

# **ORF307 — Optimization**

## **14. Duality II**

# Ed Forum

- how exactly do we apply Lagrange multipliers in the context of linear programming to find the best lower bounds?
- can we interpret the primal as the dual problem and vice versa? Especially since they are solving for the same thing as stated in the Strong duality theorem.

**Recap**

**Weak and strong duality**

# Optimal objective values

## Primal

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax \leq b \end{aligned}$$

$p^*$  is the primal optimal value

Primal infeasible:  $p^* = +\infty$

Primal unbounded:  $p^* = -\infty$

## Dual

$$\begin{aligned} &\text{maximize} && -b^T y \\ &\text{subject to} && A^T y + c = 0 \\ &&& y \geq 0 \end{aligned}$$

$d^*$  is the dual optimal value

Dual infeasible:  $d^* = -\infty$

Dual unbounded:  $d^* = +\infty$

# Weak duality

## Theorem

If  $x, y$  satisfy:

- $x$  is a feasible solution to the primal problem
  - $y$  is a feasible solution to the dual problem
- $\longrightarrow -b^T y \leq c^T x$

## Proof

We know that  $Ax \leq b$ ,  $A^T y + c = 0$  and  $y \geq 0$ . Therefore,

$$0 \leq y^T (b - Ax) = b^T y - y^T Ax = c^T x + b^T y \quad \blacksquare$$

## Remark

- Any dual feasible  $y$  gives a **lower bound** on the primal optimal value
- Any primal feasible  $x$  gives an **upper bound** on the dual optimal value
- $c^T x + b^T y$  is the **duality gap**

# Weak duality

## Corollaries

### Unboundedness vs feasibility

- Primal unbounded ( $p^* = -\infty$ )  $\Rightarrow$  dual infeasible ( $d^* = -\infty$ )
- Dual unbounded ( $d^* = +\infty$ )  $\Rightarrow$  primal infeasible ( $p^* = +\infty$ )

### Optimality condition

If  $x, y$  satisfy:

- $x$  is a feasible solution to the primal problem
- $y$  is a feasible solution to the dual problem
- The duality gap is zero, *i.e.*,  $c^T x + b^T y = 0$

Then  $x$  and  $y$  are **optimal solutions** to the primal and dual problem respectively

# Strong duality

## Theorem

If a linear optimization problem has an optimal solution, so does its dual, and the optimal value of primal and dual are equal

$$d^* = p^*$$



# Exception to strong duality

## Primal

$$\begin{array}{ll} \text{minimize} & x \\ \text{subject to} & 0 \cdot x \leq -1 \end{array}$$

Optimal value is  $p^* = +\infty$

## Dual

$$\begin{array}{ll} \text{maximize} & y \\ \text{subject to} & 0 \cdot y + 1 = 0 \\ & y \geq 0 \end{array}$$

Optimal value is  $d^* = -\infty$

Both **primal** and **dual infeasible**

# Relationship between primal and dual

	$p^* = +\infty$	$p^*$ finite	$p^* = -\infty$
$d^* = +\infty$	primal inf. dual unb.		
$d^*$ finite		optimal values equal	
$d^* = -\infty$	exception		primal unb. dual inf

- Upper-right excluded by **weak duality**
- (1, 1) and (3, 3) proven by **weak duality**
- (3, 1) and (2, 2) proven by **strong duality**

**Example**

# Production problem

maximize  $x_1 + 2x_2$  ← Profits  
subject to  $x_1 \leq 100$   
 $2x_2 \leq 200$  ← Resources  
 $x_1 + x_2 \leq 150$   
 $x_1, x_2 \geq 0$

## Dualize

1. Transform in inequality form

minimize  $c^T x$   
subject to  $Ax \leq b$

2. Derive dual

maximize  $-b^T y$   
subject to  $A^T y + c = 0$   
 $y \geq 0$

$$c = (-1, -2)$$
$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$b = (100, 200, 150, 0, 0)$$

# Production problem

## Dualized

$$\begin{aligned} &\text{maximize} && -b^T y \\ &\text{subject to} && A^T y + c = 0 \\ &&& y \geq 0 \end{aligned}$$

$$c = (-1, -2)$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$b = (100, 200, 150, 0, 0)$$

## Fill-in data

$$\begin{aligned} &\text{minimize} && 100y_1 + 200y_2 + 150y_3 \\ &\text{subject to} && y_1 + y_3 - y_4 = 1 \\ &&& 2y_2 + y_3 - y_5 = 2 \\ &&& y_1, y_2, y_3, y_4, y_5 \geq 0 \end{aligned}$$



## Eliminate variables

$$\begin{aligned} &\text{minimize} && 100y_1 + 200y_2 + 150y_3 \\ &\text{subject to} && y_1 + y_3 \geq 1 \\ &&& 2y_2 + y_3 \geq 2 \\ &&& y_1, y_2, y_3 \geq 0 \end{aligned}$$

# Production problem

## The dual

$$\text{minimize } 100y_1 + 200y_2 + 150y_3$$

$$\text{subject to } y_1 + y_3 \geq 1$$

$$2y_2 + y_3 \geq 2$$

$$y_1, y_2, y_3 \geq 0$$

## Interpretation

- **Sell all your resources** at a fair (minimum) price
- Selling must be **more convenient than producing**:
  - Product 1 (price 1, needs 1× resource 1 and 3):  $y_1 + y_3 \geq 1$
  - Product 2 (price 2, needs 2× resource 2 and 1× resource 3):  $2y_2 + y_3 \geq 2$

# Today's agenda

## More on duality

- Two-person zero-sum games
- Farkas lemma
- Complementary slackness
- KKT conditions

# Two-person games



# Rock paper scissors

## Rules

At count to three declare one of: Rock, Paper, or Scissors

## Winners

Identical selection is a draw, otherwise:

- Rock beats (“dulls”) scissors
- Scissors beats (“cuts”) paper
- Paper beats (“covers”) rock

Extremely popular: world RPS society, USA RPS league, etc.

# Two-person zero-sum game

- Player 1 (P1) chooses a number  $i \in \{1, \dots, m\}$  (one of  $m$  actions)
- Player 2 (P2) chooses a number  $j \in \{1, \dots, n\}$  (one of  $n$  actions)

Two players make their choice independently

## Rule

Player 1 pays  $A_{ij}$  to player 2

$A \in \mathbf{R}^{m \times n}$  is the **payoff matrix**

## Rock, Paper, Scissors

$$A = \begin{array}{c} \text{R} \\ \text{P} \\ \text{S} \end{array} \begin{array}{ccc} \text{R} & \text{P} & \text{S} \\ \left[ \begin{array}{ccc} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{array} \right] \end{array}$$

# Mixed (randomized) strategies

**Deterministic strategies can be systematically defeated**

## Randomized strategies

- P1 chooses randomly according to distribution  $x$ :

$x_i =$  probability that P1 selects action  $i$

- P2 chooses randomly according to distribution  $y$ :

$y_j =$  probability that P2 selects action  $j$

**Expected payoff** (from P1 P2), if they use mixed-strategies  $x$  and  $y$ ,

$$\sum_{i=1}^m \sum_{j=1}^n x_i y_j A_{ij} = x^T A y$$

# Mixed strategies and probability simplex

**Probability simplex in  $\mathbf{R}^k$**

$$P_k = \{p \in \mathbf{R}^k \mid p \geq 0, \quad \mathbf{1}^T p = 1\}$$

## Mixed strategy

For a game player, a mixed strategy is a distribution over all possible deterministic strategies.

The **set of all mixed strategies** is the probability simplex  $\longrightarrow x \in P_m, \quad y \in P_n$

# Optimal mixed strategies

P1: optimal strategy  $x^*$  is the solution of

$$\begin{array}{ll} \text{minimize} & \max_{y \in P_n} x^T A y \\ \text{subject to} & x \in P_m \end{array}$$



$$\begin{array}{ll} \text{minimize} & \max_{j=1, \dots, n} (A^T x)_j \\ \text{subject to} & x \in P_m \end{array}$$

Inner problem over  
deterministic  
strategies (**vertices**)

P2: optimal strategy  $y^*$  is the solution of

$$\begin{array}{ll} \text{maximize} & \min_{x \in P_m} x^T A y \\ \text{subject to} & y \in P_n \end{array}$$



$$\begin{array}{ll} \text{maximize} & \min_{i=1, \dots, m} (A y)_i \\ \text{subject to} & y \in P_n \end{array}$$

Optimal strategies  $x^*$  and  $y^*$  can be computed using **linear optimization**

# Minmax theorem

## Theorem

$$\max_{y \in P_n} \min_{x \in P_m} x^T A y = \min_{x \in P_m} \max_{y \in P_n} x^T A y$$

## Proof

The optimal  $x^*$  is the solution of

$$\begin{aligned} &\text{minimize} && t \\ &\text{subject to} && A^T x \leq t \mathbf{1} \\ &&& \mathbf{1}^T x = 1 \\ &&& x \geq 0 \end{aligned}$$

The optimal  $y^*$  is the solution of

$$\begin{aligned} &\text{maximize} && w \\ &\text{subject to} && A y \geq w \mathbf{1} \\ &&& \mathbf{1}^T y = 1 \\ &&& y \geq 0 \end{aligned}$$

The two LPs are **duals** and by **strong duality** the equality follows. ■

# Nash equilibrium

## Theorem

$$\max_{y \in P_n} \min_{x \in P_m} x^T A y = \min_{x \in P_m} \max_{y \in P_n} x^T A y$$

## Consequence

The pair of mixed strategies  $(x^*, y^*)$  attains the **Nash equilibrium** of the two-person matrix game, i.e.,

$$x^T A y^* \geq x^{*T} A y^* \geq x^{*T} A y, \quad \forall x \in P_m, \forall y \in P_n$$

# Example

$$A = \begin{bmatrix} 4 & 2 & 0 & -3 \\ -2 & -4 & -3 & 3 \\ -2 & -3 & 4 & 1 \end{bmatrix}$$

## Optimal deterministic strategies

$$\min_i \max_j A_{ij} = 3 > -2 = \max_j \min_i A_{ij}$$

## Optimal mixed strategies

$$x^* = (0.37, 0.33, 0.3), \quad y^* = (0.4, 0, 0.13, 0.47)$$

## Expected payoff

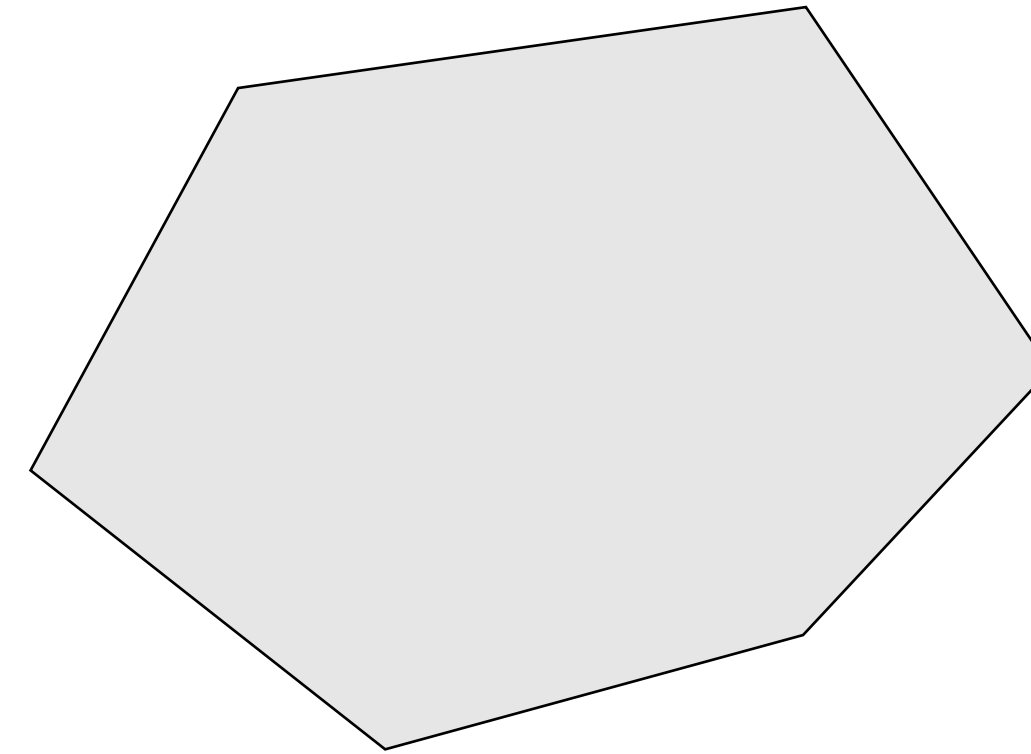
$$x^{*T} A y^* = 0.2$$



# Farkas lemma

# Feasibility of polyhedra

$$P = \{x \mid Ax = b, \quad x \geq 0\}$$



How to show that  $P$  is **feasible**?

Easy: we just need to provide an  $x \in P$ , i.e., a **certificate**

How to show that  $P$  is **infeasible**?

# Farkas lemma

## Theorem

Given  $A$  and  $b$ , exactly one of the following statements is true:

1. There exists an  $x$  with  $Ax = b, x \geq 0$
2. There exists a  $y$  with  $A^T y \geq 0, b^T y < 0$

# Farkas lemma

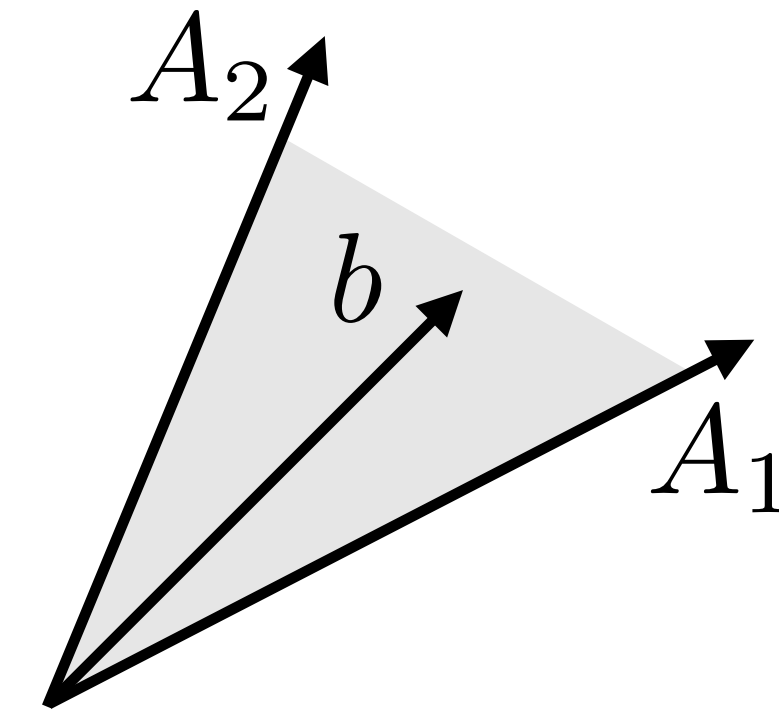
## Geometric interpretation

### 1. First alternative

There exists an  $x$  with  $Ax = b, x \geq 0$

$$b = \sum_{i=1}^n x_i A_i, \quad x_i \geq 0, \quad i = 1, \dots, n$$

$b$  is in the cone generated by the columns of  $A$

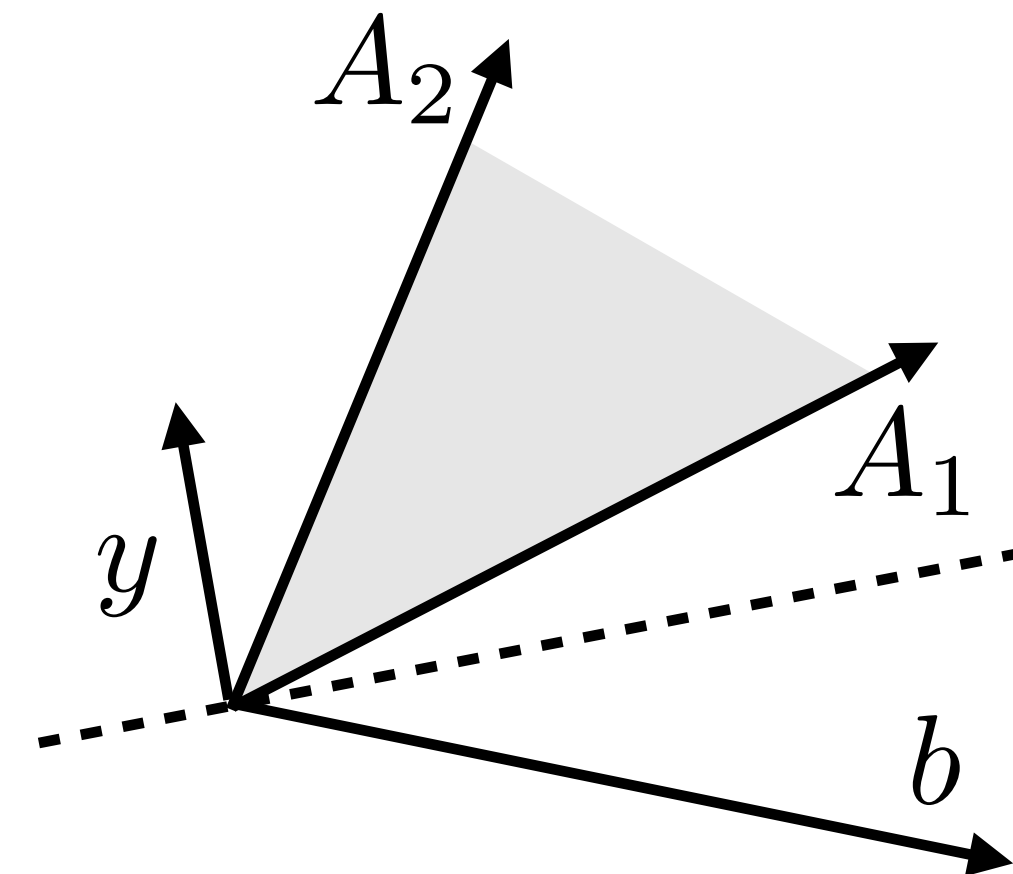


### 2. Second alternative

There exists a  $y$  with  $A^T y \geq 0, b^T y < 0$

$$y^T A_i \geq 0, \quad i = 1, \dots, m, \quad y^T b < 0$$

The hyperplane  $y^T z = 0$   
separates  $b$  from  $A_1, \dots, A_n$



# Farkas lemma

There exists  $x$  with  $Ax = b$ ,  $x \geq 0$     **OR**    There exists  $y$  with  $A^T y \geq 0$ ,  $b^T y < 0$

## Proof

**1 and 2 cannot be both true (easy)**

$$x \geq 0, Ax = b \text{ and } y^T A \geq 0 \quad \longrightarrow \quad y^T b = y^T Ax \geq 0$$

# Farkas lemma

There exists  $x$  with  $Ax = b, x \geq 0$       **OR**      There exists  $y$  with  $A^T y \geq 0, b^T y < 0$

## Proof

**1 and 2 cannot be both false (duality)**

### Primal

minimize    0  
subject to    $Ax = b$   
               $x \geq 0$

### Dual

maximize     $-b^T y$   
subject to    $A^T y \geq 0$



$y = 0$  always feasible

**Strong duality holds**

$$d^* \neq -\infty, \quad p^* = d^*$$

# Farkas lemma

There exists  $x$  with  $Ax = b$ ,  $x \geq 0$       **OR**      There exists  $y$  with  $A^T y \geq 0$ ,  $b^T y < 0$

## Proof

**1 and 2 cannot be both false (duality)**

### Primal

minimize    0  
subject to    $Ax = b$   
               $x \geq 0$

### Dual

maximize     $-b^T y$   
subject to    $A^T y \geq 0$

**Alternative 1:** primal feasible  $p^* = d^* = 0$

$b^T y \geq 0$  for all  $y$  such that  $A^T y \geq 0$

# Farkas lemma

There exists  $x$  with  $Ax = b$ ,  $x \geq 0$       **OR**      There exists  $y$  with  $A^T y \geq 0$ ,  $b^T y < 0$

## Proof

**1 and 2 cannot be both false (duality)**

### Primal

minimize    0  
subject to    $Ax = b$   
               $x \geq 0$

### Dual

maximize     $-b^T y$   
subject to    $A^T y \geq 0$

**Alternative 2:** primal infeasible  $p^* = d^* = +\infty$

There exists  $y$  such that  $A^T y \geq 0$  and  $b^T y < 0$

$y$  is an  
**infeasibility  
certificate**



# Farkas lemma

## Many variations

There exists  $x$  with  $Ax = b, x \geq 0$

**OR**

There exists  $y$  with  $A^T y \geq 0, b^T y < 0$

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There exists  $x$  with  $Ax \leq b, x \geq 0$

**OR**

There exists  $y$  with  $A^T y \geq 0, b^T y < 0, y \geq 0$

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There exists  $x$  with  $Ax \leq b$

**OR**

There exists  $y$  with  $A^T y = 0, b^T y < 0, y \geq 0$

**Complementary slackness**

# Optimality conditions

## Primal

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax \leq b \end{aligned}$$

## Dual

$$\begin{aligned} &\text{maximize} && -b^T y \\ &\text{subject to} && A^T y + c = 0 \\ &&& y \geq 0 \end{aligned}$$

$x$  and  $y$  are **primal** and **dual** optimal if and only if

- $x$  is **primal feasible**:  $Ax \leq b$
- $y$  is **dual feasible**:  $A^T y + c = 0$  and  $y \geq 0$
- The **duality gap** is zero:  $c^T x + b^T y = 0$

Can we **relate**  $x$  and  $y$  (not only the objective)?

# Complementary slackness

## Primal

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax \leq b \end{aligned}$$

## Dual

$$\begin{aligned} &\text{maximize} && -b^T y \\ &\text{subject to} && A^T y + c = 0 \\ &&& y \geq 0 \end{aligned}$$

## Theorem

Primal, dual feasible  $x, y$  are optimal if and only if

$$y_i(b_i - a_i^T x) = 0, \quad i = 1, \dots, m$$

i.e., at optimum,  $b - Ax$  and  $y$  have a **complementary sparsity** pattern:

$$\begin{aligned} y_i > 0 &\Rightarrow a_i^T x = b_i \\ a_i^T x < b_i &\Rightarrow y_i = 0 \end{aligned}$$

# Complementary slackness

## Primal

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax \leq b \end{aligned}$$

## Dual

$$\begin{aligned} &\text{maximize} && -b^T y \\ &\text{subject to} && A^T y + c = 0 \\ &&& y \geq 0 \end{aligned}$$

## Proof

The duality gap at primal feasible  $x$  and dual feasible  $y$  can be written as

$$c^T x + b^T y = (-A^T y)^T x + b^T y = (b - Ax)^T y = \sum_{i=1}^m y_i (b_i - a_i^T x) = 0$$

Since all the elements of the sum are nonnegative, they must all be 0 ■

For **feasible**  $x$  and  $y$  **complementary slackness = zero duality gap**

# Example

$$\begin{array}{ll} \text{minimize} & -4x_1 - 5x_2 \\ \text{subject to} & \begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 3 \\ 0 \\ 3 \end{bmatrix} \end{array}$$

Let's **show** that feasible  $x = (1, 1)$  is optimal

Second and fourth constraints are active at  $x \longrightarrow y = (0, y_2, 0, y_4)$

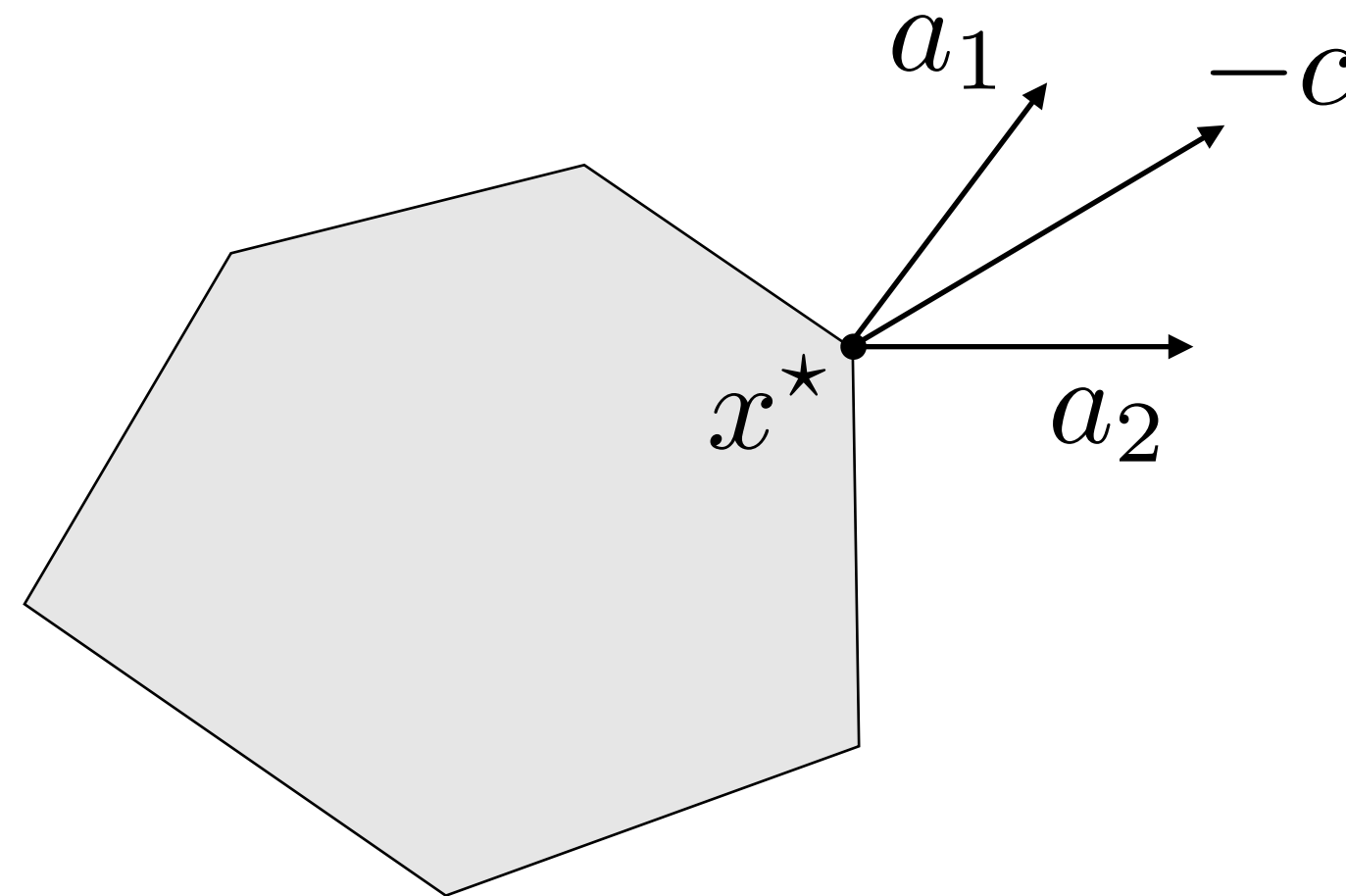
$$A^T y = -c \quad \Rightarrow \quad \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_2 \\ y_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \quad \text{and} \quad y_2 \geq 0, \quad y_4 \geq 0$$

$y = (0, 1, 0, 2)$  satisfies these conditions and proves that  $x$  is optimal

**Complementary slackness** is useful to recover  $y^*$  from  $x^*$

# Geometric interpretation

Example in  $\mathbb{R}^2$



Two active constraints at optimum:  $a_1^T x^* = b_1$ ,  $a_2^T x^* = b_2$

Optimal dual solution  $y$  satisfies:

$$A^T y + c = 0, \quad y \geq 0, \quad y_i = 0 \text{ for } i \neq \{1, 2\}$$

In other words,  $-c = a_1 y_1 + a_2 y_2$  with  $y_1, y_2 \geq 0$

# KKT Conditions



# Lagrangian and duality

## Primal

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax \leq b \end{aligned}$$

## Dual

$$\begin{aligned} &\text{maximize} && -b^T y \\ &\text{subject to} && A^T y + c = 0 \\ &&& y \geq 0 \end{aligned}$$

## Dual function

$$\begin{aligned} g(y) &= \text{minimize}_x (c^T x + y^T (Ax - b)) \\ &= -b^T y + \text{minimize}_x (c + A^T y)^T x \\ &= \begin{cases} -b^T y & \text{if } c + A^T y = 0 \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

## Lagrangian

$$L(x, y) = c^T x + y^T (Ax - b)$$

$$\nabla_x L(x, y) = c + A^T y = 0$$

# Karush-Kuhn-Tucker conditions

## Optimality conditions for linear optimization

### Primal

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax \leq b \end{aligned}$$

### Dual

$$\begin{aligned} &\text{maximize} && -b^T y \\ &\text{subject to} && A^T y + c = 0 \\ &&& y \geq 0 \end{aligned}$$

### Primal feasibility

$$Ax \leq b$$

### Dual feasibility

$$\nabla_x L(x, y) = A^T y + c = 0 \quad \text{and} \quad y \geq 0$$

### Complementary slackness

$$y_i (Ax - b)_i = 0, \quad i = 1, \dots, m$$

# Karush-Kuhn-Tucker conditions

## Solving linear optimization problems

### Primal

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax \leq b \end{aligned}$$

### Dual

$$\begin{aligned} &\text{maximize} && -b^T y \\ &\text{subject to} && A^T y + c = 0 \\ &&& y \geq 0 \end{aligned}$$

We can solve our optimization problem by solving a system of equations

$$\begin{aligned} \nabla_x L(x, y) &= A^T y + c = 0 \\ b - Ax &\geq 0 \\ y &\geq 0 \\ y^T (b - Ax) &= 0 \end{aligned}$$

# Linear optimization duality

Today, we learned to:

- **Interpret** linear optimization duality using game theory
- **Prove** Farkas lemma using duality
- **Geometrically link** primal and dual solutions with complementary slackness
- **Derive** KKT optimality conditions

# References

- Bertsimas and Tsitsiklis: Introduction to Linear Optimization
  - Chapter 4: Duality theory
- R. Vanderbei: Linear Programming — Foundations and Extensions
  - Chapter 11: Game Theory

# Next lecture

- Sensitivity analysis