ORF307 – Optimization

15. Sensitivity analysis

Bartolomeo Stellato — Spring 2024

Ed Forum

- what is complementary slackness? can you clarify the geometric interpretation of complementary slackness?

Recap

Optimal objective values

Primal

minimize	$c^T x$
subject to	$Ax \leq b$

Dual

maximize	$-b^T y$
subject to	$A^T y + c = 0$
	$y \geq 0$

$p^\star$ is the primal optimal value

$d^\star$ is the dual optimal value

Primal infeasible: $p^\star = +\infty$
Primal unbounded: $p^\star = -\infty$

Dual infeasible: $d^\star = -\infty$
Dual unbounded: $d^\star = +\infty$

Weak duality

Theorem

If x, y satisfy:

- x is a feasible solution to the primal problem
- y is a feasible solution to the dual problem

$\longrightarrow$ $-b^T y \leq c^T x$

Proof

We know that $Ax \leq b$, $A^T y + c = 0$ and $y \geq 0$. Therefore,

$$0 \leq y^T(b - Ax) = b^T y - y^T Ax = c^T x + b^T y \qquad \blacksquare$$

Remark

- Any dual feasible y gives a **lower bound** on the primal optimal value
- Any primal feasible x gives an **upper bound** on the dual optimal value
- $c^T x + b^T y$ is the **duality gap**

Strong duality

Theorem
If a linear optimization problem has an optimal solution, so does its dual, and the optimal value of primal and dual are equal

$$d^\star = p^\star$$

Complementary slackness

Primal

minimize $\quad c^T x$

subject to $\quad Ax \leq b$

Dual

maximize $\quad -b^T y$

subject to $\quad A^T y + c = 0$

$\qquad y \geq 0$

Theorem

Primal,dual feasible x, y are optimal if and only if

$$y_i(b_i - a_i^T x) = 0, \quad i = 1, \ldots, m$$

i.e., at optimum, $b - Ax$ and y have a **complementary sparsity** pattern:

$$\begin{aligned} y_i > 0 \quad &\Rightarrow \quad a_i^T x = b_i \\ a_i^T x < b_i \quad &\Rightarrow \quad y_i = 0 \end{aligned}$$

Complementary slackness

Primal

minimize $c^T x$

subject to $Ax \leq b$

Dual

maximize $-b^T y$

subject to $A^T y + c = 0$

$y \geq 0$

Proof

The duality gap at primal feasible x and dual feasible y can be written as

$$c^T x + b^T y = (-A^T y)^T x + b^T y = (b - Ax)^T y = \sum_{i=1}^{m} y_i (b_i - a_i^T x) = 0$$

Since all the elements of the sum are nonnegative, they must all be 0 ■

For **feasible** x and y **complementary slackness** = **zero duality gap**

Example

$$\begin{array}{ll} \text{minimize} & -4x_1 - 5x_2 \\ \text{subject to} & \begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 3 \\ 0 \\ 3 \end{bmatrix} \end{array}$$

Let's **show** that feasible $x = (1, 1)$ is optimal

Second and fourth constraints are active at x $\longrightarrow$ $y = (0, y_2, 0, y_4)$

$$A^T y = -c \quad \Rightarrow \quad \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_2 \\ y_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

and

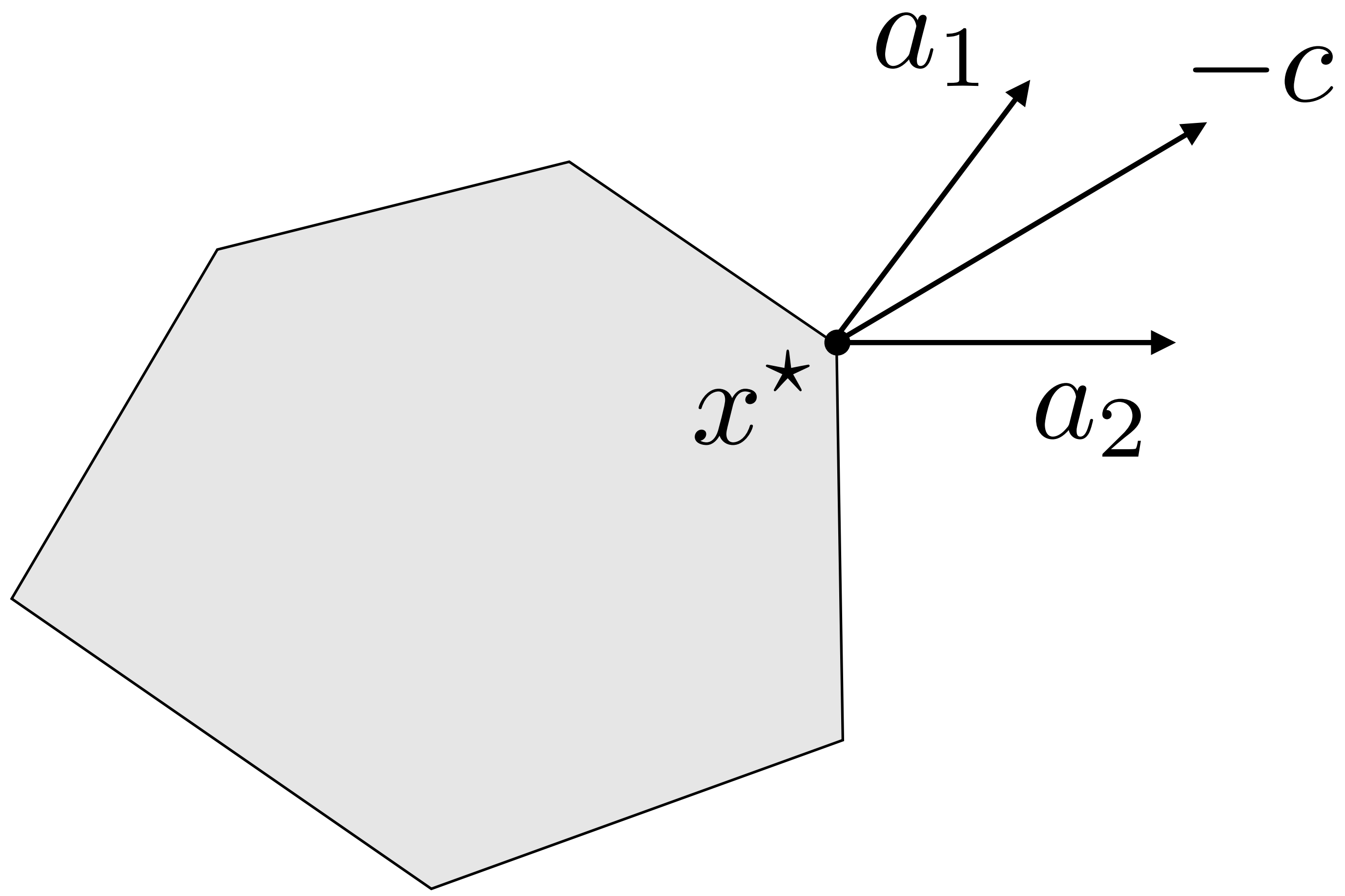
$$y_2 \geq 0, \quad y_4 \geq 0$$

$y = (0, 1, 0, 2)$ satisfies these conditions and proves that x is optimal

Complementary slackness is useful to recover $y^\star$ from $x^\star$

Geometric interpretation

Example in $\mathbb{R}^2$



Two active constraints at optimum: $a_1^T x^\star = b_1, \quad a_2^T x^\star = b_2$

Optimal dual solution y satisfies:

$$A^T y + c = 0, \quad y \geq 0, \quad y_i = 0 \text{ for } i \neq \{1, 2\}$$

In other words, $-c = a_1 y_1 + a_2 y_2$ with $y_1, y_2 \geq 0$

Lagrangian and duality

Primal

minimize $c^T x$

subject to $Ax \leq b$

Dual

maximize $-b^T y$

subject to $A^T y + c = 0$

$y \geq 0$

Dual function

$$g(y) = \underset{x}{\text{minimize}} \left(c^T x + y^T (Ax - b)\right)$$

$$= -b^T y + \underset{x}{\text{minimize}} \left(c + A^T y\right)^T x$$

$$= \begin{cases} -b^T y & \text{if } c + A^T y = 0 \\ -\infty & \text{otherwise} \end{cases}$$

→

Lagrangian

$$L(x, y) = c^T x + y^T (Ax - b)$$

$$\nabla_x L(x, y) = c + A^T y = 0$$

Karush-Kuhn-Tucker conditions

Optimality conditions for linear optimization

Primal

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

Dual

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c = 0 \\ & y \geq 0 \end{array}$$

Primal feasibility $\quad Ax \leq b$

Dual feasibility $\quad \nabla_x L(x, y) = A^T y + c = 0$ and $y \geq 0$

Complementary slackness $\quad y_i(Ax - b)_i = 0, \quad i = 1, \ldots, m$

Karush-Kuhn-Tucker conditions

Solving linear optimization problems

Primal

$$
\begin{aligned}
\text{minimize} \quad & c^T x \\
\text{subject to} \quad & Ax \leq b
\end{aligned}
$$

Dual

$$
\begin{aligned}
\text{maximize} \quad & -b^T y \\
\text{subject to} \quad & A^T y + c = 0 \\
& y \geq 0
\end{aligned}
$$

We can solve our optimization problem by solving a system of equations

$$
\begin{aligned}
\nabla_x L(x, y) = A^T y + c &= 0 \\
b - Ax &\geq 0 \\
y &\geq 0 \\
y^T (b - Ax) &= 0
\end{aligned}
$$

Today's lecture

Sensitivity analysis and game theory

- Primal and dual simplex
- Adding variables and constraints
- Global sensitivity
- Local sensitivity

Primal and dual simplex

Optimality conditions

Primal problem

minimize $c^T x$
subject to $Ax = b$
$x \geq 0$

Dual problem

maximize $-b^T y$
subject to $A^T y + c \geq 0$

x and y are **primal** and **dual** optimal if and only if

- x is **primal feasible**: $Ax = b$ and $x \geq 0$
- y is **dual feasible**: $A^T y + c \geq 0$
- The **duality gap** is zero: $c^T x + b^T y = 0$

Primal and dual basic feasible solutions

Primal problem

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Dual problem

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c \geq 0 \end{array}$$

Given a **basis** matrix A_B

Reduced costs

Primal feasible: $Ax = b,\ x \geq 0 \quad \Rightarrow \quad x_B = A_B^{-1} b \geq 0$

Dual feasible: $A^T y + c \geq 0$. Set $y = -A_B^{-T} c_B$. Dual feasible if $\bar{c} = c + A^T y \geq 0$

Zero duality gap: $c^T x + b^T y = c_B^T x_B - b^T A_B^{-T} c_B = c_B^T x_B - c_B^T A_B^{-1} b = 0$

(by construction)

The primal (dual) simplex method

Primal problem

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Dual problem

$$\begin{aligned} \text{maximize} \quad & -b^T y \\ \text{subject to} \quad & A^T y + c \geq 0 \end{aligned}$$

Primal simplex

- Primal feasibility
- Zero duality gap

↓

Dual feasibility

Dual simplex
(solve dual instead)

- Dual feasibility
- Zero duality gap

↓

Primal feasibility

Adding new constraints and variables

Adding new variables

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & c^T x + c_{n+1} x_{n+1} \\ \text{subject to} & Ax + A_{n+1} x_{n+1} = b \\ & x, x_{n+1} \geq 0 \end{array}$$

Solution $x^\star, y^\star$

Is the solution $(x^\star, 0), y^\star$ **optimal** for the new problem?

Adding new variables
Optimality conditions

$$\begin{array}{ll} \text{minimize} & c^T x + c_{n+1} x_{n+1} \\ \text{subject to} & Ax + A_{n+1} x_{n+1} = b \\ & x, x_{n+1} \geq 0 \end{array}$$

$\longrightarrow$ Solution $(x^\star, 0)$ is still **primal feasible**

Is $y^\star$ still **dual feasible**?

$$A_{n+1}^T y^\star + c_{n+1} \geq 0$$

Yes

$(x^\star, 0)$ still **optimal** for new problem

Otherwise

Primal simplex

Adding new variables

Example

$$\begin{aligned} \text{minimize} \quad & -60x_1 - 30x_2 - 20x_3 && \text{-profit} \\ \text{subject to} \quad & 8x_1 + 6x_2 + x_3 \leq 48 && \text{material} \\ & 4x_1 + 2x_2 + 1.5x_3 \leq 20 && \text{production} \\ & 2x_1 + 1.5x_2 + 0.5x_3 \leq 8 && \text{quality control} \\ & x \geq 0 \end{aligned}$$

$$c = (-60, -30, -20, 0, 0, 0)$$

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned} \qquad A = \begin{bmatrix} 8 & 6 & 1 & 1 & 0 & 0 \\ 4 & 2 & 1.5 & 0 & 1 & 0 \\ 2 & 1.5 & 0.5 & 0 & 0 & 1 \end{bmatrix}$$

$$b = (48, 20, 8)$$

$x^\star = (2, 0, 8, 24, 0, 0), \quad y^\star = (0, 10, 10), \quad c^T x^\star = -280,$ basis $\{1, 3, 4\}$

Adding new variables

Example: add new product?

$$c = (-60, -30, -20, 0, 0, 0, -15)$$

$$A = \begin{bmatrix} 8 & 6 & 1 & 1 & 0 & 0 & 1 \\ 4 & 2 & 1.5 & 0 & 1 & 0 & 1 \\ 2 & 1.5 & 0.5 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$b = (48, 20, 8)$$

minimize $\quad c^T x + c_{n+1} x_{n+1}$

subject to $\quad Ax + A_{n+1} x_{n+1} = b$

$\quad x, x_{n+1} \geq 0$

Previous solution

$x^\star = (2, 0, 8, 24, 0, 0), \quad y^\star = (0, 10, 10), \quad c^T x^\star = -280,$ basis $\{1, 3, 4\}$

$(x^\star, 0)$ **is still optimal**

$$A_{n+1}^T y^\star + c_{n+1} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 10 \\ 10 \end{bmatrix} - 15 = 5 \geq 0$$

Shall we add a new product?

Adding new constraints

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax = b \\
& x \geq 0
\end{array}
\longrightarrow
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax = b \\
& a_{m+1}^T x = b_{m+1} \\
& x \geq 0
\end{array}
$$

Solution $x^\star, y^\star$

Dual

$$
\begin{array}{ll}
\text{maximize} & -b^T y \\
\text{subject to} & A^T y + a_{m+1} y_{m+1} + c \geq 0
\end{array}
$$

Is the solution $x^\star, (y^\star, 0)$ **optimal** for the new problem?

Adding new constraints

Optimality conditions

maximize $\quad -b^T y$

subject to $\quad A^T y + a_{m+1} y_{m+1} + c \geq 0$ ⟶ Solution $(y^\star, 0)$ is still **dual feasible**

Is $x^\star$ still **primal feasible**?

$$\begin{aligned} &Ax = b \\ &a_{m+1}^T x = b_{m+1} \\ &x \geq 0 \end{aligned}$$

Yes

$x^\star$ still **optimal** for new problem

Otherwise

Dual simplex

Adding new constraints

Example

Global sensitivity analysis

Changes in problem data

Goal: extract information from $x^\star, y^\star$ about their sensitivity with respect to changes in problem data

Modified LP

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax = b + u \\
& x \geq 0
\end{array}
$$

Optimal cost $p^\star(u)$

Global sensitivity

Dual of modified LP

$$\begin{array}{ll} \text{maximize} & -(b+u)^T y \\ \text{subject to} & A^T y + c \geq 0 \end{array}$$

Global lower bound

Given $y^\star$ a dual optimal solution for $u = 0$, then

$$\begin{aligned} p^\star(u) &\geq -(b+u)^T y^\star \\ &= p^\star(0) - u^T y^\star \end{aligned}$$

(from weak duality and dual feasibility)

It holds for any u

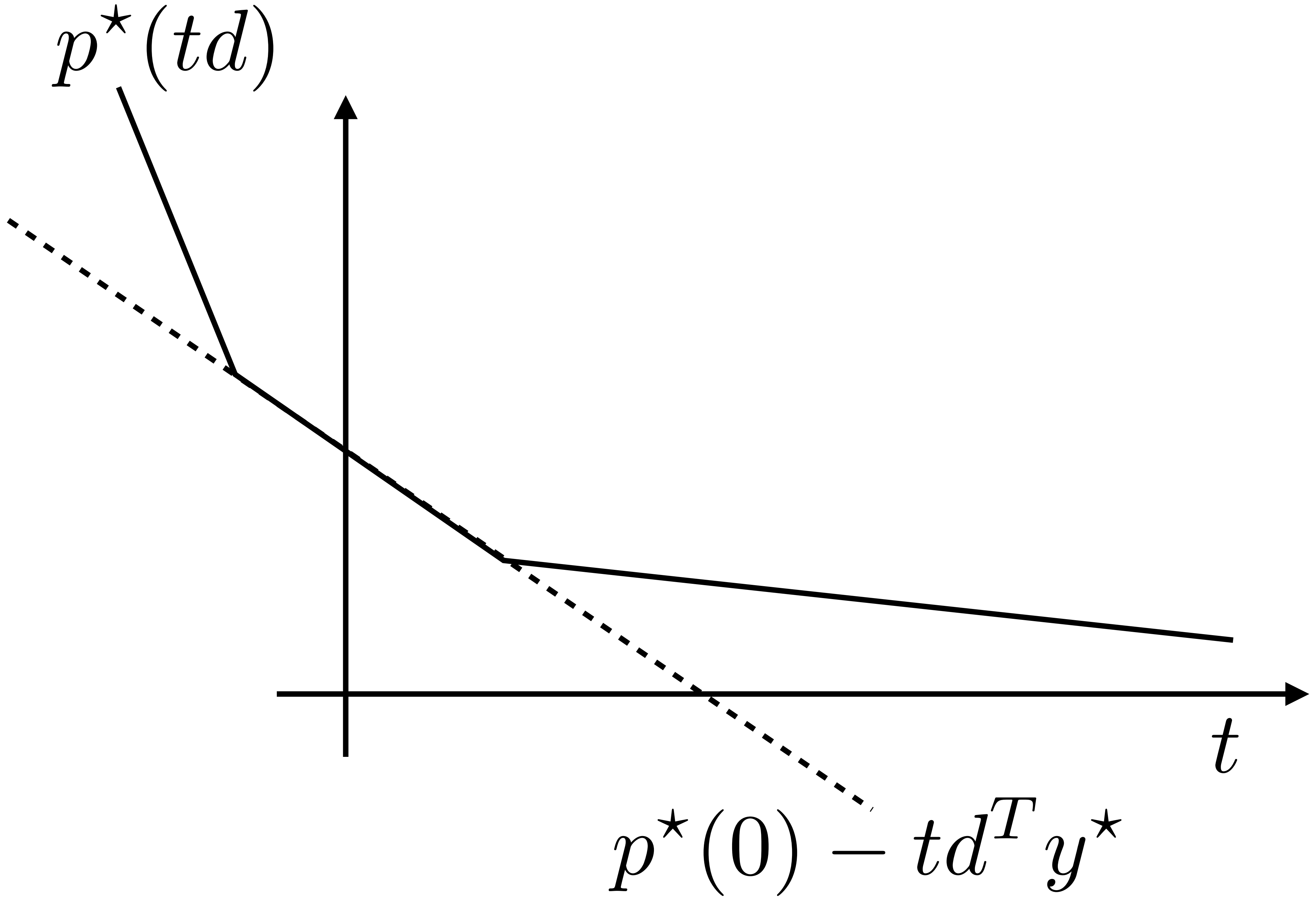
Global sensitivity

Example

Take $u = td$ with $d \in \mathbf{R}^m$ fixed

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b + td \\ & x \geq 0 \end{array}$$

$p^\star(td)$ is the optimal value as a function of t



Sensitivity information (assuming $d^T y^\star \geq 0$)

- $t < 0$ the optimal value increases
- $t > 0$ the optimal value decreases

Optimal value function

$$p^\star(u) = \min\{c^T x \mid Ax = b + u,\ x \geq 0\}$$

Assumption: $p^\star(0)$ is finite

Properties

- $p^\star(u) > -\infty$ everywhere (from global lower bound)
- $p^\star(u)$ is piecewise-linear on its domain

Optimal value function is piecewise linear

Proof

Dual feasible set

$$p^\star(u) = \min\{c^T x \mid Ax = b + u,\ x \geq 0\}$$

$$D = \{y \mid A^T y + c \geq 0\}$$

Assumption: $p^\star(0)$ is finite

If $p^\star(u)$ finite

$$p^\star(u) = \max_{y \in D} -(b+u)^T y = \max_{k=1,\dots,r} -y_k^T u - b^T y_k$$

$y_1, \dots, y_r$ are the extreme points of D

Local sensitivity analysis

Local sensitivity

u in neighborhood of the origin

Original LP

minimize $c^T x$

subject to $Ax = b$

$x \geq 0$

→

Optimal solution

Primal
$$x_i^\star = 0, \quad i \notin B$$
$$x_B^\star = A_B^{-1} b$$

Dual
$$y^\star = -A_B^{-T} c_B$$

Modified LP

minimize $c^T x$

subject to $Ax = b + u$

$x \geq 0$

Modified dual

maximize $-(b+u)^T y$

subject to $A^T y + c \geq 0$

Optimal basis does not change

Modified optimal solution

$$x_B^\star(u) = A_B^{-1}(b+u) = x_B^\star + A_B^{-1} u$$
$$y^\star(u) = y^\star$$

Derivative of the optimal value function

Modified optimal solution

$$x_B^\star(u) = A_B^{-1}(b+u) = x_B^\star + A_B^{-1}u$$
$$y^\star(u) = y^\star$$

Optimal value function

$$\begin{aligned} p^\star(u) &= c^T x^\star(u) \\ &= c^T x^\star + c_B^T A_B^{-1} u \\ &= p^\star(0) - y^{\star T} u \end{aligned}$$

(affine for small u)

Local derivative

$$\nabla p^\star(u) = -y^\star$$

($y^\star$ are the **shadow prices**)

Sensitivity example

minimize	$-60x_1 - 30x_2 - 20x_3$	-profit
subject to	$8x_1 + 6x_2 + x_3 \leq 48$	material
	$4x_1 + 2x_2 + 1.5x_3 \leq 20$	production
	$2x_1 + 1.5x_2 + 0.5x_3 \leq 8$	quality control
	$x \geq 0$	

$x^\star = (2, 0, 8, 24, 0, 0), \quad y^\star = (0, 10, 10), \quad c^T x^\star = -280,$ basis $\{1, 3, 4\}$

What does $y_3^\star = 10$ mean?

Let's increase the quality control budget by 1, i.e., $u = (0, 0, 1)$

$$p^\star(u) = p^\star(0) - y^{\star T} u = -280 - 10 = -290$$

Sensitivity analysis

Today, we learned to:

- **Reuse** primal and dual solutions when variables or constraints are added
- **Analyze** value function as problem parameters change
- **Compute** local sensitivity to parameter perturbations

References

- D. Bertsimas and J. Tsitsiklis: Introduction to Linear Optimization
  - Chapter 5: Sensitivity analysis
- R. Vanderbei: “Linear Programming
  - Chapter 7: Sensitivity and parametric analysis

Next lecture

- Network optimization