ORF307 – Optimization

19. Linear optimization review

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Today’s lecture

Linear optimization review

- Formulations
- Piecewise linear optimization
- Duality
- Sensitivity analysis
- Simplex method
- Interior point methods

Formulations

Linear optimization

$a^T x \geq b \Rightarrow -a^T x \leq -b$

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \\ & Dx = f \end{array}$$

- Minimization
- Less-than ineq. constraints
- Equality constraints

x is **feasible** if it satisfies the constraints $Ax \leq b$ and $Dx = f$

The **feasible set** is the set of all feasible points

$x^\star$ is **optimal** if it is feasible and $c^T x^\star \leq c^T x$ for all feasible x

The **optimal value** is $p^\star = c^T x^\star$

Unbounded problem: $c^T x$ is unbounded below on the feasible set ($p^\star = -\infty$)

Infeasible problem: feasible set is empty ($p^\star = +\infty$)

Feasibility problems

$$\begin{array}{ll} \text{find} & x \\ \text{subject to} & Ax \leq b \\ & Dx = f \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & 0 \\ \text{subject to} & Ax \leq b \\ & Dx = f \end{array}$$

Possible results

- $p^\star = 0$ if constraints are feasible (consistent). (Every feasible x is optimal)
- $p^\star = \infty$ otherwise

Standard form

Definition

$$Ax \le b \rightarrow Ax + s = b$$
$$s \ge 0$$

$$x_i = x_i^+ - x_i^-$$
$$x_i^+ \ge 0$$
$$x_i^- \ge 0$$

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \ge 0 \end{array}$$

- Minimization
- Equality constraints
- Nonnegative variables

Useful to

- develop **algorithms**
- **algebraic** manipulations

Piecewise linear optimization

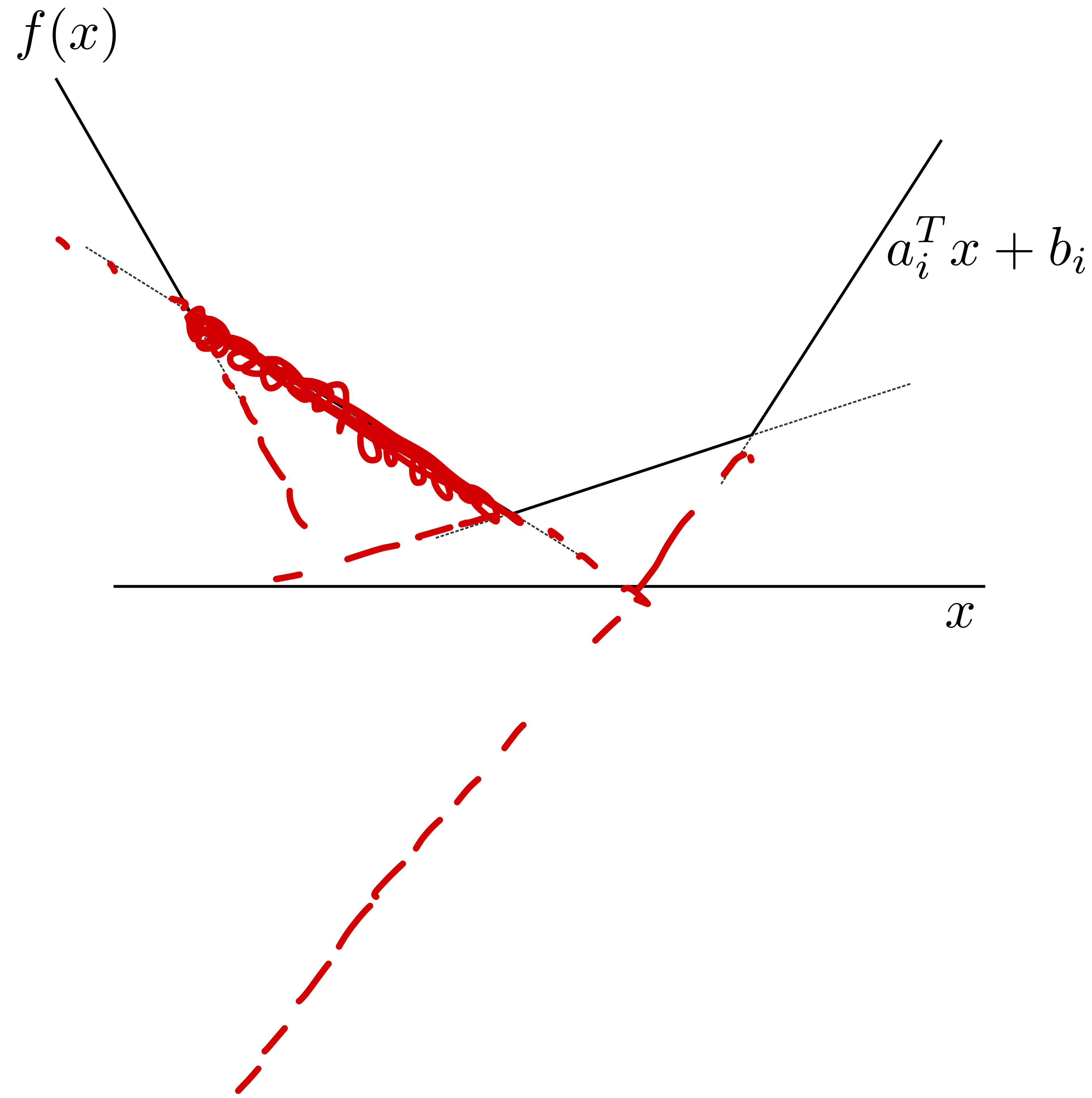
Piecewise-linear minimization

minimize $\quad f(x) = \max_{i=1,\ldots,m} (a_i^T x + b_i)$

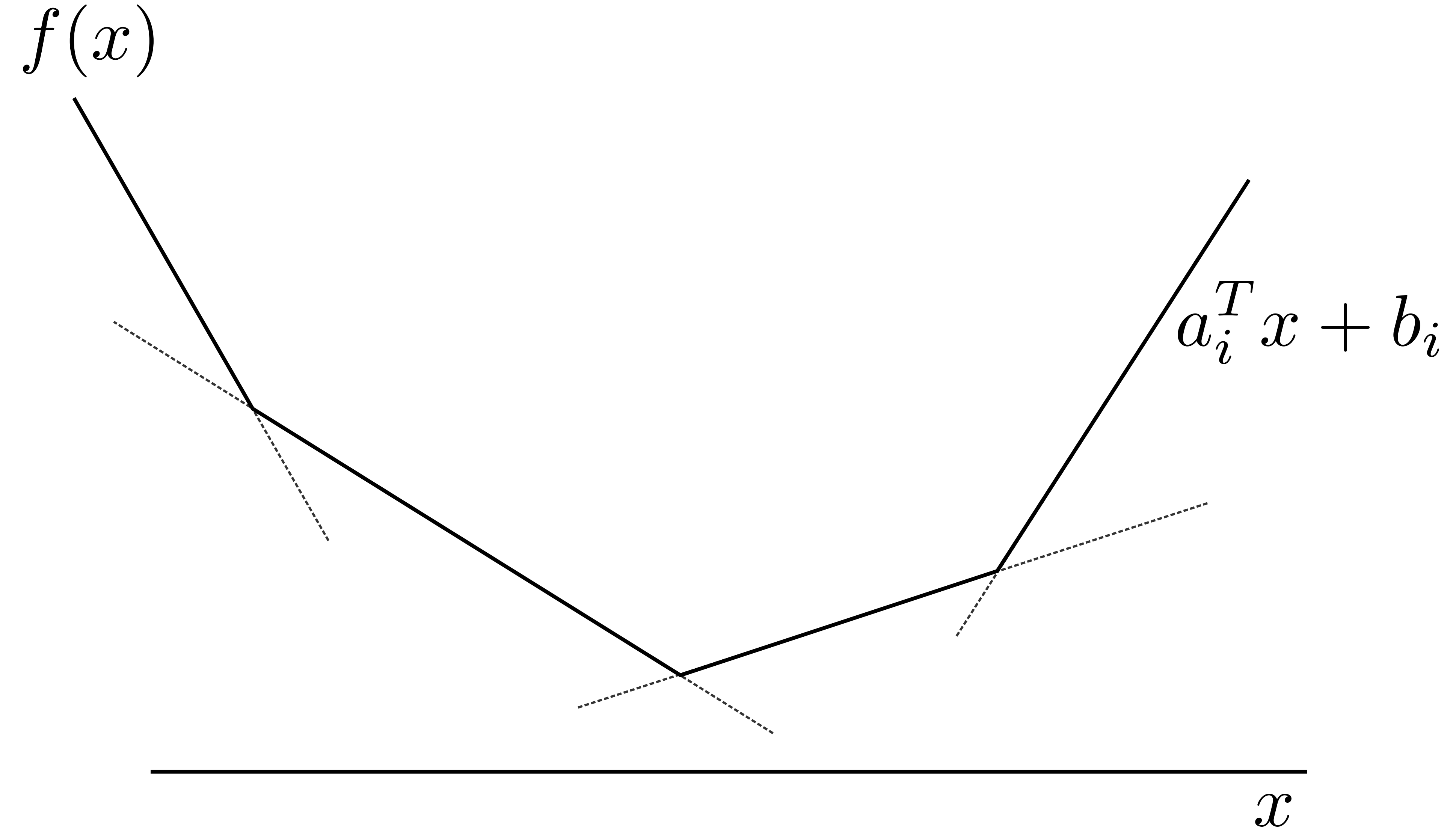
$\downarrow$

minimize $\quad t$

subject to $\quad a_i^T x + b_i \leq t, \quad i = 1, \ldots, m$



Piecewise-linear minimization



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$\downarrow$

minimize $\quad t$

subject to $\quad a_i^T x + b_i \leq t, \quad i = 1, \ldots, m$

$a_i^T x - t \leq -b_i$

$\begin{bmatrix} a_i^T & -1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix}$

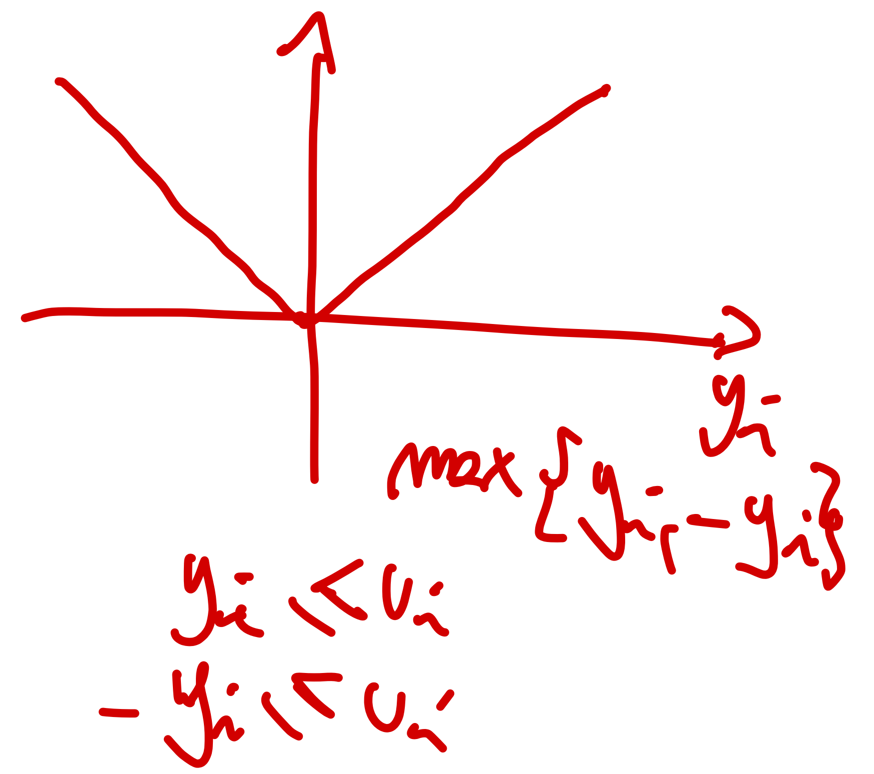
Matrix notation

minimize $\quad \tilde{c}^T \tilde{x}$

subject to $\quad \tilde{A}\tilde{x} \leq \tilde{b}$

$$\tilde{x} = \begin{bmatrix} x \\ t \end{bmatrix}, \quad \tilde{c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} a_1^T & -1 \\ \vdots & \vdots \\ a_m^T & -1 \end{bmatrix}, \quad \tilde{b} = \begin{bmatrix} -b_1 \\ \vdots \\ -b_m \end{bmatrix}$$

1 and infinity norms reformulations



1-norm minimization:

minimize $\|Ax - b\|_1 = \sum_i |(Ax - b)_i|$

Equivalent to:

minimize $\mathbf{1}^T u$

subject to $-u \leq Ax - b \leq u$

Absolute value of every element $(Ax - b)_i$ is bounded by a component of the **vector** u

∞-norm minimization:

minimize $\|Ax - b\|_\infty = \max_i |(Ax - b_i)_i|$

Equivalent to:

minimize t

subject to $-t\mathbf{1} \leq Ax - b \leq t\mathbf{1}$

Absolute value of every element $(Ax - b)_i$ is bounded by the same **scalar** t

Duality

Lagrangian and duality

Primal

minimize $c^T x$

subject to $Ax \leq b$ (y)

Dual

maximize $-b^T y$

subject to $A^T y + c = 0$

$y \geq 0$

Lagrangian and duality

Primal

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \le b \end{array}$$

Dual

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c = 0 \\ & y \ge 0 \end{array}$$

Dual function

$$\begin{aligned} g(y) &= \underset{x}{\text{minimize}} \left(c^T x + y^T (Ax - b)\right) \\ &= -b^T y + \underset{x}{\text{minimize}} \left(c + A^T y\right)^T x \\ &= \begin{cases} -b^T y & \text{if } c + A^T y = 0 \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

Lagrangian and duality

Primal

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

Dual

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Lagrangian

$$L(x, y) = c^T x + y^T (Ax - b)$$

Lagrangian and duality

Primal

minimize $c^T x$

subject to $Ax \leq b$

Dual

maximize $-b^T y$

subject to $A^T y + c = 0$

$y \geq 0$

Dual function

$$g(y) = \underset{x}{\text{minimize}} \left(c^T x + y^T (Ax - b)\right)$$

$$= -b^T y + \underset{x}{\text{minimize}} \left(c + A^T y\right)^T x$$

$$= \begin{cases} -b^T y & \text{if } c + A^T y = 0 \\ -\infty & \text{otherwise} \end{cases}$$

→

Lagrangian

$$L(x, y) = c^T x + y^T (Ax - b)$$

$$\nabla_x L(x, y) = c + A^T y = 0$$

Karush-Kuhn-Tucker conditions

Optimality conditions for linear optimization

Primal		**Dual**	
minimize	$c^T x$	maximize	$-b^T y$
subject to	$Ax \leq b$	subject to	$A^T y + c = 0$
			$y \geq 0$

Primal feasibility $\quad Ax \leq b$

Dual feasibility $\quad \nabla_x L(x, y) = A^T y + c = 0$ and $y \geq 0$

Complementary slackness $\quad y_i(Ax - b)_i = 0, \quad i = 1, \ldots, m$

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \le b \\ & Cx = d \\ & x \le g \\ & Dx \ge f \end{aligned} \quad \longrightarrow \quad \begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \le b && (y) \\ & Cx = d && (z) \\ & x \le g && (w) \\ & -Dx \le -f && (s) \end{aligned}$$

$$L(x, y, z, w, s) = c^T x + y^T(Ax - b) + z^T(Cx - d) + w^T(x - g) + s^T(-Dx + f)$$

$$L(\bar{x}, y, z, w, s) = c^T \bar{x} + \underbrace{y^T}_{\ge 0}\underbrace{(A\bar{x} - b)}_{\le 0} + z^T\underbrace{(C\bar{x} - d)}_{= 0} + \underbrace{w^T}_{\ge 0}\underbrace{(\bar{x} - g)}_{\le 0} + \underbrace{s^T}_{\ge 0}\underbrace{(-D\bar{x} + f)}_{\le 0} \le c^T \bar{x}$$

$\bar{x}$ ↑ FEASIBLE PRIMAL POINT; z ↑ FREE

$$L(x, y, z, w, s) = (c + A^T y + C^T z + w - D^T s)^T x - b^T y - d^T z - g^T w + f^T s$$

$$g(y, z, w, s) = \min_x L(x, y, z, w, s) \le c^T x \quad \text{if } x \text{ FEASIBLE for PRIMAL}$$

$$\nabla_x L(x, y, z, w, s) = c + A^T y + C^T z + w - D^T s = 0$$

$$\begin{aligned} \max \quad & -b^T y - d^T z - g^T w + f^T s \\ \text{s.t.} \quad & c + A^T y + C^T z + w - D^T s = 0 \\ & y \ge 0, \; w \ge 0, \; s \ge 0 \end{aligned}$$

$$\min \; c^T x \quad \text{st.} \quad Ax = b, \; x \geq 0 \quad \rightarrow \quad \min \; c^T x \quad \text{st.} \quad Ax = b \;\; (y), \quad -x \leq 0 \;\; (w)$$

$$L(x,y,w) = c^T x + y^T(Ax - b) + w^T(-x) = (c + A^T y - w)^T x - b^T y$$

$$\nabla_x L(x,y,w) = c + A^T y - w = 0$$

$$\max \; -b^T y \quad \text{st.} \quad c + A^T y - w = 0, \quad w \geq 0$$

$$c + A^T y = w \geq 0$$

$$\max \; -b^T y \quad \text{st.} \quad c + A^T y \geq 0$$

$$f(x) \quad \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$Df(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_m}{\partial x_1} & \cdots & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

$$m = 1$$

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \end{bmatrix}$$

$$\nabla f = (Df)^T$$

General forms

Inequality form LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \le b \end{array}$$

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c = 0 \\ & y \ge 0 \end{array}$$

Standard form LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \ge 0 \end{array}$$

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c \ge 0 \end{array}$$

LP with inequalities and equalities

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \le b \\ & Dx = f \end{array}$$

$$\begin{array}{ll} \text{maximize} & -b^T y - f^T z \\ \text{subject to} & A^T y + D^T z + c = 0 \\ & y \ge 0 \end{array}$$

Weak duality

Theorem

If x, y satisfy:

- x is a feasible solution to the primal problem
- y is a feasible solution to the dual problem

$\longrightarrow$ $-b^T y \leq c^T x$

Weak duality

Theorem

If x, y satisfy:

- x is a feasible solution to the primal problem
- y is a feasible solution to the dual problem

$\longrightarrow$ $-b^T y \leq c^T x$

Proof

We know that $Ax \leq b$, $A^T y + c = 0$ and $y \geq 0$. Therefore,

$$0 \leq y^T(b - Ax) = b^T y - y^T Ax = c^T x + b^T y \qquad \blacksquare$$

$y^T(Ax - b) \leq 0$

Weak duality

Theorem

If x, y satisfy:

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We know that $Ax \leq b$, $A^T y + c = 0$ and $y \geq 0$. Therefore,

$$0 \leq y^T (b - Ax) = b^T y - y^T Ax = c^T x + b^T y \qquad \blacksquare$$

Remark

- Any dual feasible y gives a **lower bound** on the primal optimal value
- Any primal feasible x gives an **upper bound** on the dual optimal value
- $c^T x + b^T y$ is the **duality gap**

Weak duality

Corollaries

Unboundedness vs feasibility

- Primal unbounded ($p^\star = -\infty$) $\Rightarrow$ dual infeasible ($d^\star = -\infty$)
- Dual unbounded ($d^\star = +\infty$) $\Rightarrow$ primal infeasible ($p^\star = +\infty$)

Weak duality

Corollaries

Unboundedness vs feasibility

- Primal unbounded ($p^\star = -\infty$) $\Rightarrow$ dual infeasible ($d^\star = -\infty$)
- Dual unbounded ($d^\star = +\infty$) $\Rightarrow$ primal infeasible ($p^\star = +\infty$)

Optimality condition

If x, y satisfy:

- x is a feasible solution to the primal problem
- y is a feasible solution to the dual problem
- The duality gap is zero, *i.e.*, $c^T x + b^T y = 0$

Then x and y are **optimal solutions** to the primal and dual problem respectively

Strong duality

Primal

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Dual

$$\begin{aligned} \text{maximize} \quad & -b^T y \\ \text{subject to} \quad & A^T y + c \geq 0 \end{aligned}$$

Theorem

If a linear optimization problem has an optimal solution, then

- so does its dual
- the optimal values of the primal and dual are equal

Relationship between primal and dual

	$p^\star = +\infty$	$p^\star$ finite	$p^\star = -\infty$
$d^\star = +\infty$	primal inf. dual unb.		
$d^\star$ finite		optimal values equal	
$d^\star = -\infty$	exception		primal unb. dual inf

Complementary slackness

Primal

minimize $c^T x$

subject to $Ax \leq b$

Dual

maximize $-b^T y$

subject to $A^T y + c = 0$

$y \geq 0$

Theorem

Primal,dual feasible x, y are optimal if and only if

$$y_i(b_i - a_i^T x) = 0, \quad i = 1, \dots, m$$

i.e., at optimum, $b - Ax$ and y have a **complementary sparsity** pattern:

$$y_i > 0 \quad \Rightarrow \quad a_i^T x = b_i$$
$$a_i^T x < b_i \quad \Rightarrow \quad y_i = 0$$

Complementary slackness

Primal

minimize $c^T x$

subject to $Ax \leq b$

Dual

maximize $-b^T y$

subject to $A^T y + c = 0$ → $c = -A^T y$

$y \geq 0$

Proof

The duality gap at primal feasible x and dual feasible y can be written as

$$c^T x + b^T y = (-A^T y)^T x + b^T y = (b - Ax)^T y = \sum_{i=1}^{m} y_i (b_i - a_i^T x) = 0$$

Complementary slackness

Primal

minimize $c^T x$
subject to $Ax \leq b$

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Since all the elements of the sum are nonnegative, they must all be 0 ∎

Complementary slackness

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Since all the elements of the sum are nonnegative, they must all be 0 ■

For **feasible** x and y **complementary slackness** = **zero duality gap**

Example

$$\text{minimize} \quad -4x_1 - 5x_2$$

$$\text{subject to} \quad \begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 3 \\ 0 \\ 3 \end{bmatrix}$$

A   b

Let's **show** that feasible $x = (1, 1)$ is optimal

$c = (-4, -5)$

$$\begin{aligned} \max \quad & -b^T y \\ \text{s.t.} \quad & A^T y + c = 0 \\ & y \geq 0 \end{aligned}$$

$-1 \leq 0$

$3 \leq 3$ ACTIVE

$-1 \leq 0$

$3 \leq 3$ ACTIVE

Example

max $-b^T y$
s.t. $A^T y + c = 0$
$y \geq 0$

$$\begin{array}{ll} \text{minimize} & -4x_1 - 5x_2 \\ \text{subject to} & \begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 3 \\ 0 \\ 3 \end{bmatrix} \end{array}$$

Let's **show** that feasible $x = (1, 1)$ is optimal

Second and fourth constraints are active at x $\longrightarrow$ $y = (0, y_2, 0, y_4)$

$$A^T y = -c \quad \Rightarrow \quad \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_2 \\ y_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \qquad \text{and} \qquad y_2 \geq 0, \quad y_4 \geq 0$$

Example

minimize $\quad -4x_1 - 5x_2$

subject to
$$\begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 3 \\ 0 \\ 3 \end{bmatrix}$$

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and

$$y_2 \geq 0, \quad y_4 \geq 0$$

$y = (0, 1, 0, 2)$ satisfies these conditions and proves that x is optimal

Example

minimize $\quad -4x_1 - 5x_2$

subject to $\quad \begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 3 \\ 0 \\ 3 \end{bmatrix}$

$y_i(b_i - a_i^T x) = 0 \quad \forall i$

max $-b^T y$
s.t. $A^T y + c = 0$
$y \geq 0$

Let's **show** that feasible $x = (1, 1)$ is optimal

Second and fourth constraints are active at $x \longrightarrow y = (0, y_2, 0, y_4)$

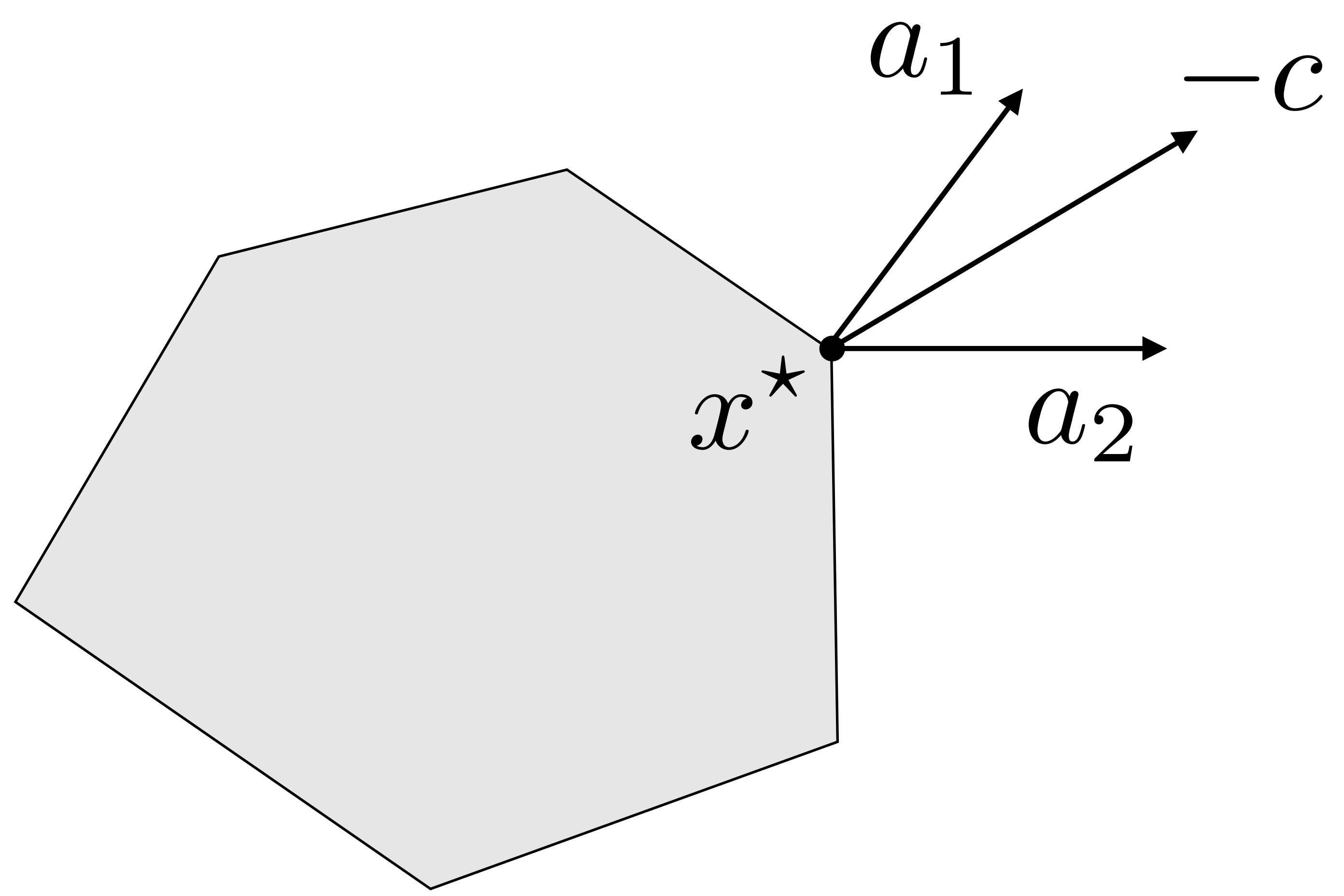
$$A^T y = -c \quad \Rightarrow \quad \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_2 \\ y_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \quad \text{and} \quad y_2 \geq 0, \quad y_4 \geq 0$$

$y = (0, 1, 0, 2)$ satisfies these conditions and proves that x is optimal

Complementary slackness is useful to recover $y^\star$ from $x^\star$

Geometric interpretation

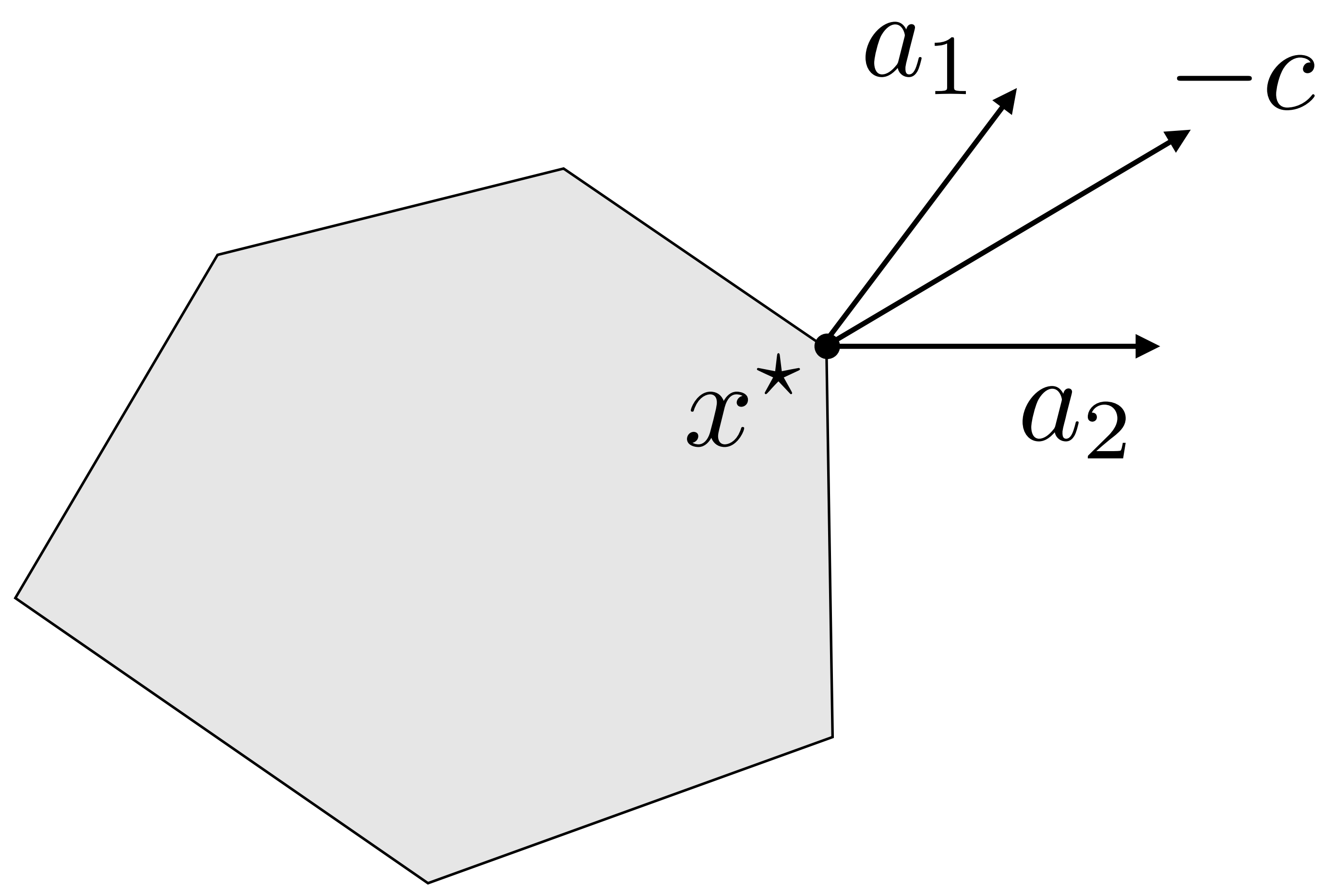
Example in $\mathbf{R}^2$



Two active constraints at optimum: $a_1^T x^\star = b_1, \quad a_2^T x^\star = b_2$

Geometric interpretation

Example in $\mathbf{R}^2$



Two active constraints at optimum: $a_1^T x^\star = b_1, \quad a_2^T x^\star = b_2$

Optimal dual solution y satisfies:

$$A^T y + c = 0, \quad y \geq 0, \quad y_i = 0 \text{ for } i \neq \{1, 2\}$$

In other words, $-c = a_1 y_1 + a_2 y_2$ with $y_1, y_2 \geq 0$

Sensitivity analysis

Changes in problem data

Goal: extract information from $x^\star, y^\star$ about their sensitivity with respect to changes in problem data

Modified LP

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax = b + u \\
& x \geq 0
\end{array}
$$

Optimal value function

$$p^\star(u) = \min\{c^T x \mid Ax = b + u,\ x \geq 0\}$$

Changes in problem data

Goal: extract information from $x^\star, y^\star$ about their sensitivity with respect to changes in problem data

Modified LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b + u \\ & x \geq 0 \end{array}$$

Optimal value function

$$p^\star(u) = \min\{c^T x \mid Ax = b + u,\ x \geq 0\}$$

Assumption: $p^\star(0)$ is finite

Properties

- $p^\star(u) > -\infty$ everywhere (from global lower bound)
- $p^\star(u)$ is piecewise-linear on its domain

Global sensitivity

Dual of modified LP

$$\begin{array}{ll} \text{maximize} & -(b+u)^T y \\ \text{subject to} & A^T y + c \geq 0 \end{array}$$

Global sensitivity

Dual of modified LP

$$\begin{array}{ll} \text{maximize} & -(b+u)^T y \\ \text{subject to} & A^T y + c \geq 0 \end{array}$$

Global lower bound

Given $y^\star$ a dual optimal solution for $u = 0$, then

$$\begin{aligned} p^\star(u) &\geq -(b+u)^T y^\star \\ &= p^\star(0) - u^T y^\star \end{aligned}$$

(from weak duality and dual feasibility)

Global sensitivity

Dual of modified LP

$$\begin{array}{ll} \text{maximize} & -(b+u)^T y \\ \text{subject to} & A^T y + c \geq 0 \end{array}$$

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Given $y^\star$ a dual optimal solution for $u = 0$, then

$$\begin{aligned} p^\star(u) &\geq -(b+u)^T y^\star \\ &= p^\star(0) - u^T y^\star \end{aligned}$$

(from weak duality and dual feasibility)

It holds for any u

Local sensitivity

u in neighborhood of the origin

Original LP

minimize $c^T x$

subject to $Ax = b$

$x \geq 0$

$\longrightarrow$

Optimal solution

Primal
$$x_i = 0, \quad i \notin B$$
$$x_B^\star = A_B^{-1} b$$

Dual
$$y^\star = -A_B^{-T} c_B$$

Local sensitivity

u in neighborhood of the origin

Original LP

minimize $c^T x$

subject to $Ax = b$

$x \geq 0$

→

Optimal solution

Primal $x_i = 0, \quad i \notin B$

$x_B^\star = A_B^{-1} b$

Dual $y^\star = -A_B^{-T} c_B$

Modified LP

minimize $c^T x$

subject to $Ax = b + u$

$x \geq 0$

Modified dual

maximize $-(b + u)^T y$

subject to $A^T y + c \geq 0$

Optimal basis does not change

Local sensitivity

u in neighborhood of the origin

Original LP

minimize $c^T x$

subject to $Ax = b$

$x \geq 0$

$\longrightarrow$

Optimal solution

Primal $x_i = 0, \quad i \notin B$

$x_B^\star = A_B^{-1} b$

Dual $y^\star = -A_B^{-T} c_B$

Modified LP

minimize $c^T x$

subject to $Ax = b + u$

$x \geq 0$

Modified dual

maximize $-(b+u)^T y$

subject to $A^T y + c \geq 0$

Optimal basis does not change

Modified optimal solution

$$x_B^\star(u) = A_B^{-1}(b+u) = x_B^\star + A_B^{-1} u$$

$$y^\star(u) = y^\star$$

Derivative of the optimal value function

Modified optimal solution

$$x_B^\star(u) = A_B^{-1}(b+u) = x_B^\star + A_B^{-1}u$$
$$y^\star(u) = y^\star$$

Derivative of the optimal value function

Modified optimal solution

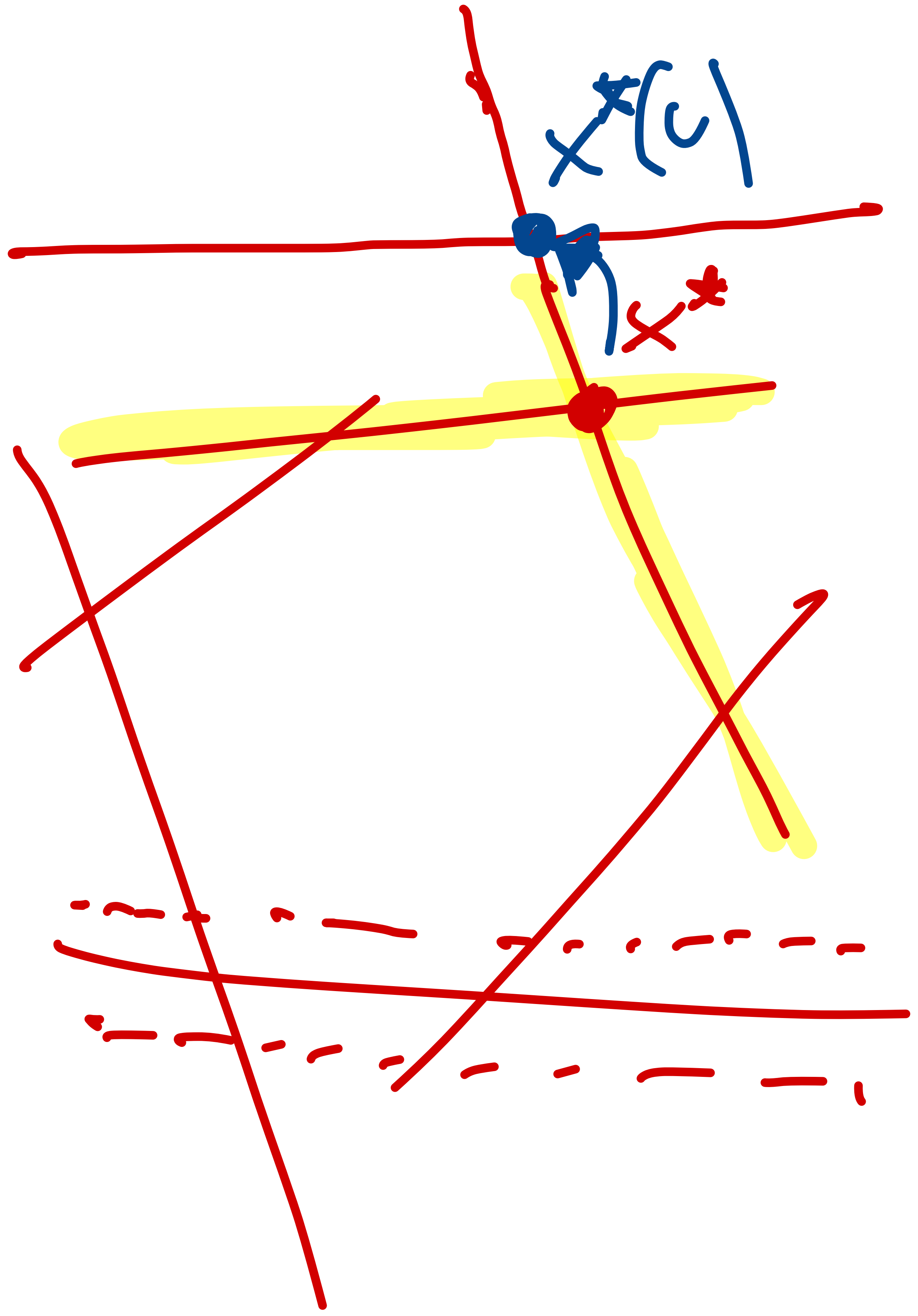
$$x_B^\star(u) = A_B^{-1}(b+u) = x_B^\star + A_B^{-1}u$$

$$y^\star(u) = y^\star$$

Optimal value function

$$\begin{aligned} p^\star(u) &= c^T x^\star(u) \\ &= c^T x^\star + c_B^T A_B^{-1} u \\ &= p^\star(0) - y^{\star T} u \qquad \text{(affine for small } u\text{)} \end{aligned}$$

Derivative of the optimal value function



Modified optimal solution

$$x_B^\star(u) = A_B^{-1}(b+u) = x_B^\star + A_B^{-1}u$$

$$y^\star(u) = y^\star$$

Optimal value function

$$\begin{aligned} p^\star(u) &= c^T x^\star(u) \\ &= c^T x^\star + c_B^T A_B^{-1} u \\ &= p^\star(0) - y^{\star T} u \end{aligned}$$

(affine for small u)

Local derivative

$$\nabla p^\star(u) = -y^\star$$

($y^\star$ are the **shadow prices**)

Network flow optimization

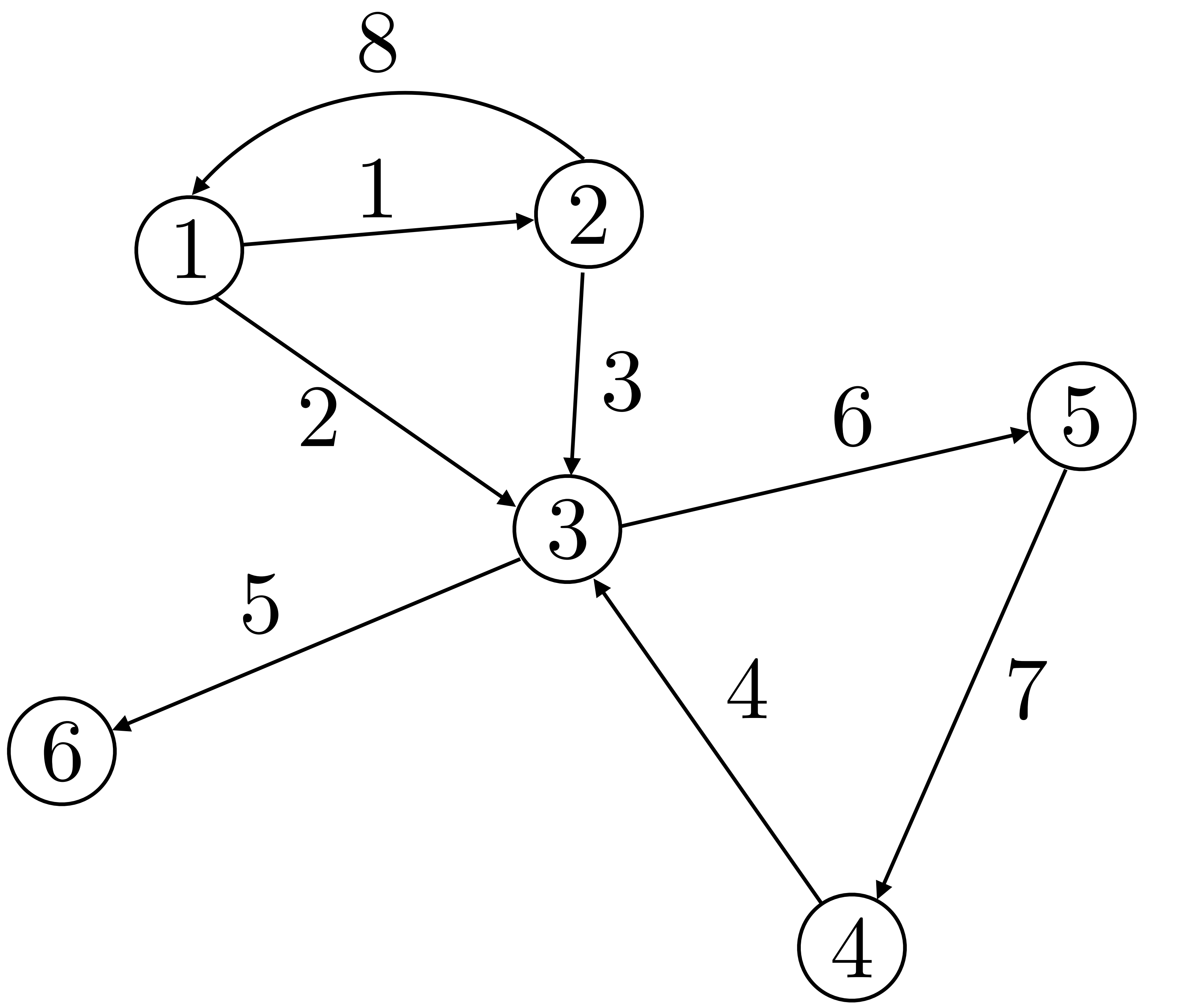
Arc-node incidence matrix

$m \times n$ matrix A with entries

$$A_{ij} = \begin{cases} 1 & \text{if arc } j \text{ starts at node } i \\ -1 & \text{if arc } j \text{ ends at node } i \\ 0 & \text{otherwise} \end{cases}$$

Note Each column has one -1 and one 1

Arc-node incidence matrix



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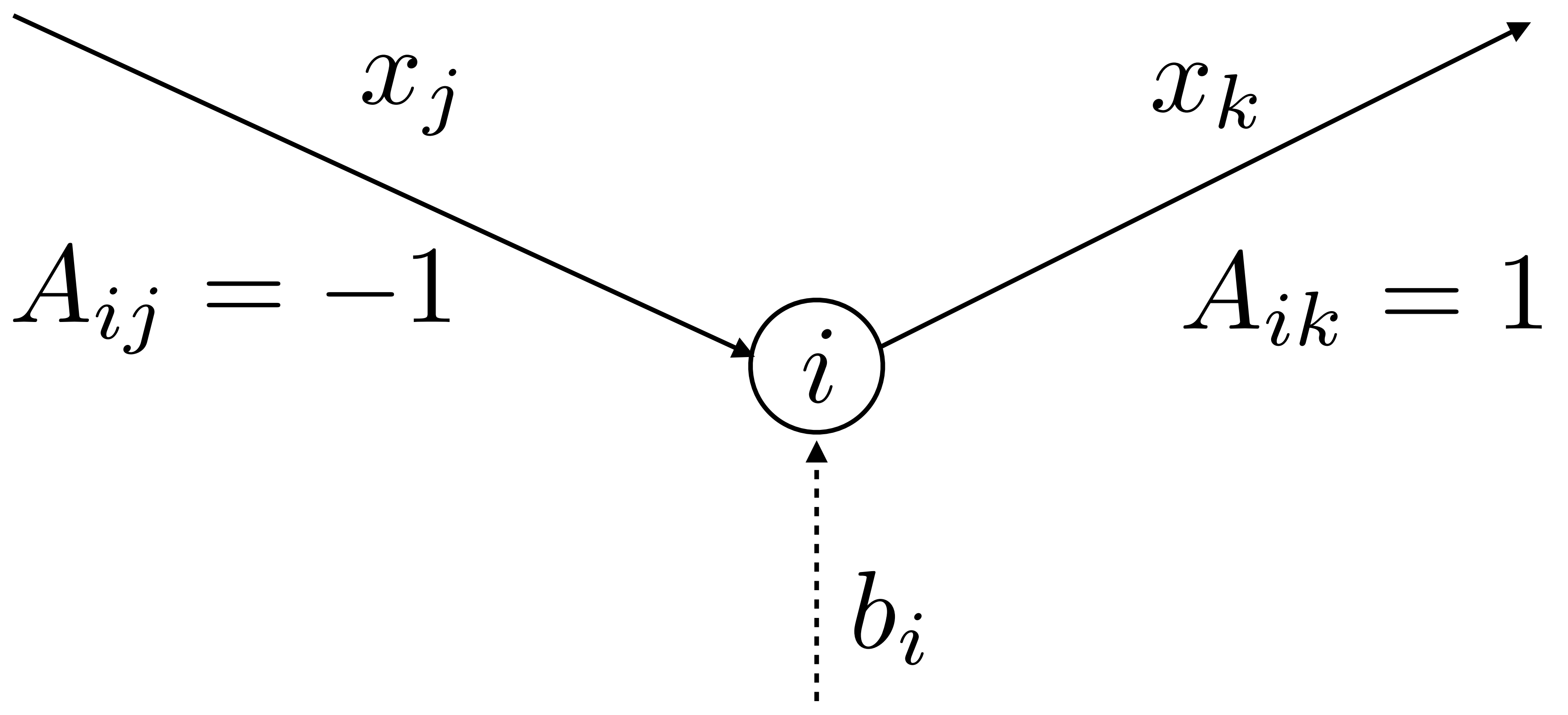
$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}$$

External supply

supply vector $b \in \mathbf{R}^m$

- b_i is the external supply at node i (if $b_i < 0$, it represents demand)
- We must have $\mathbf{1}^T b = 0$ (total supply = total demand)



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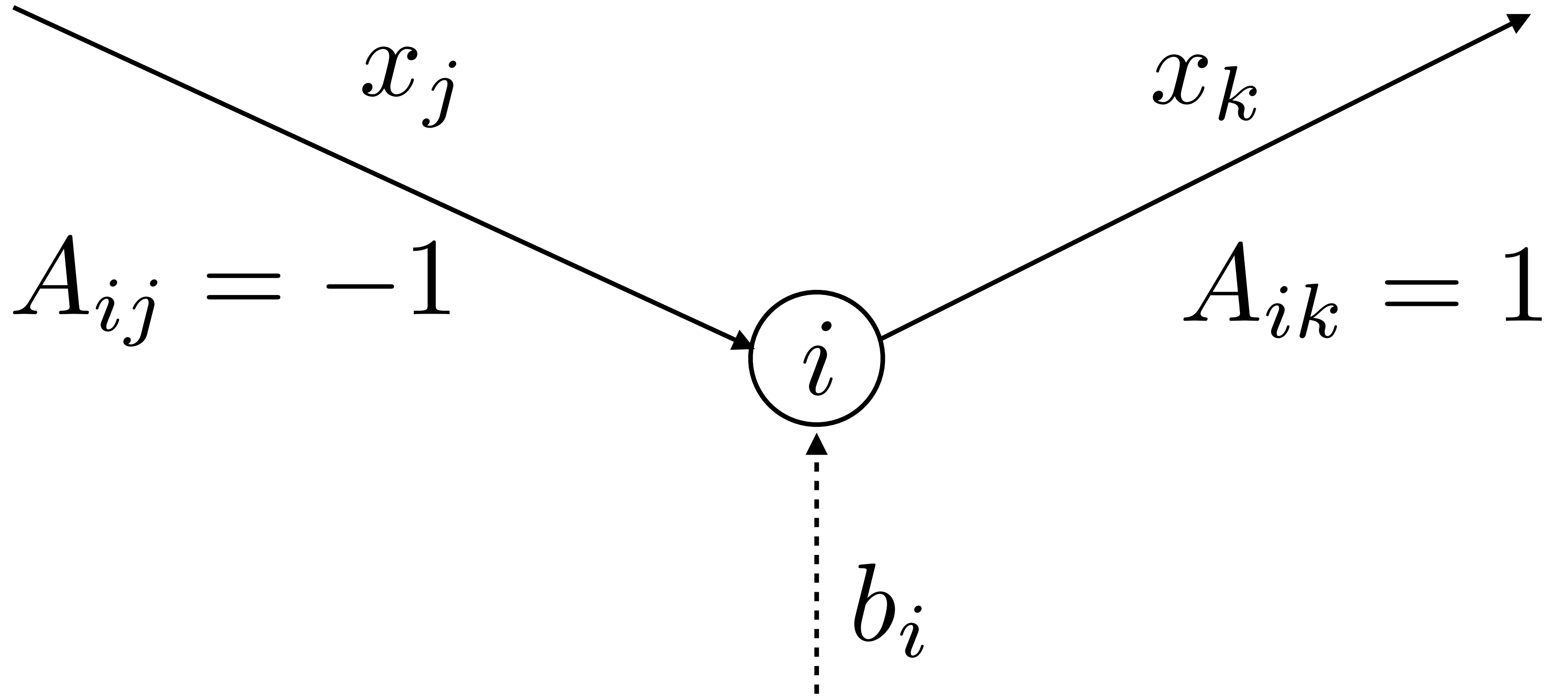

Balance equations

$$\sum_{j=1}^{n} A_{ij} x_j = (Ax)_i = b_i, \quad \text{for all } i$$

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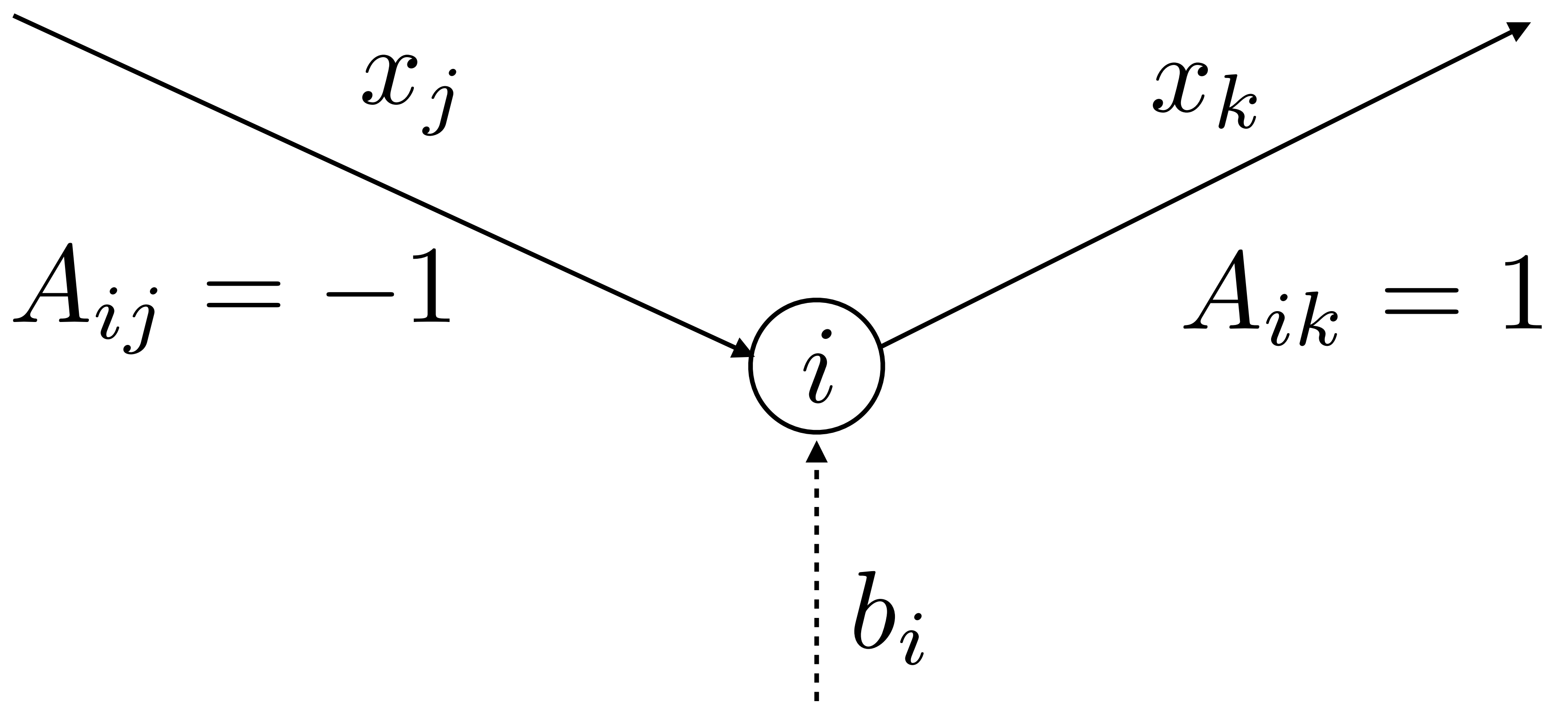

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Total leaving flow

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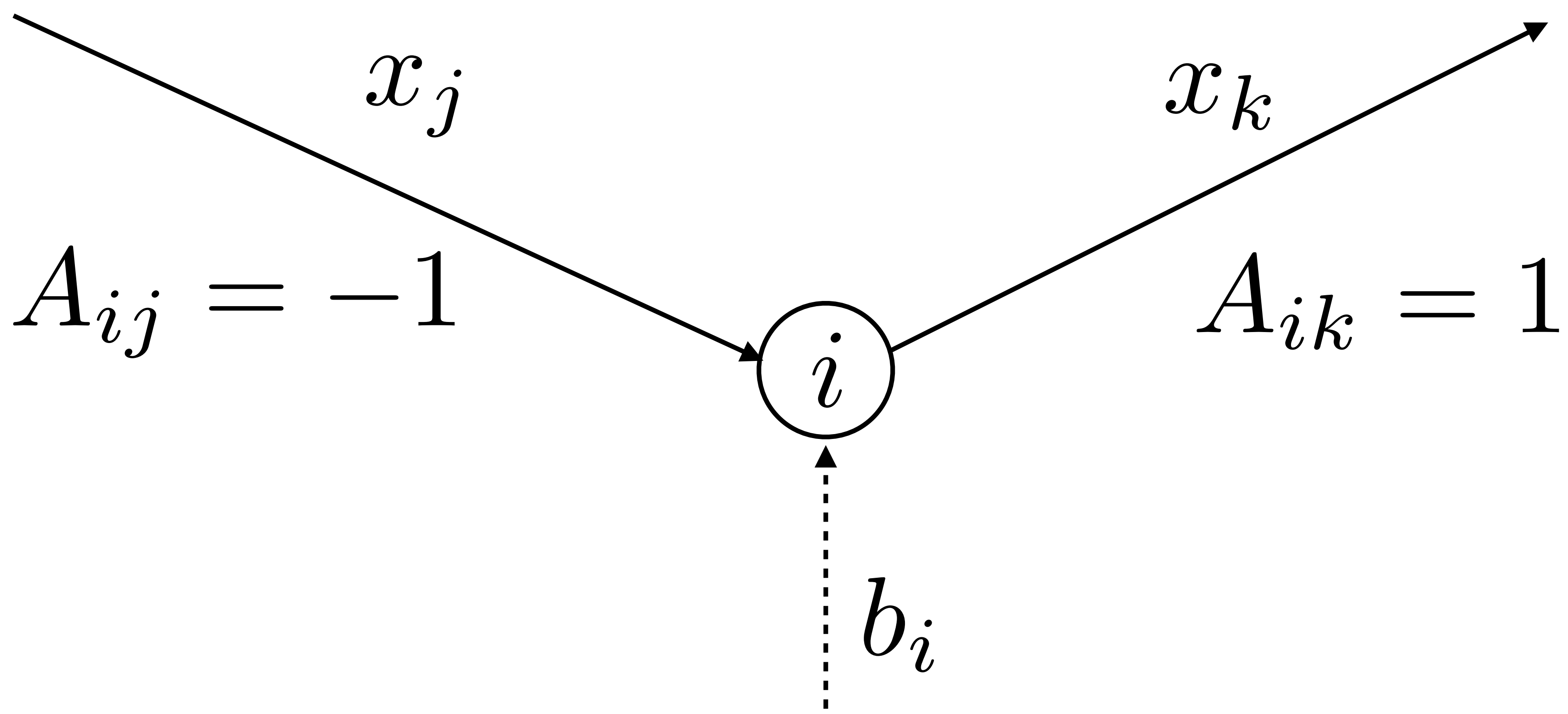

Balance equations

$$\sum_{j=1}^{n} A_{ij} x_j = (Ax)_i = b_i, \quad \text{for all } i$$

Total leaving flow

Supply

External supply

supply vector $b \in \mathbf{R}^m$

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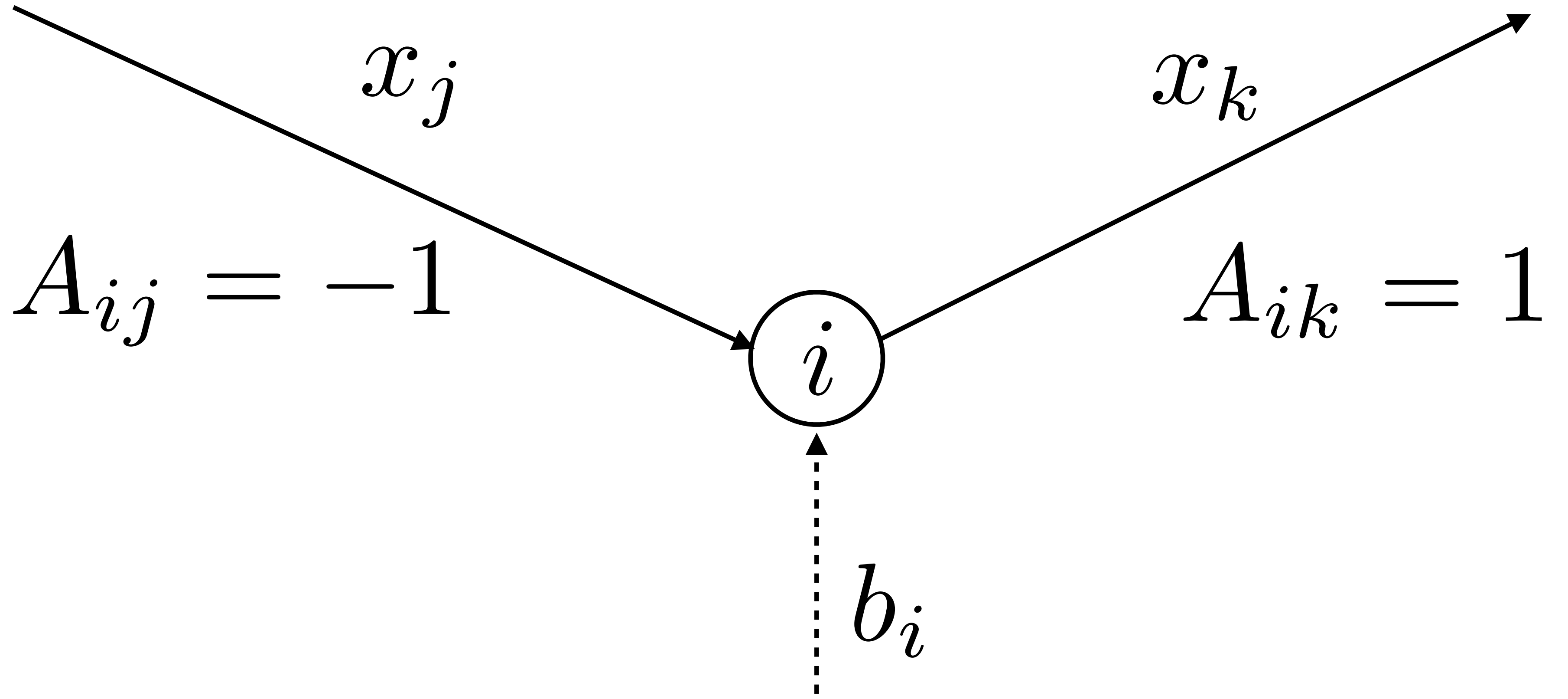

Balance equations

$$\sum_{j=1}^{n} A_{ij} x_j = (Ax)_i = b_i, \quad \text{for all } i \longrightarrow Ax = b$$

$(Ax)_i$: Total leaving flow

b_i: Supply

Minimum cost network flow problem

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax = b \\
& 0 \leq x \leq u
\end{array}
$$

- c_i is unit cost of flow through arc i
- Flow x_i must be nonnegative
- u_i is the maximum flow capacity of arc i
- Many network optimization problems are just special cases

Integrality theorem

Given a polyhedron $P = \{x \in \mathbf{R}^n \mid Ax = b, \quad x \geq 0\}$

where

- A is totally unimodular
- b is an integer vector

$\longrightarrow$ all the extreme points of P are integer vectors.

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$\longrightarrow$ all the extreme points of P are integer vectors.

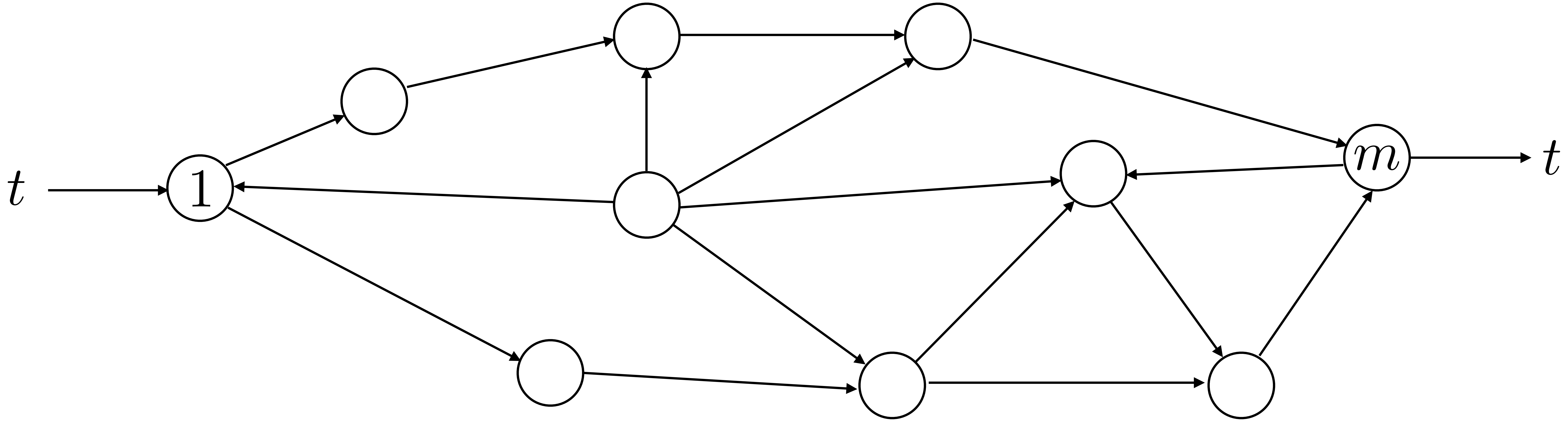
Proof

- All extreme points are basic feasible solutions with $x_B = A_B^{-1}b$ and $x_i = 0,\ i \neq B$
- A_B^{-1} has integer components because of total unimodularity of A
- b has also integer components
- Therefore, also x is integral

$\blacksquare$

Maximum flow problem

Goal maximize flow from node 1 (source)
to node m (sink) through the network

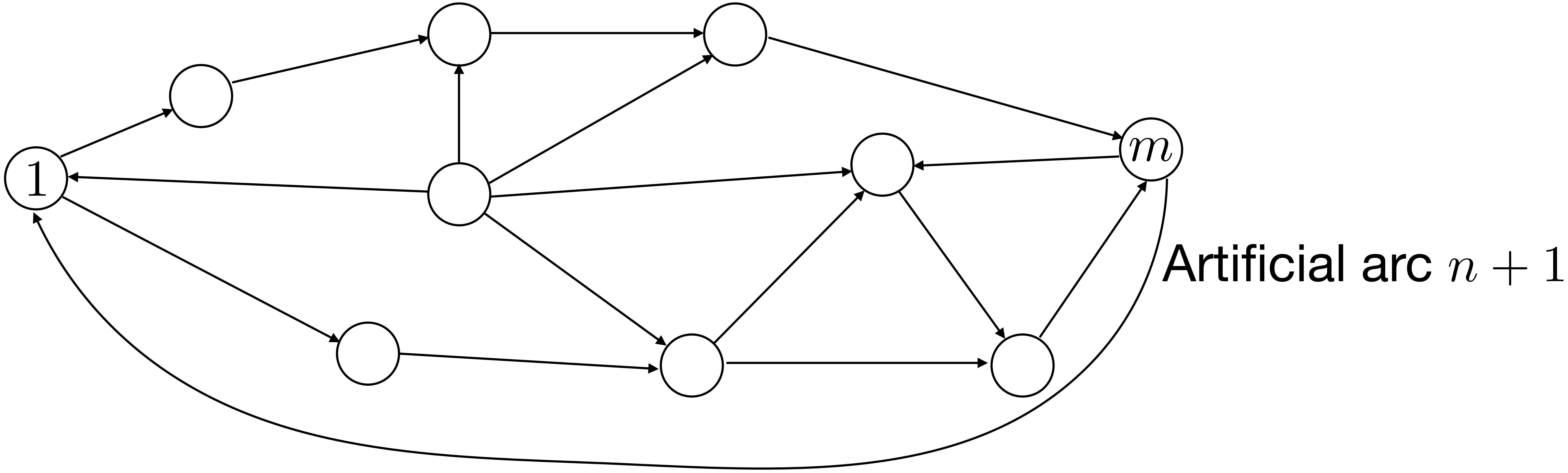


maximize $\quad t$

subject to $\quad Ax = te \qquad e = (1, 0, \ldots, 0, -1)$

$\quad 0 \leq x \leq u$

Maximum flow as minimum cost flow



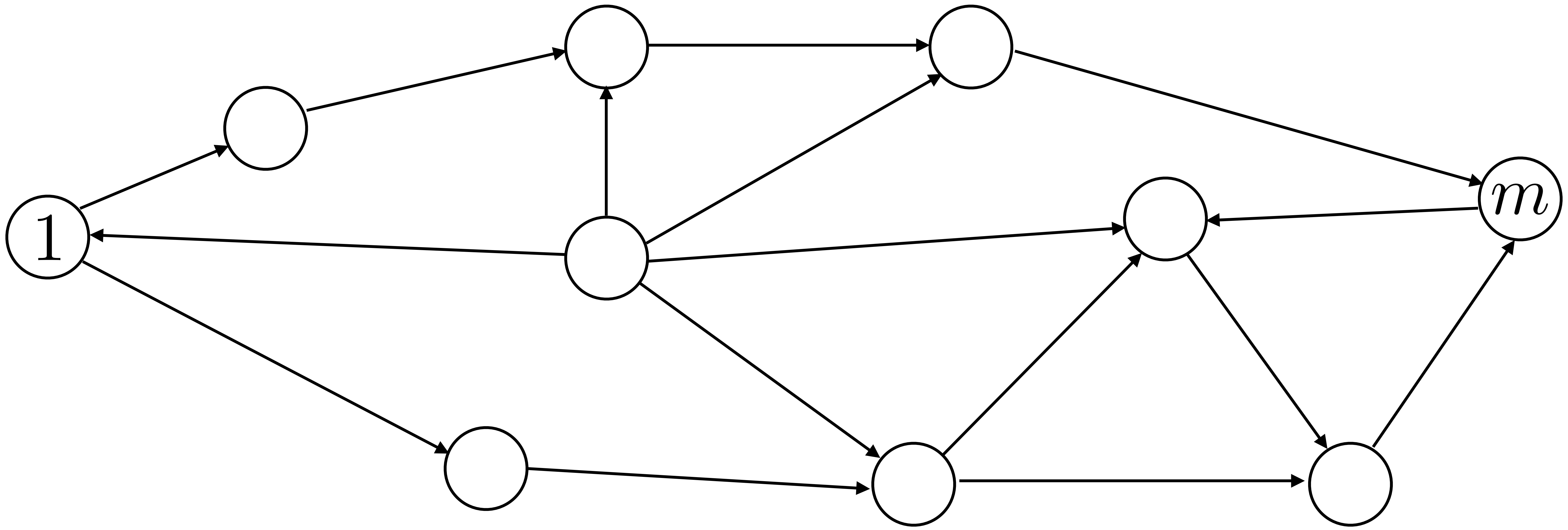
minimize $\quad -t$

subject to $\quad \begin{bmatrix} A & -e \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} = 0$

$$0 \leq \begin{bmatrix} x \\ t \end{bmatrix} \leq \begin{bmatrix} u \\ \infty \end{bmatrix}$$

Shortest path problem

Goal Find the shortest path between nodes 1 and m



paths can be represented as vectors $x \in \{0,1\}^n$

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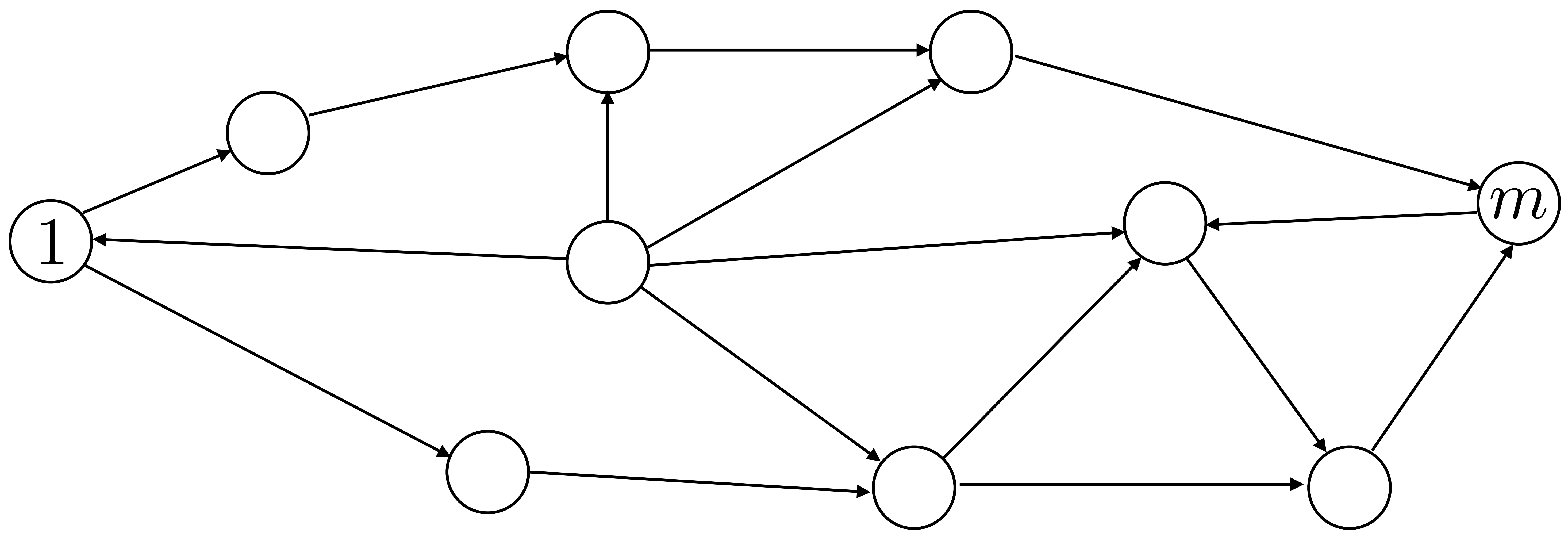

paths can be represented as vectors $x \in \{0,1\}^n$

Formulation

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax = e \\
& x \in \{0,1\}^n
\end{array}
$$

- c_j is the "length" of arc j
- $e = (1, 0, \ldots, 0, -1)$
- Variables are binary (include or not arc in path)

Shortest path as minimum cost flow

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = e \\ & x \in \{0,1\}^n \end{array}$$

Shortest path as minimum cost flow

minimize $c^T x$
subject to $Ax = e$
$x \in \{0,1\}^n$

$\longrightarrow$

Relaxation

minimize $c^T x$
subject to $Ax = e$
$0 \leq x \leq \mathbf{1}$

Shortest path as minimum cost flow

minimize $c^T x$

subject to $Ax = e$

$x \in \{0,1\}^n$

→

Relaxation

minimize $c^T x$

subject to $Ax = e$

$0 \leq x \leq \mathbf{1}$

Extreme points satisfy $x_i \in \{0, 1\}$

Shortest path as minimum cost flow

minimize	$c^T x$
subject to	$Ax = e$
	$x \in \{0, 1\}^n$

→

Relaxation

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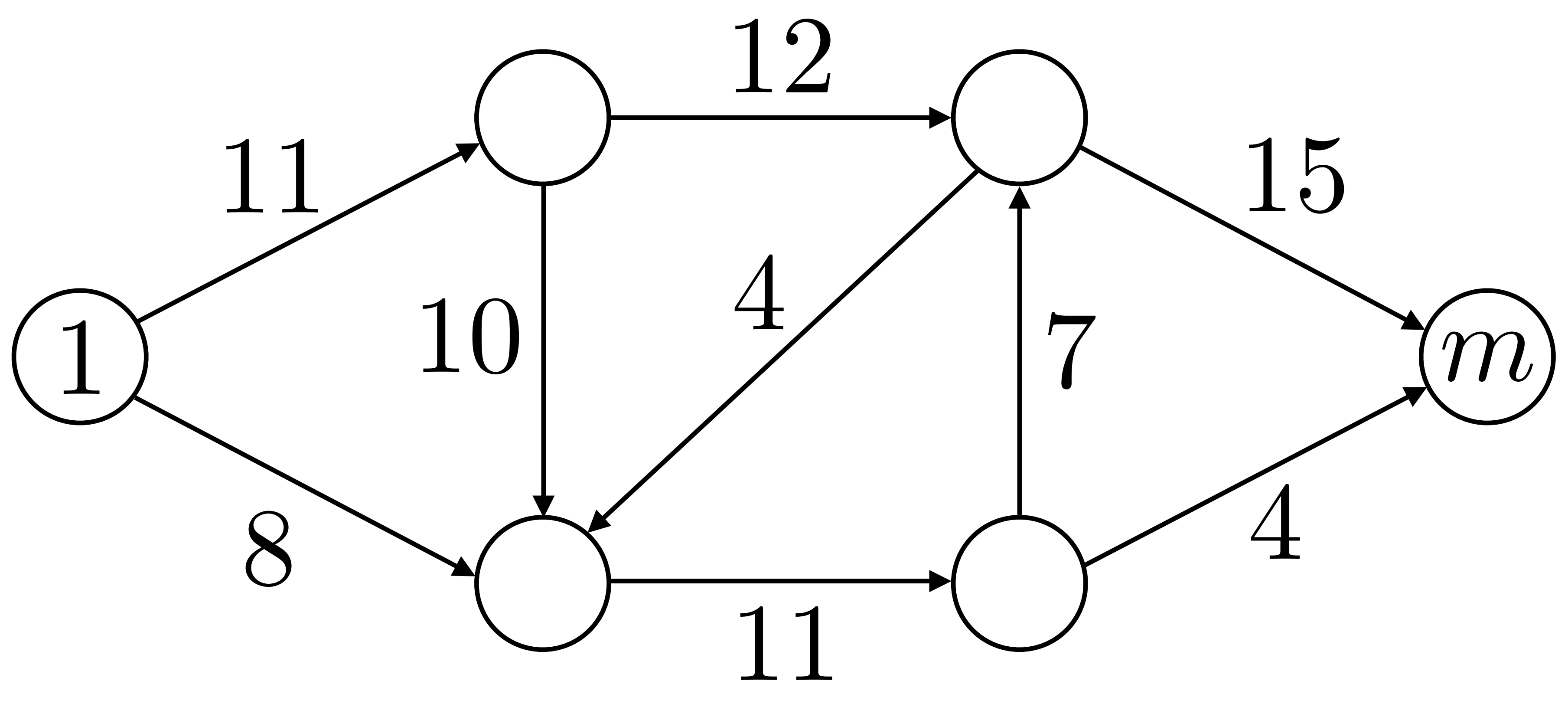
Example (arc costs shown)



Shortest path as minimum cost flow

minimize	$c^T x$
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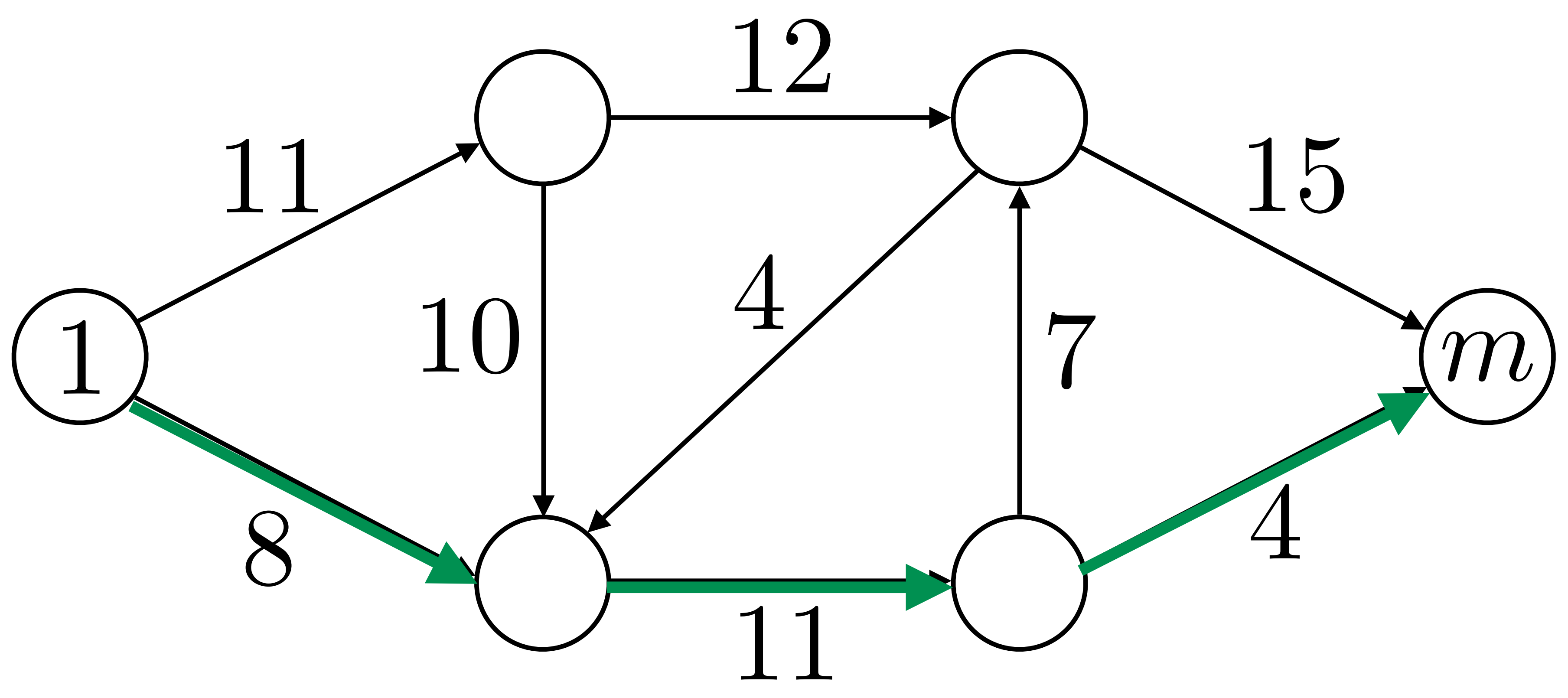
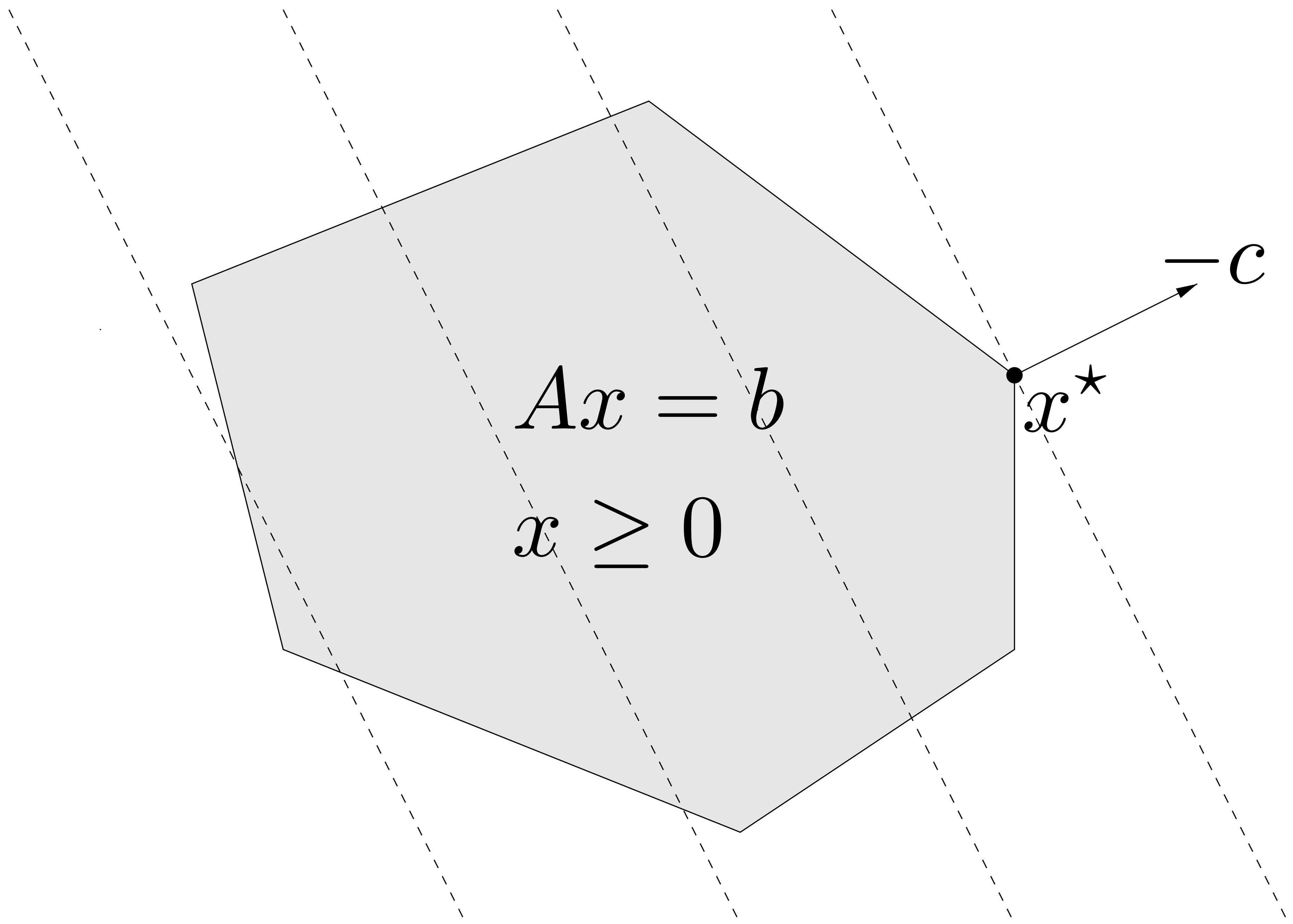


$$c = (11, 8, 10, 12, 4, 11, 7, 15, 4)$$
$$x^\star = (0, 1, 0, 0, 0, 1, 0, 0, 1)$$
$$c^T x^\star = 24$$

Simplex method

Optimality of extreme points



$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

If
- P has at least one extreme point
- There exists an optimal solution $x^\star$

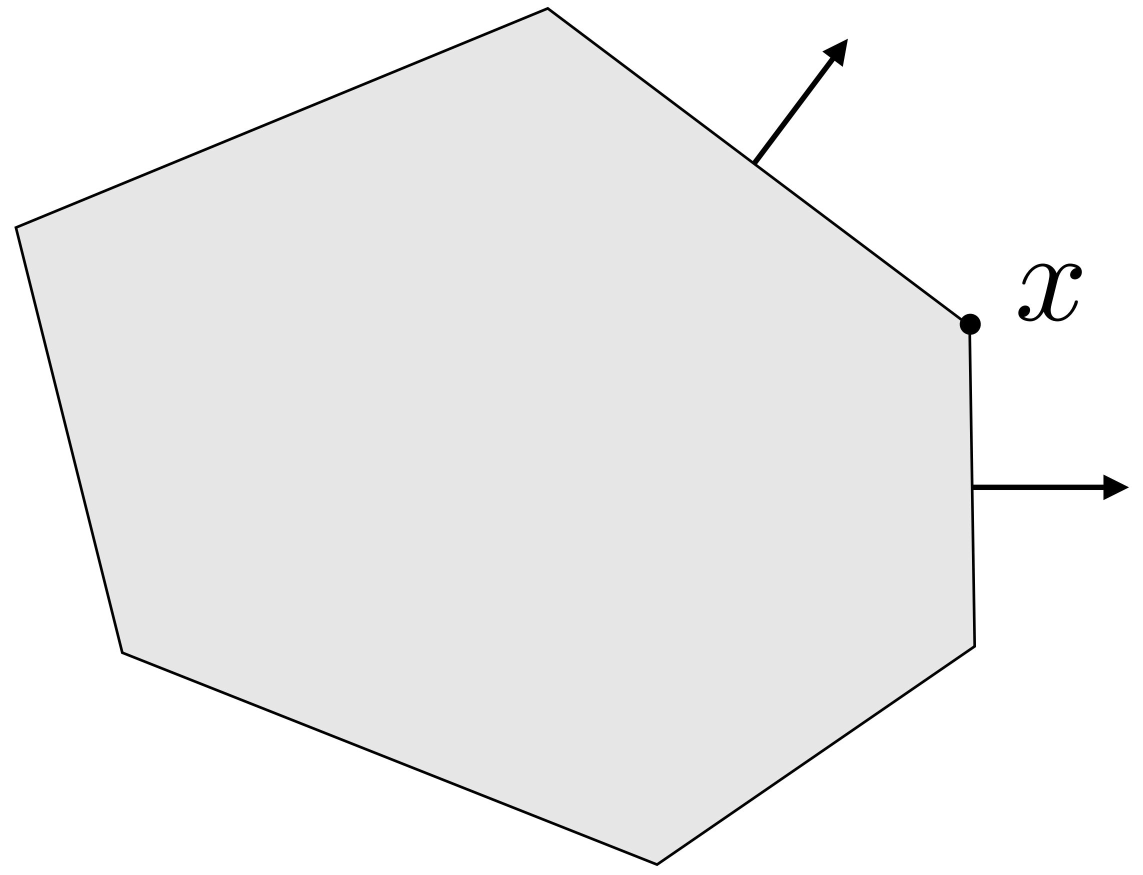
Then, there exists an optimal solution which is an **extreme point** of P

We only need to search between **extreme points**

Equivalence

Theorem

Given a nonempty polyhedron $P = \{x \mid Ax = b,\ x \geq 0\}$



Let $x \in P$

x is a **vertex** $\iff$ x is an **extreme point** $\iff$ x is a **basic feasible solution**

Constructing basic solution

1. Choose any m independent columns of A: $A_{B(1)}, \ldots, A_{B(m)}$
2. Let $x_i = 0$ for all $i \neq B(1), \ldots, B(m)$
3. Solve $Ax = b$ for the remaining $x_{B(1)}, \ldots, x_{B(m)}$

Basis matrix

Basis columns

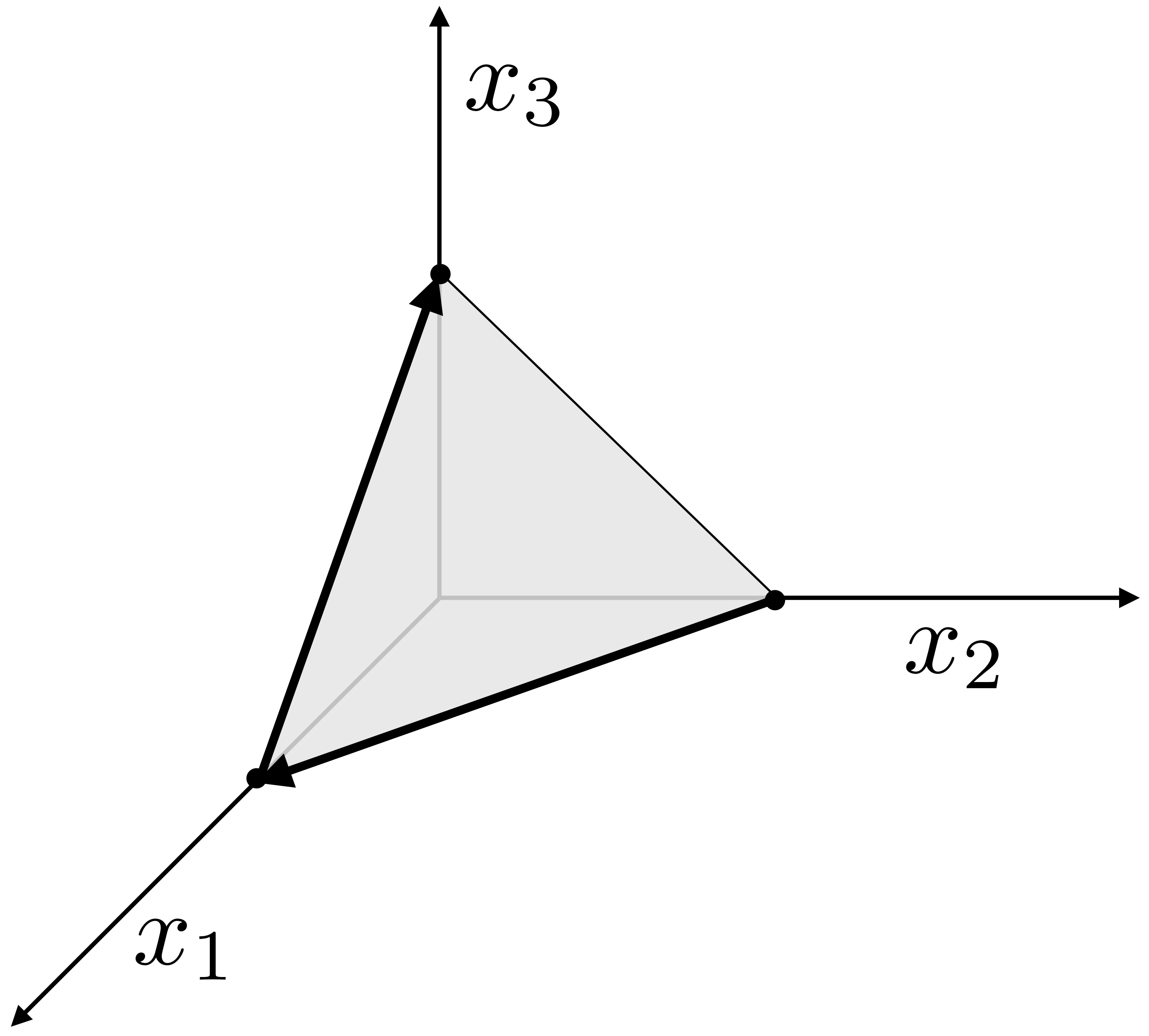
Basic variables

$$A_B = \begin{bmatrix} | & | & & | \\ A_{B(1)} & A_{B(2)} & \ldots & A_{B(m)} \\ | & | & & | \end{bmatrix}, \quad x_B = \begin{bmatrix} x_{B(1)} \\ \vdots \\ x_{B(m)} \end{bmatrix} \longrightarrow \text{Solve} \quad A_B x_B = b$$

If $x_B \geq 0$, then x is a **basic feasible solution**

Conceptual algorithm

- Start at corner
- Visit neighboring corner that improves the objective



How does the cost change?

Cost improvement

$$c^T(x + \theta d) - c^T x = \theta c^T d$$

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New cost

How does the cost change?

Cost improvement

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New cost

Old cost

How does the cost change?

Cost improvement

$$c^T(x + \theta d) - c^T x = \theta c^T d$$

New cost **Old cost**

We call $\bar{c}_j$ the **reduced cost** of (introducing) variable x_j in the basis

$$\bar{c}_j = c^T d = \sum_{i=1}^{n} c_j d_j = c_j + c_B^T d_B = c_j - c_B^T A_B^{-1} A_j$$

Optimality conditions

Theorem

Let x be a basic feasible solution associated with basis B

Let $\bar{c}$ be the vector of reduced costs.

If $\bar{c} \geq 0$, then x is **optimal**

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Remark

This is a **stopping criterion** for the simplex algorithm.

If the **neighboring solutions** do not improve the cost, we are done

Single simplex iteration

1. Compute the reduced costs $\bar{c}$
   - Solve $A_B^T p = c_B$
   - $\bar{c} = c - A^T p$
2. If $\bar{c} \geq 0$, x **optimal. break**
3. Choose j such that $\bar{c}_j < 0$
4. Compute search direction d with $d_j = 1$ and $A_B d_B = -A_j$
5. If $d_B \geq 0$, the problem is **unbounded** and the optimal value is $-\infty$. **break**
6. Compute step length $\theta^\star = \min_{\{i \in B | d_i < 0\}} \left(-\frac{x_i}{d_i} \right)$
7. Define y such that $y = x + \theta^\star d$
8. Get new basis $\bar{B}$ (i exits and j enters)

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Bottleneck

Two linear systems

Single simplex iteration

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Two linear systems

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Matrix inversion lemma trick
$\approx n^2$ per iteration
(very cheap)

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Two linear systems

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How many iterations do we need?

Complexity of the simplex method

We do **not know any polynomial version of the simplex method**, no matter which pivoting rule we pick. → Still **open research question!**

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Worst-case

There are problem instances where the simplex method will run an **exponential number of iterations** in terms of the dimensions, e.g. 2^n

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Good news: average-case
Practical performance is very good. On average, it stops in n iterations.

Interior point method

Optimality conditions

Primal

minimize $c^T x$

subject to $Ax + s = b$

$s \geq 0$

Dual

maximize $-b^T y$

subject to $A^T y + c = 0$

$y \geq 0$

KKT conditions

$$
\begin{aligned}
Ax + s - b &= 0 \\
A^T y + c &= 0 \\
s_i y_i &= 0, \quad i = 1, \ldots, m \\
s, y &\geq 0
\end{aligned}
$$

$$
S = \begin{bmatrix} s_1 & & & \\ & s_2 & & \\ & & \ddots & \\ & & & s_m \end{bmatrix} \qquad Y = \begin{bmatrix} y_1 & & & \\ & y_2 & & \\ & & \ddots & \\ & & & y_m \end{bmatrix}
$$

$$
\implies SY\mathbf{1} = 0
$$

Main idea

$$h(x, s, y) = \begin{bmatrix} Ax + s - b \\ A^T y + c \\ SY\mathbf{1} \end{bmatrix} = 0$$

$$s, y \geq 0$$

$$S = \mathbf{diag}(s)$$
$$Y = \mathbf{diag}(y)$$

- Apply variants of Newton's method to solve $h(x, s, y) = 0$
- Enforce $s, y > 0$ (strictly) at every iteration
- **Motivation** avoid getting stuck in "corners"

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- **Motivation** avoid getting stuck in "corners"

Issue

Pure **Newton's step** does not allow significant progress towards $h(x, s, y) = 0$ **and** $x, y \geq 0$.

Smoothed optimality conditions

Optimality conditions

$$
\begin{aligned}
Ax + s - b &= 0 \\
A^T y + c &= 0 \\
s_i y_i &= \tau \\
s, y &\geq 0
\end{aligned}
$$

← Same τ for every pair

Same optimality conditions for a “smoothed” version of our problem

Central path

minimize	$c^T x - \tau \sum_{i=1}^m \log(s_i)$
subject to	$Ax + s = b$

Set of points $(x^\star(\tau), s^\star(\tau), y^\star(\tau))$
with $\tau > 0$ such that

$$
\begin{aligned}
Ax + s - b &= 0 \\
A^T y + c &= 0 \\
s_i y_i &= \tau \\
s, y &\geq 0
\end{aligned}
$$

Central path

minimize $\quad c^T x - \tau \sum_{i=1}^m \log(s_i)$

subject to $\quad Ax + s = b$

Set of points $(x^\star(\tau), s^\star(\tau), y^\star(\tau))$
with $\tau > 0$ such that

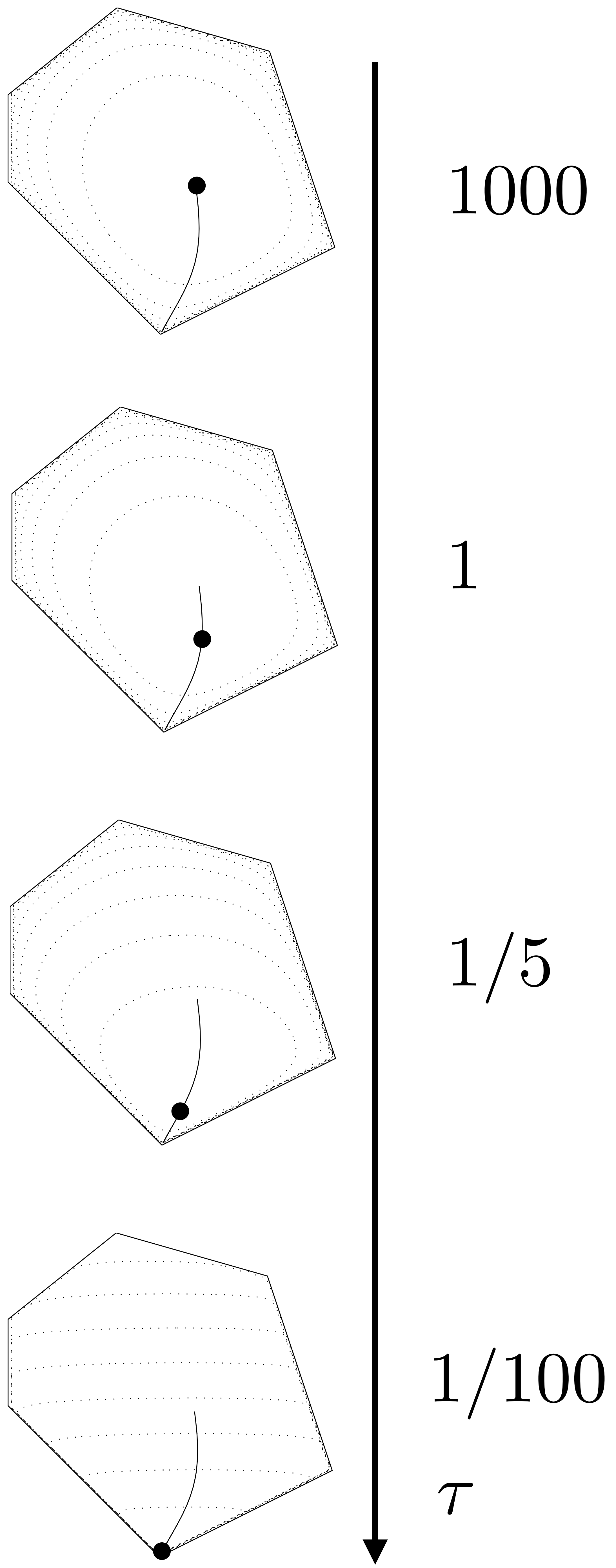
$$Ax + s - b = 0$$

$$A^T y + c = 0$$

$$s_i y_i = \tau$$

$$s, y \geq 0$$

Analytic Center
$\tau \to \infty$



Main idea

Follow central path as $\tau \to 0$

Newton's method for smoothed optimality conditions

Smoothed optimality conditions

$$h_\tau(x, s, y) = \begin{bmatrix} Ax + s - b \\ A^T y + c \\ SY\mathbf{1} - \tau\mathbf{1} \end{bmatrix} = 0$$

$$s, y \geq 0$$

Newton's method for smoothed optimality conditions

Smoothed optimality conditions

$$h_\tau(x, s, y) = \begin{bmatrix} Ax + s - b \\ A^T y + c \\ SY\mathbf{1} - \tau\mathbf{1} \end{bmatrix} = 0$$

$$s, y \geq 0$$

Linear system

$$\begin{bmatrix} 0 & A & I \\ A^T & 0 & 0 \\ S & 0 & Y \end{bmatrix} \begin{bmatrix} \Delta y \\ \Delta x \\ \Delta s \end{bmatrix} = \begin{bmatrix} -r_p \\ -r_d \\ -SY + \tau\mathbf{1} \end{bmatrix}$$

Line search to enforce $x, s > 0$

$$(x, s, y) \leftarrow (x, s, y) + \alpha(\Delta x, \Delta s, \Delta y)$$

Algorithm step

Linear system

$$\begin{bmatrix} 0 & A & I \\ A^T & 0 & 0 \\ S & 0 & Y \end{bmatrix} \begin{bmatrix} \Delta y \\ \Delta x \\ \Delta s \end{bmatrix} = \begin{bmatrix} -r_p \\ -r_d \\ -SY\mathbf{1} + \sigma\mu\mathbf{1} \end{bmatrix}$$

Duality measure

$$\mu = \frac{s^T y}{m}$$

Centering parameter

$$\sigma \in [0, 1]$$

Algorithm step

Linear system

$$\begin{bmatrix} 0 & A & I \\ A^T & 0 & 0 \\ S & 0 & Y \end{bmatrix} \begin{bmatrix} \Delta y \\ \Delta x \\ \Delta s \end{bmatrix} = \begin{bmatrix} -r_p \\ -r_d \\ -SY\mathbf{1} + \sigma\mu\mathbf{1} \end{bmatrix}$$

Duality measure

$$\mu = \frac{s^T y}{m}$$

Centering parameter

$\sigma \in [0, 1]$

$\sigma = 0 \quad \Rightarrow$ Newton step

$\sigma = 1 \quad \Rightarrow$ Centering step towards $(x^\star(\mu), s^\star(\mu), y^\star(\mu))$

Algorithm step

Linear system

$$\begin{bmatrix} 0 & A & I \\ A^T & 0 & 0 \\ S & 0 & Y \end{bmatrix} \begin{bmatrix} \Delta y \\ \Delta x \\ \Delta s \end{bmatrix} = \begin{bmatrix} -r_p \\ -r_d \\ -SY\mathbf{1} + \sigma\mu\mathbf{1} \end{bmatrix}$$

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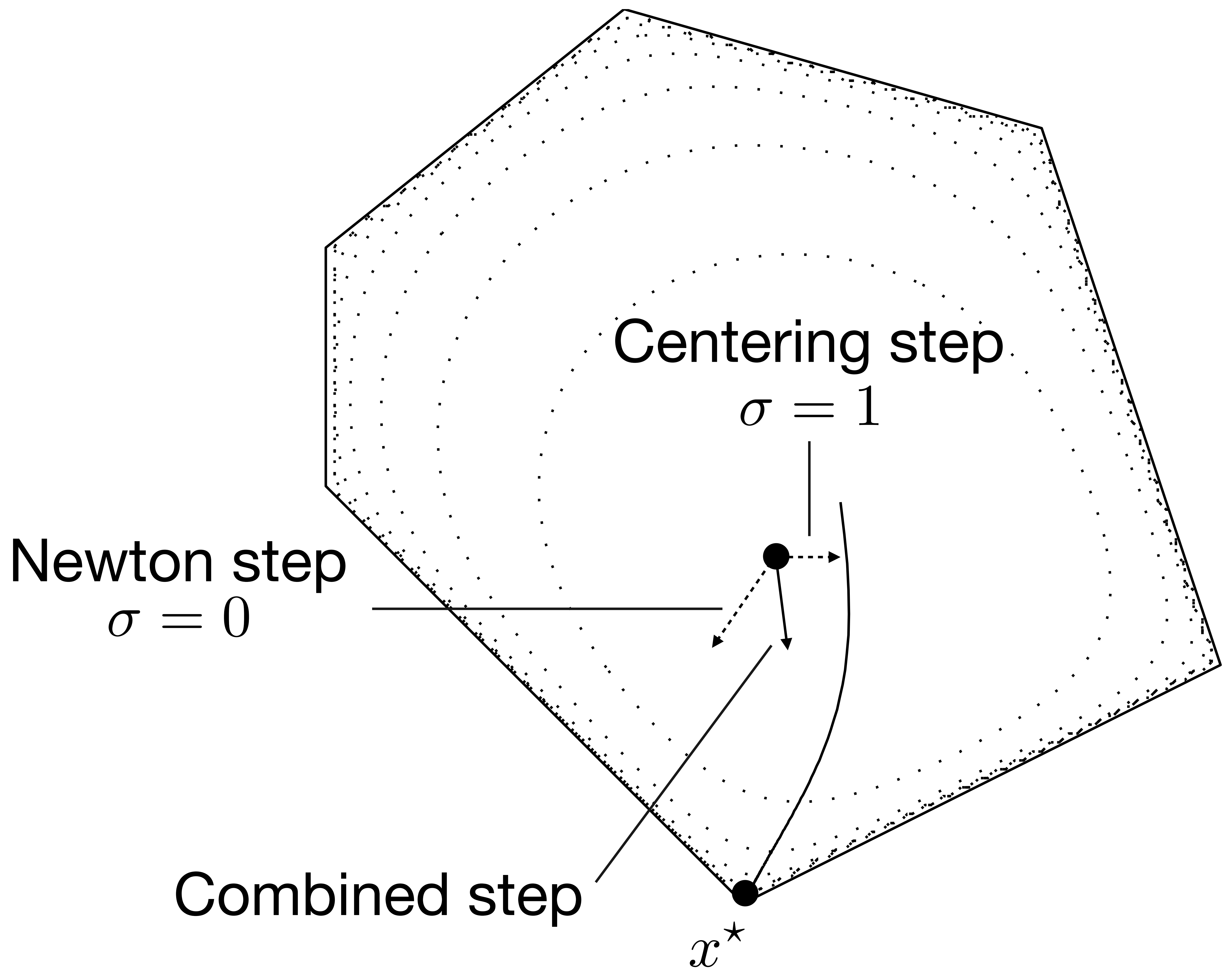
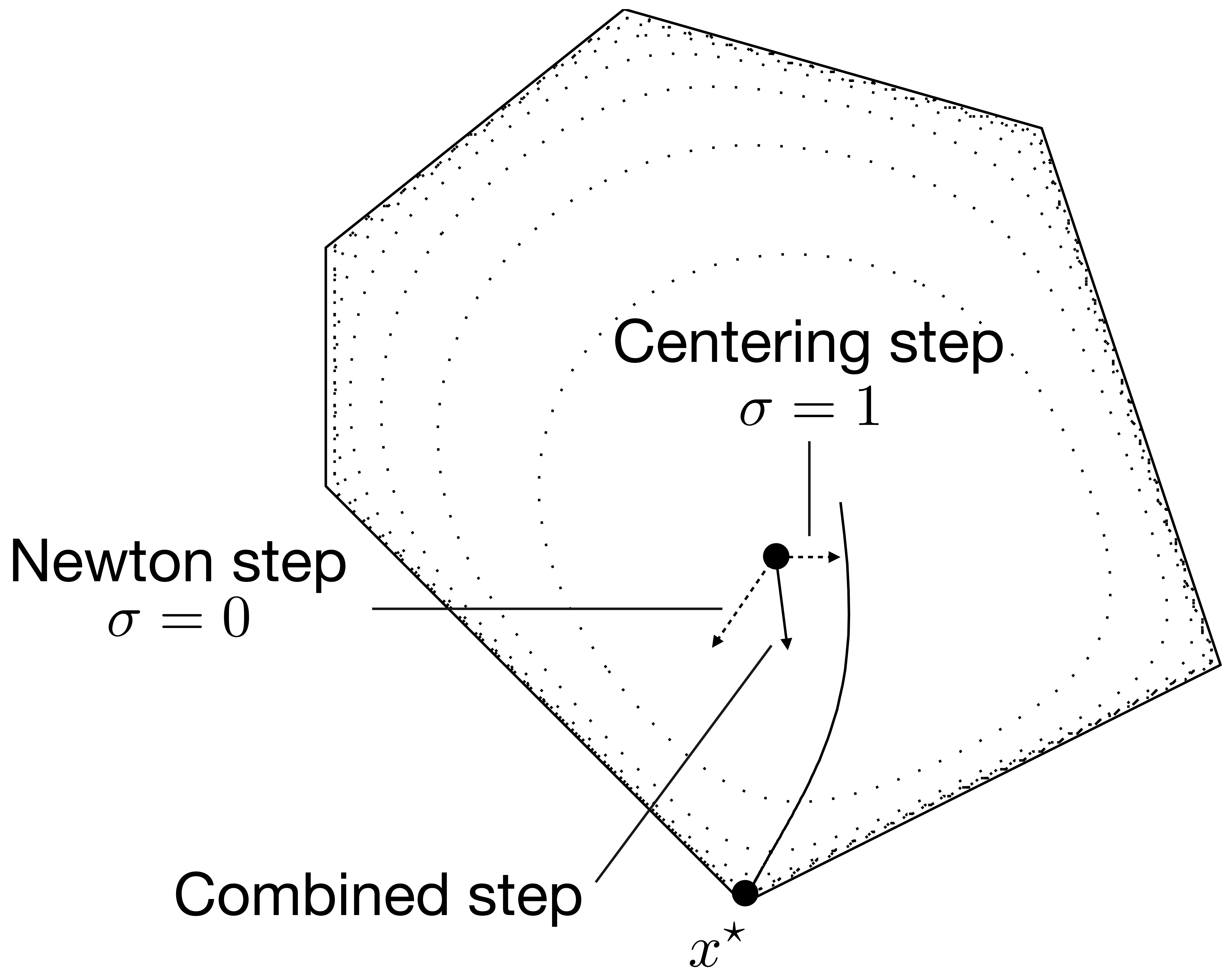
Line search to enforce $x, s > 0$

$$(x, s, y) \leftarrow (x, s, y) + \alpha(\Delta x, \Delta s, \Delta y)$$

Path-following algorithm idea



Path-following algorithm idea



Centering step

Moves towards the **central path** and is usually biased towards $s, y > 0$.
No progress on duality measure μ

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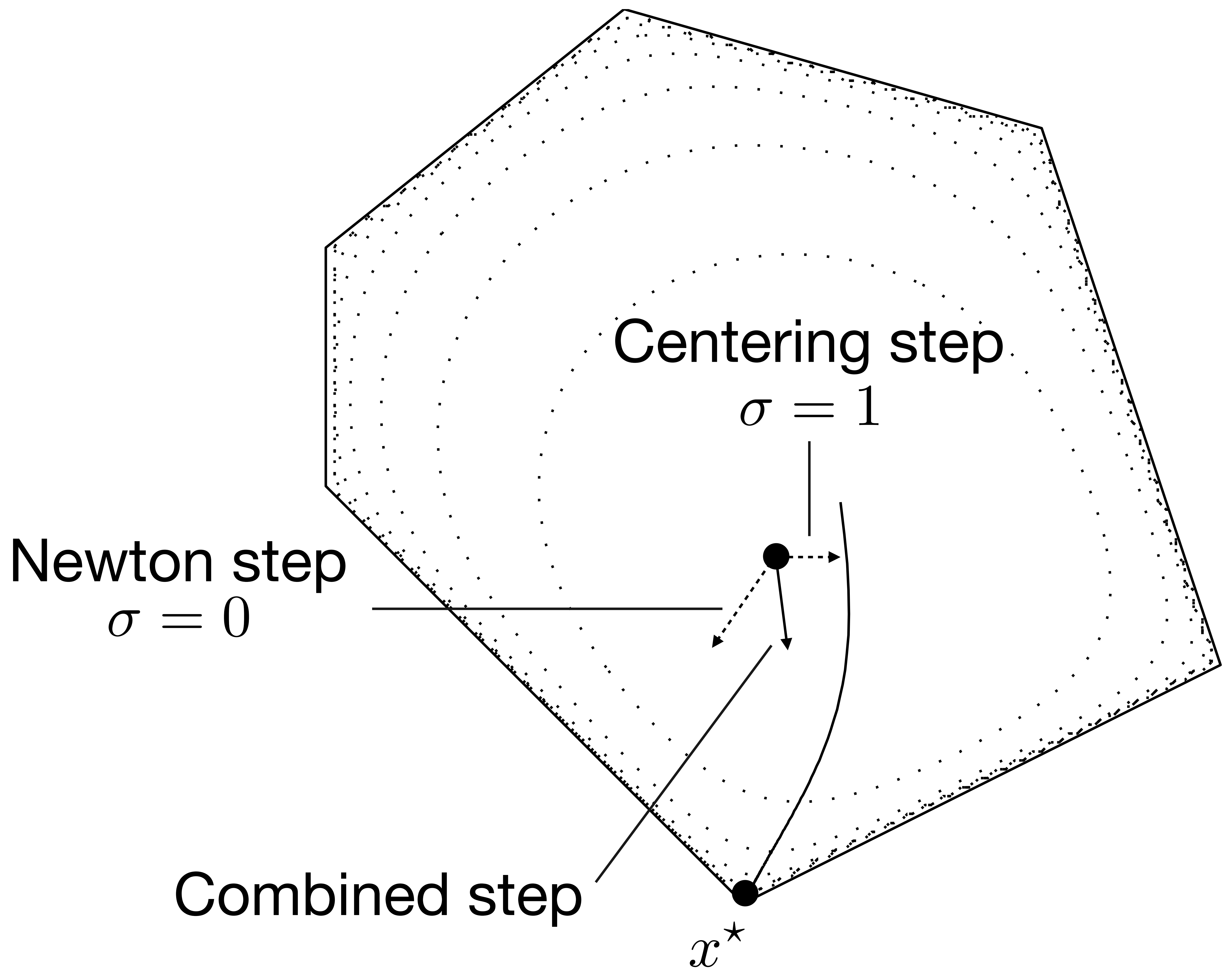

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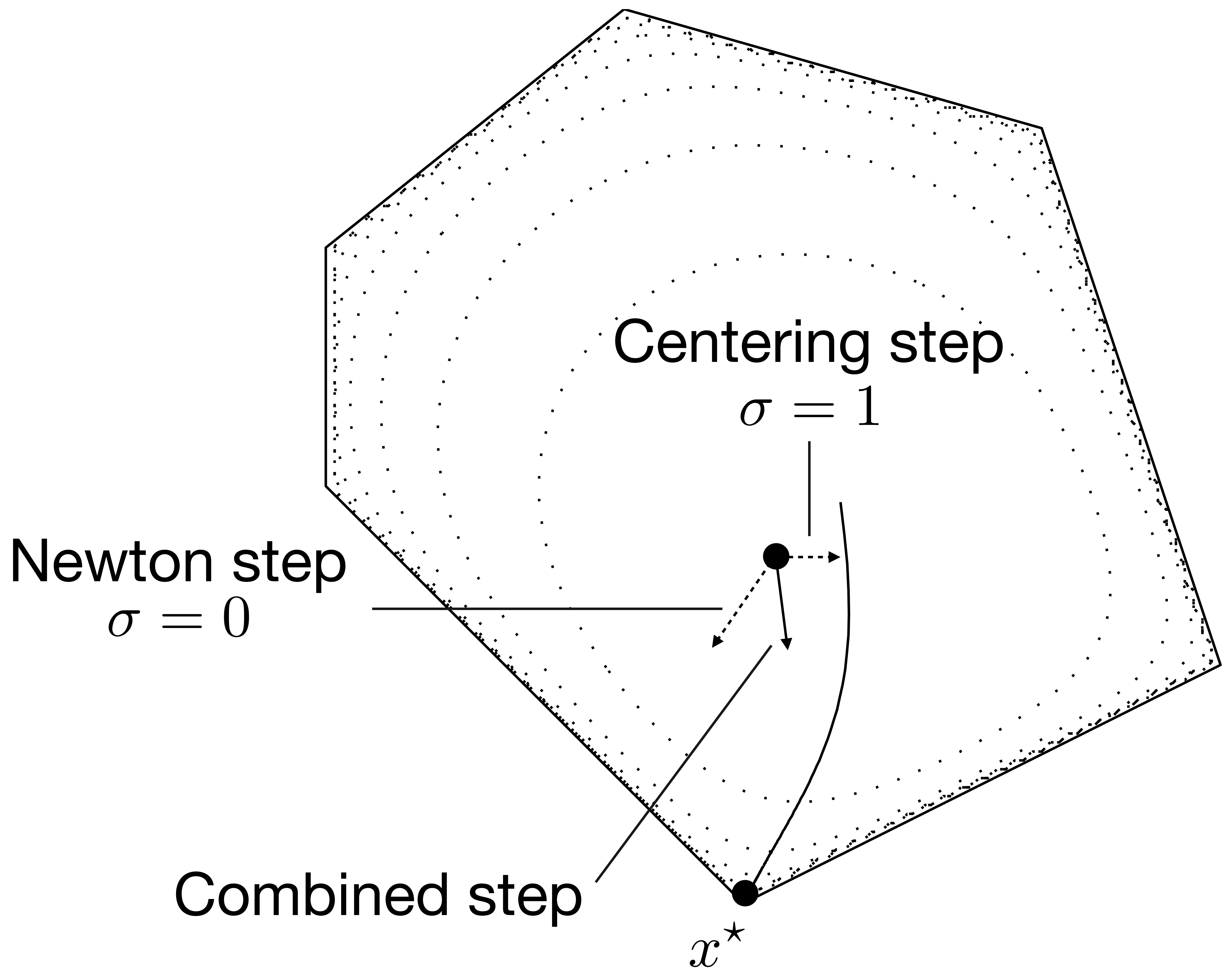

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Combined step

Best of both, with longer steps.

Convergence

Mehrotra's algorithm

No convergence theory → Examples where it **diverges** (rare!)

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Theoretical iteration complexity

Alternative versions (slower than Mehrotra) converge in $O(\sqrt{n})$ iterations

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Average iteration complexity

Average iterations complexity is $O(\log n)$

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Theoretical iteration complexity

Alternative versions (slower than Mehrotra) converge in $O(\sqrt{n})$ iterations → **Operations** $O(n^{3.5})$

Average iteration complexity

Average iterations complexity is $O(\log n)$ → $O(n^3 \log n)$

Interior-point vs simplex

Comparison between interior-point method and simplex

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Primal simplex

- Primal feasibility

↓

- Zero duality gap
- Dual feasibility

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Requires feasible point

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Exponential worst-case complexity

Requires feasible point

Can be warm-started

Polynomial worst-case complexity

Allows infeasible start

Cannot be warm-started

Which algorithm should I use?

Dual simplex

- Small-to-medium problems
- Repeated solves with varying constraints

Interior-point (barrier)

- Medium-to-large problems
- Sparse structured problems

How do solvers with multiple options decide?

<u>Concurrent Optimization</u>

Why not both? (crossover)

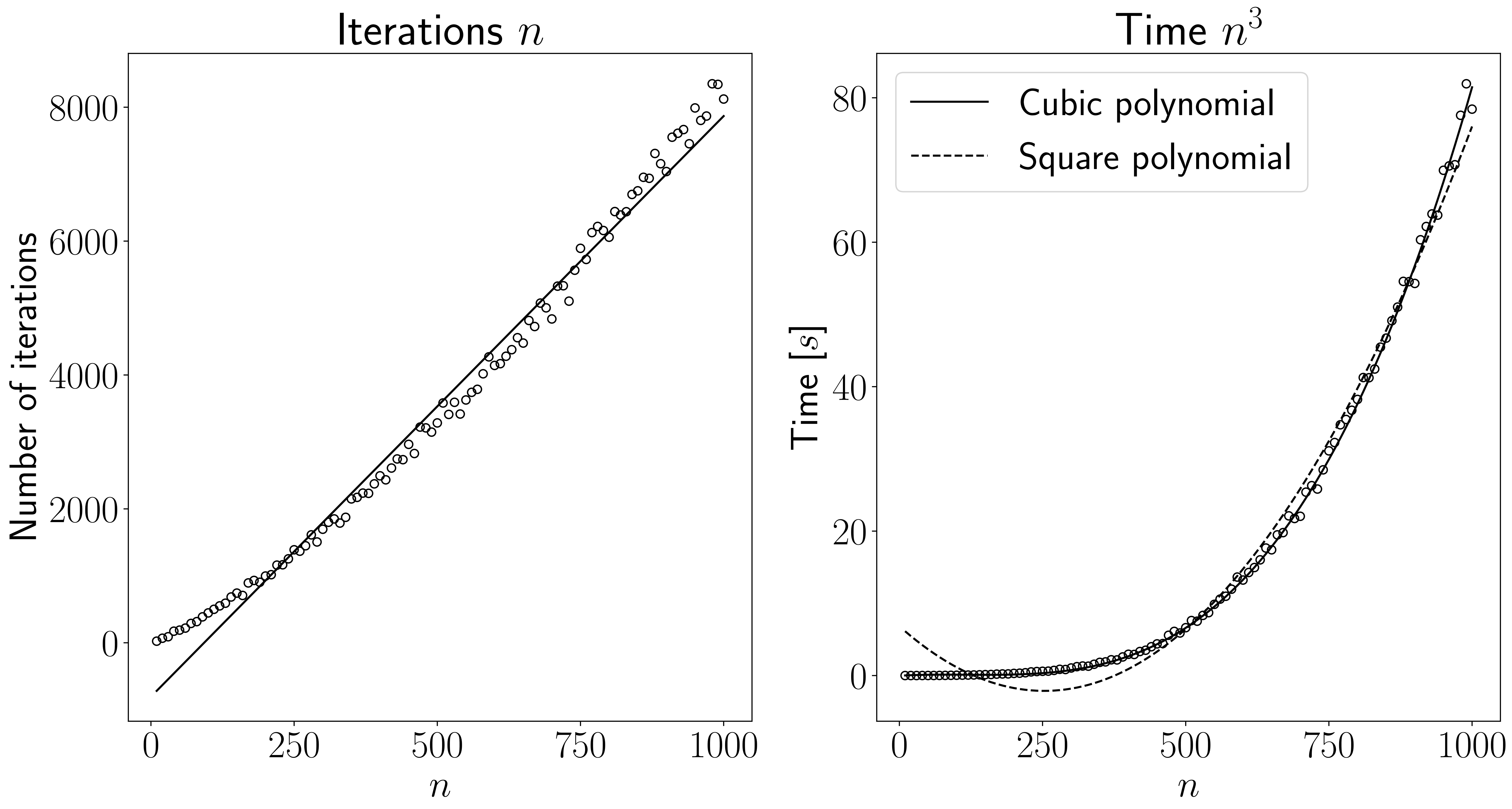
Interior-point → Few simplex steps

Average simplex complexity

Random LPs

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

n variables
$3n$ constraints



Average interior-point complexity

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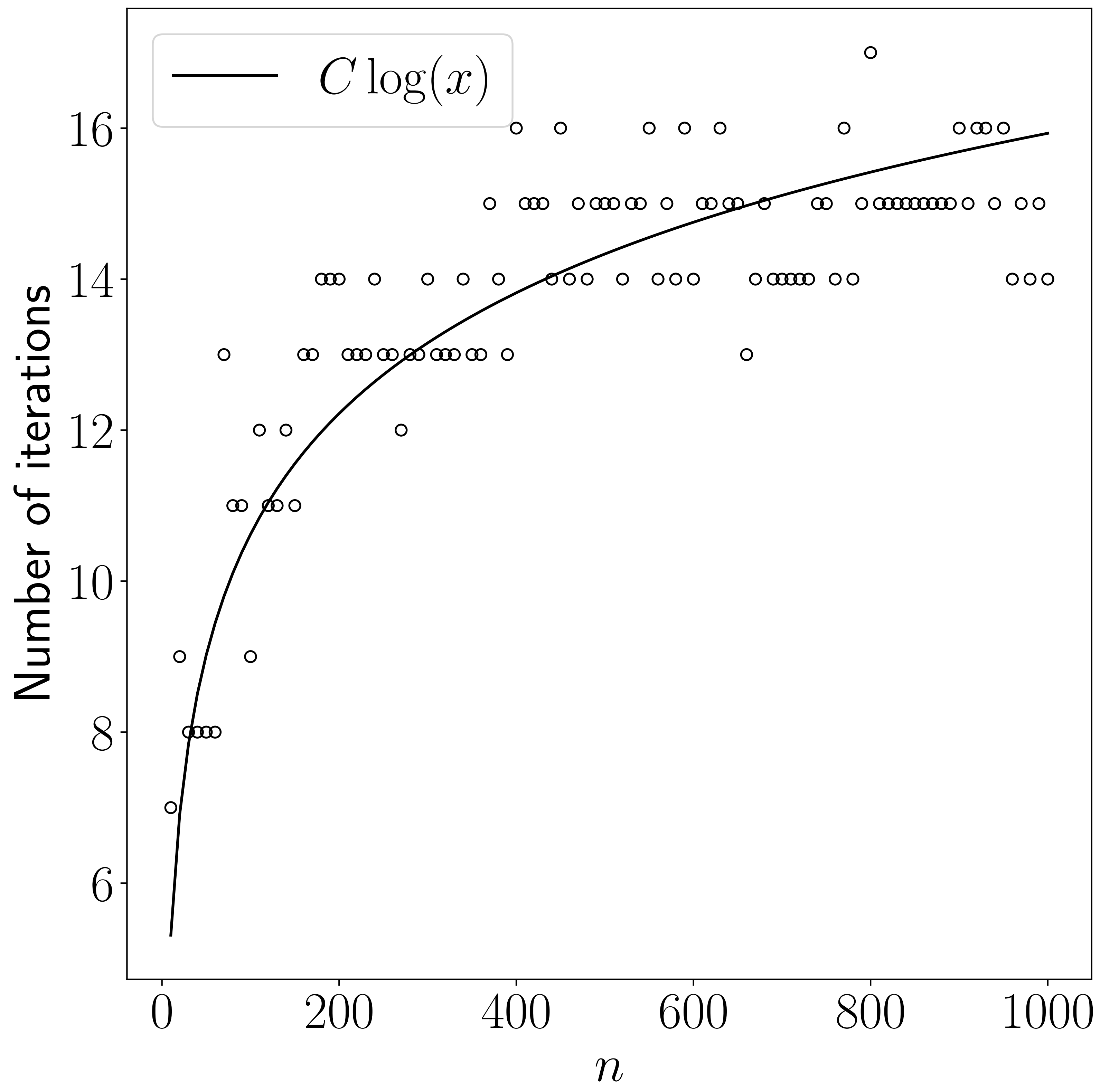
Iterations: $O(\log n)$



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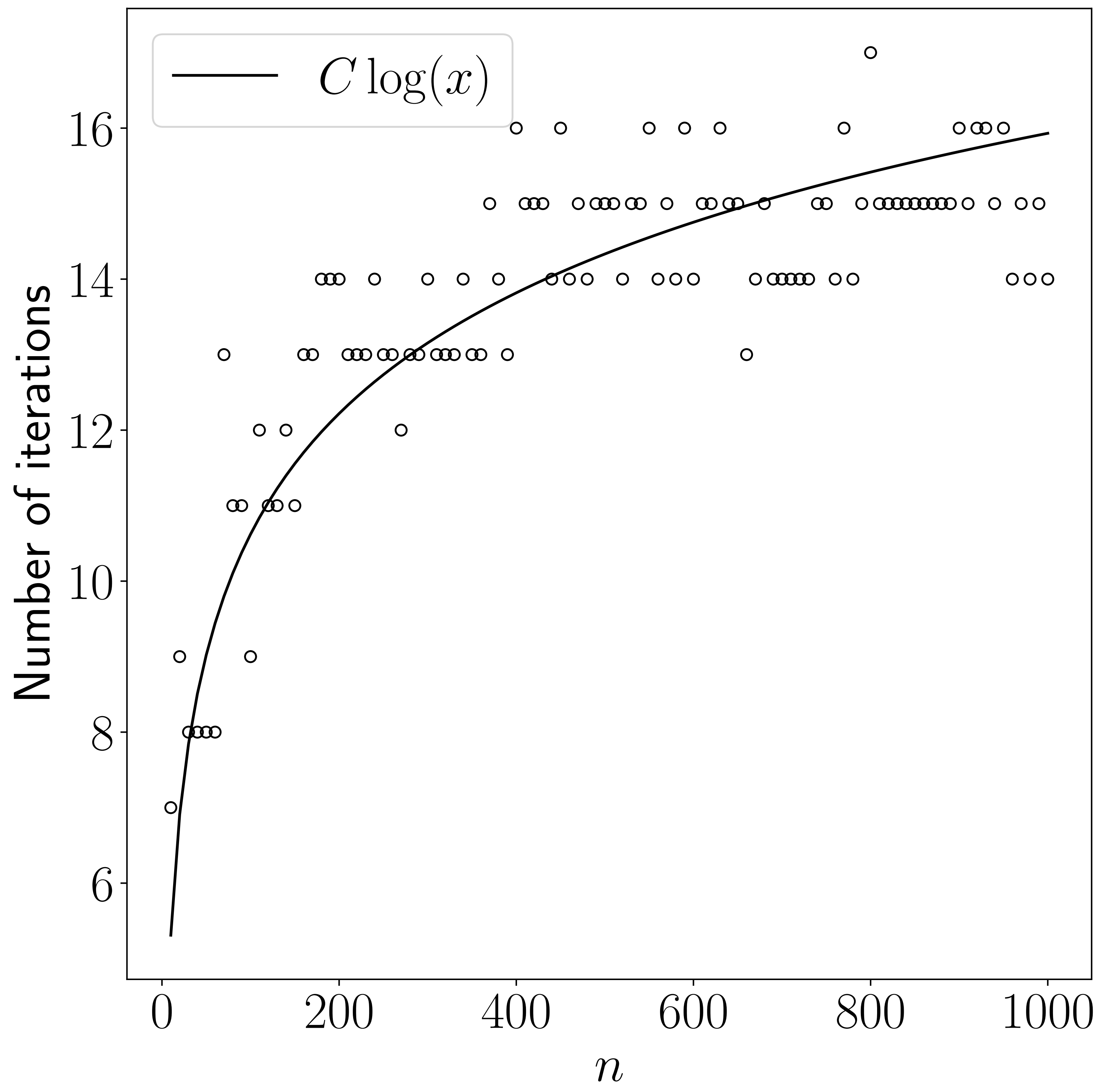
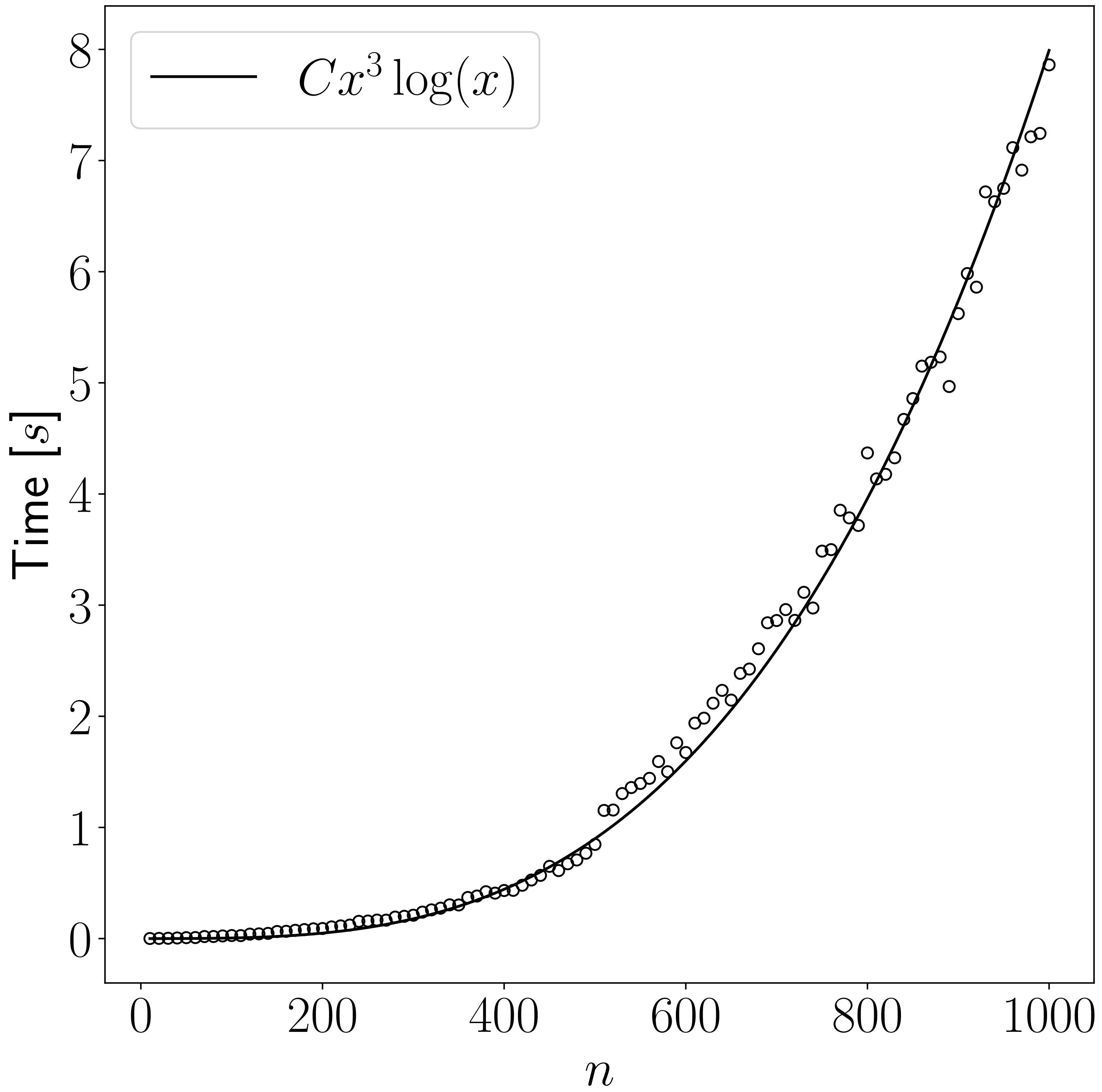


Time: $O(n^3 \log n)$



Questions

Next lecture

- Integer optimization