 \geq 2 \]
\[ y_1, y_2, y_3 \geq 0 \]
Production problem

The dual

minimize \[ 100y_1 + 200y_2 + 150y_3 \]

subject to
\[ y_1 + y_3 \geq 1 \]
\[ 2y_2 + y_3 \geq 2 \]
\[ y_1, y_2, y_3 \geq 0 \]

Interpretation

• **Sell your resources** at a fair (minimum) price
• Selling must be **more convenient than producing**:
  - Product 1 (price 1, needs \(1 \times\) resource 1 and 2): \( y_1 + y_3 \geq 1 \)
  - Product 2 (price 2, needs \(2 \times\) resource 2 and \(1 \times\) resource 3): \( 2y_2 + y_3 \geq 2 \)
Linear optimization duality

Today, we learned to:

• **Dualize** linear optimization problems
• **Prove** weak and strong duality conditions
• **Interpret** simple dual optimization problems
Next lecture

More on duality:

• Game theoretic interpretation
• Complementary slackness
• Alternative systems