# ORF307 – Optimization

## 13. Duality

Bartolomeo Stellato — Spring 2024

# Ed Forum

- For phase 1 vs phase 2, I understand it finds an extreme point that is not necessarily the optimal but I do not understand how it gets this point by setting x = 0 and y = b.

- how does the simplex method's approach to handling degeneracy and cycling impact its efficiency and reliability in practical applications, such as logistics or resource allocation? Are there examples where alternative methods might be more effective due to these issues?

# Complexity

# Complexity of a single simplex iteration

1. Compute the reduced costs $\bar{c}$
   - Solve $A_B^T p = c_B$
   - $\bar{c} = c - A^T p$
2. If $\bar{c} \geq 0$, $x$ **optimal**. **break**
3. Choose $j$ such that $\bar{c}_j < 0$
4. Compute search direction $d$ with $d_j = 1$ and $A_B d_B = -A_j$
5. If $d_B \geq 0$, the problem is **unbounded** and the optimal value is $-\infty$. **break**
6. Compute step length $\theta^\star = \min_{\{i \in B | d_i < 0\}} \left( -\frac{x_i}{d_i} \right)$
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**Bottleneck**

Two linear systems

# Linear system solutions

**Very similar linear systems**

$$A_B^T p = c_B$$

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**$LU$ factorization**

$(2/3)n^3$ flops

$$A_B = PLU$$

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$\longrightarrow$

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**Easy linear systems**

$4n^2$ flops

$$U^T L^T P^T p = c_B$$
$$PLU d_B = -A_j$$

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**Easy linear systems**

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**Factorization is expensive**

**Do we need to recompute it at every iteration?**

# Basis update

**Index update**

- $j$ enters ($x_j$ becomes $\theta^\star$)
- $i = B(\ell)$ exists ($x_i$ becomes $0$)

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$$A = \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{bmatrix}$$

**Example**

$$B = \{4, 1, 6\} \quad \rightarrow \quad \bar{B} = \{4, 1, 2\}$$

- $2$ enters
- $6 = B(3)$ exists

# Basis update

### Index update

- $j$ enters ($x_j$ becomes $\theta^\star$)
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### Basis matrix change

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### Example

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$$B = \{4, 1, 6\} \quad \rightarrow \quad \bar{B} = \{4, 1, 2\}$$

- $2$ enters
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$$A_{\bar{B}} = \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}}_{A_B} + \underbrace{\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}}_{A_2 e_3^T} - \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A_6 e_3^T} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

# Smarter linear system solution

**Basis matrix change**

$$A_{\bar{B}} = A_B + \overbrace{(A_i - A_j)}^{v} e_\ell^T$$

**Matrix inversion lemma**
(from homework 2)

$$\longrightarrow \quad (A_B + ve_\ell^T)^{-1} = \left( I - \frac{1}{1 + e_\ell^T A_B^{-1} v} A_B^{-1} v e_\ell^T \right) A_B^{-1}$$

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**Solve** $A_{\bar{B}} d_{\bar{B}} = -A_j$

1. Solve $A_B z^1 = e_\ell$ ($2n^2$ flops)
2. Solve $A_B z^2 = -A_j$ ($2n^2$ flops)
3. Solve $d_{\bar{B}} = z^2 - \frac{v^T z^2}{1 + v^T z^1} z^1$

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**Remarks**

- Same complexity for $A_B^T p = c_B$ ($4n^2$ flops)
- $k$-th next iteration ($4kn^2$ flops, derive as exercise...)
- Once in a while (e.g., $k = 100$), better to refactor $A_B$

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**Matrix inversion lemma trick**
$\approx n^2$ per iteration
(very cheap)

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**How many iterations do we need?**

# Complexity of the simplex method

## Example of worst-case behavior

**Innocent-looking problem**

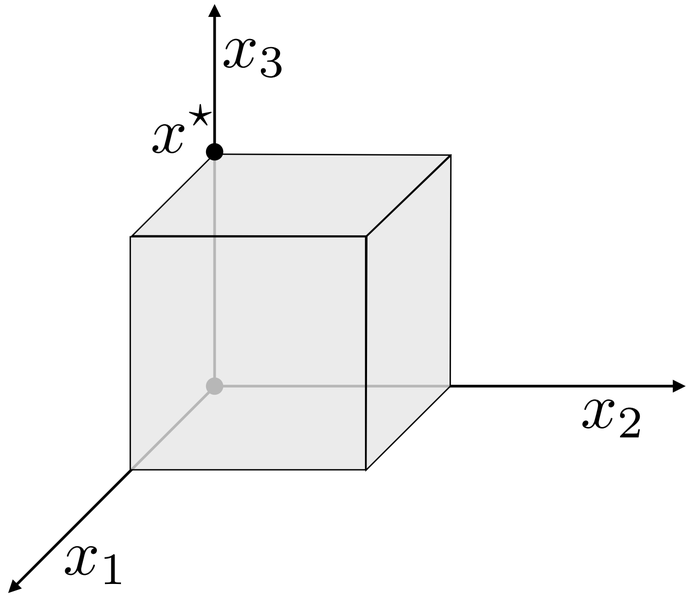
minimize $\quad -x_n$

subject to $\quad 0 \leq x \leq 1$

$2^n$ **vertices**

$2^n/2$ vertices: cost = $1$

$2^n/2$ vertices: cost = $0$



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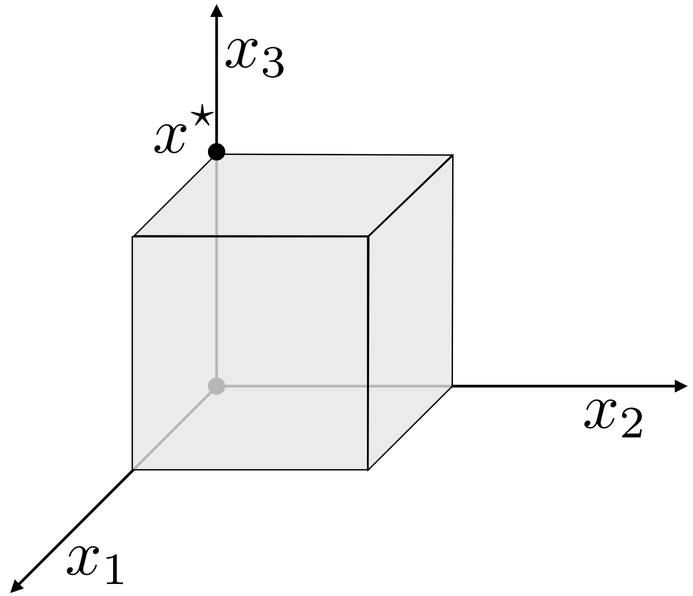


**Perturb unit cube**

minimize $\quad -x_n$

subject to $\quad \epsilon \leq x_1 \leq 1$

$$\epsilon x_{i-1} \leq x_i \leq 1 - \epsilon x_{i-1}, \quad i = 2, \ldots, n$$

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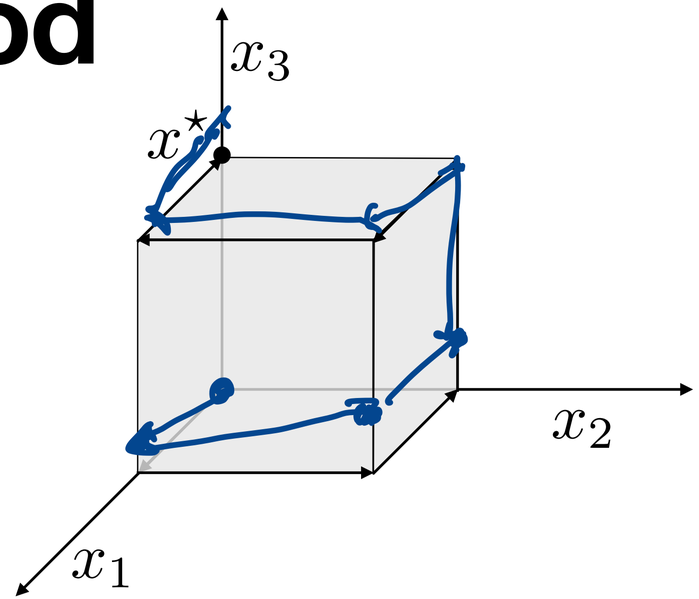

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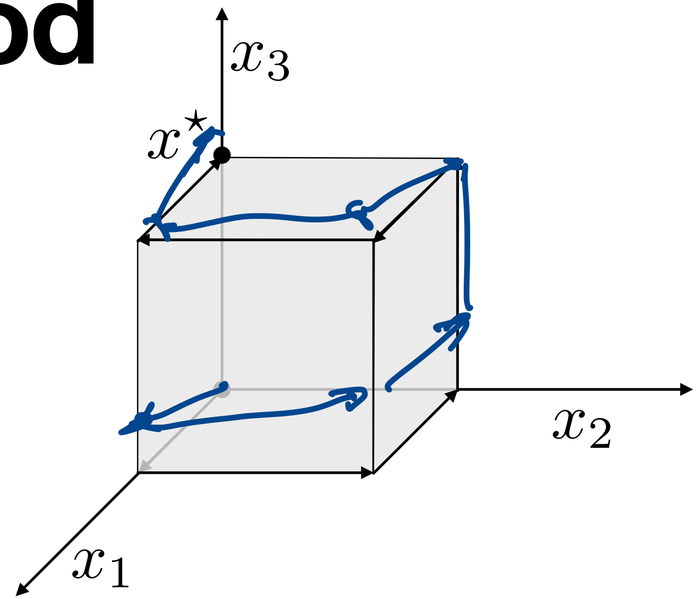


**Theorem**

- The vertices can be ordered so that each one is adjacent to and has a **lower cost than the previous one**
- There exists a pivoting rule under which the simplex method terminates after $2^n - 1$ **iterations**

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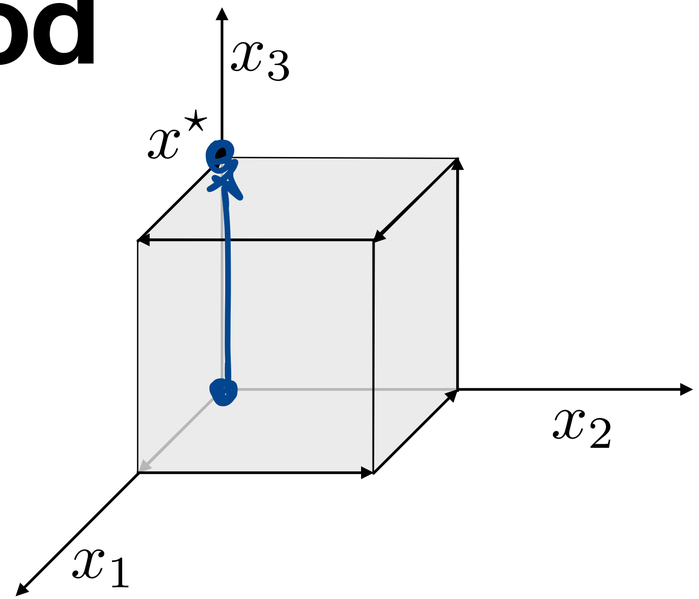


**Theorem**

- The vertices can be ordered so that each one is adjacent to and has a **lower cost than the previous one**
- There exists a pivoting rule under which the simplex method terminates after $2^n - 1$ **iterations**

**Remark**

- A **different pivot rule** would have converged in one iteration.
- We have a bad example for every pivot rule.

# Complexity of the simplex method

We do **not know any polynomial version of the simplex method**, no matter which pivoting rule we pick.

Still **open research question!**

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**Worst-case**

There are problem instances where the simplex method will run an **exponential number of iterations** in terms of the dimensions, e.g. $2^n$

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We do **not know any polynomial version of the simplex method**, no matter which pivoting rule we pick. ⟶ Still **open research question!**

**Worst-case**

There are problem instances where the simplex method will run an **exponential number of iterations** in terms of the dimensions, e.g. $2^n$

**Good news: average-case**
**Practical performance** is very good. On average, it stops in $n$ iterations.

# Average simplex complexity

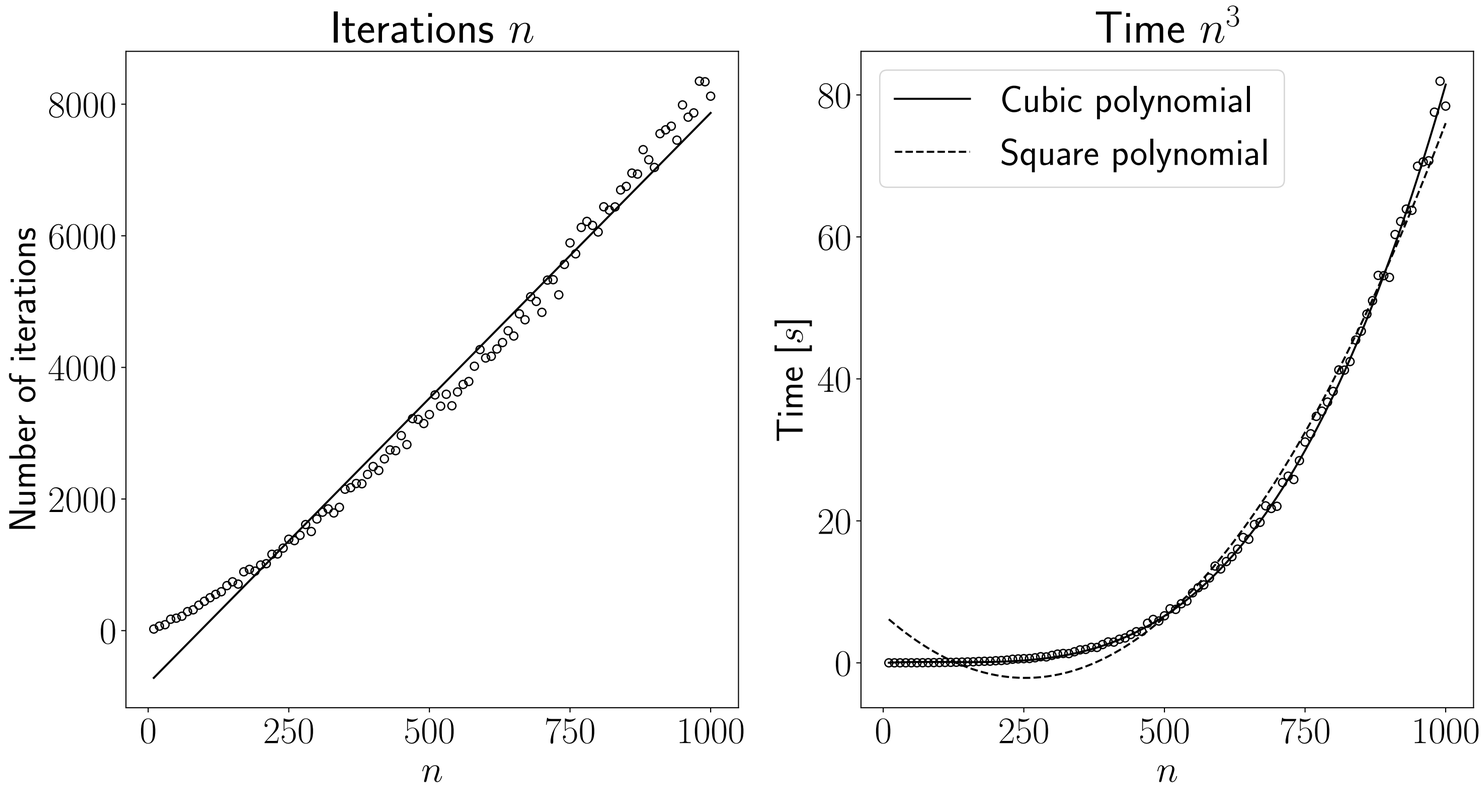
**Random LPs**

minimize $c^T x$

subject to $Ax \leq b$

$n$ variables

$3n$ constraints



# Recap

# Linear optimization formulations

**Standard form LP**

minimize $c^T x$

subject to $Ax = b$

$x \geq 0$

**Inequality form LP**

minimize $c^T x$

subject to $Ax \leq b$

# Today’s agenda

## Duality

- Obtaining lower bounds
- The dual problem
- Weak and strong duality

# Obtaining lower bounds

# Obtaining lower bounds

## A simple example

$$\begin{aligned} \text{minimize} \quad & x_1 + 3x_2 \\ \text{subject to} \quad & x_1 + 3x_2 \geq 2 \end{aligned}$$

What is a **lower bound** on the optimal cost?

# Obtaining lower bounds

## A simple example

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What is a **lower bound** on the optimal cost?

A lower bound is $2$ because $x_1 + 3x_2 \geq 2$

# Obtaining lower bounds

## Another example

$$\begin{aligned} \text{minimize} \quad & x_1 + 3x_2 \\ \text{subject to} \quad & x_1 + x_2 \geq 2 \\ & x_2 \geq 1 \end{aligned}$$

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What is a **lower bound** on the optimal cost?

Let's sum the constraints

$$\begin{aligned} & 1 \cdot (x_1 + x_2 \geq 2) \\ + \; & 2 \cdot (x_2 \geq 1) \\ = \; & x_1 + 3x_2 \geq 4 \end{aligned}$$

# Obtaining lower bounds

## Another example

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A lower bound is $4$

# Obtaining lower bounds

## A more interesting example

minimize $x_1 + 3x_2$

subject to $x_1 + x_2 \geq 2$

$x_2 \geq 1$

$x_1 - x_2 \geq 3$

How can we obtain a lower bound?

# Obtaining lower bounds

## A more interesting example

$$\begin{aligned} \text{minimize} \quad & x_1 + 3x_2 \\ \text{subject to} \quad & x_1 + x_2 \geq 2 \\ & x_2 \geq 1 \\ & x_1 - x_2 \geq 3 \end{aligned}$$

How can we obtain a lower bound?

**Add constraints**

$$\begin{aligned} & y_1 \cdot (x_1 + x_2 \geq 2) \\ + \; & y_2 \cdot (x_2 \geq 1) \\ + \; & y_3 \cdot (x_1 - x_2 \geq 3) \\ \Rightarrow \; & (y_1 + y_3)x_1 + (y_1 + y_2 - y_3)x_2 \geq 2y_1 + y_2 + 3y_3 \end{aligned}$$

# Obtaining lower bounds

## A more interesting example

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**Bound**

# Obtaining lower bounds

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minimize $x_1 + 3x_2$

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How can we obtain a lower bound?

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$$y_1 \cdot (x_1 + x_2 \geq 2)$$
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$$\Rightarrow (y_1 + y_3)x_1 + (y_1 + y_2 - y_3)x_2 \geq 2y_1 + y_2 + 3y_3$$

**Match cost coefficients**

$$y_1 + y_3 = 1$$
$$y_1 + y_2 - y_3 = 3$$
$$y_1, y_2, y_3 \geq 0$$

**Bound**

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**Bound**

**Many options**

$y = (1, 2, 0) \Rightarrow$ Bound $4$

# Obtaining lower bounds

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$y = (1, 2, 0) \Rightarrow$ Bound $4$

$y = (0, 4, 1) \Rightarrow$ Bound $7$

# Obtaining lower bounds

## A more interesting example

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**Bound**

**Many options**

$y = (1, 2, 0) \Rightarrow$ Bound $4$

$y = (0, 4, 1) \Rightarrow$ Bound $7$

How can we get the **best one**?

# Obtaining lower bounds

## A more interesting example — Best lower bound

We can obtain the **best lower bound** by solving the following problem

$$
\begin{aligned}
\text{maximize} \quad & 2y_1 + y_2 + 3y_3 \\
\text{subject to} \quad & y_1 + y_3 = 1 \\
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This linear optimization problem is called the **dual problem**

# The dual problem

# Lagrange multipliers

Consider the LP in standard form

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

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Relax the constraint

$$g(y) = \begin{array}{ll} \underset{x}{\text{minimize}} & c^T x + y^T (Ax - b) \\ \text{subject to} & x \geq 0 \end{array}$$

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Lower bound

$$g(y) \leq c^T x^\star + y^T (Ax^\star - b) = c^T x^\star$$

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Lower bound

$$g(y) \leq c^T x^\star + y^T (Ax^\star - b) = c^T x^\star$$

Best lower bound

$$\begin{array}{ll} \underset{y}{\text{maximize}} & g(y) \end{array}$$

# The dual

**Dual function**

$$g(y) = \underset{x \geq 0}{\text{minimize}} \left( c^T x + y^T (Ax - b) \right)$$

$$= -b^T y + \underset{x \geq 0}{\text{minimize}} \left( c + A^T y \right)^T x$$

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**Dual problem** (find the best bound)

$$\underset{y}{\text{maximize}} \quad g(y) \quad = \quad \begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c \geq 0 \end{array}$$

# Primal and dual problems

**Primal problem**

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

**Dual problem**

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c \geq 0 \end{array}$$

Primal variable $x \in \mathbf{R}^n$

Dual variable $y \in \mathbf{R}^m$

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Dual variable $y \in \mathbf{R}^m$

The dual problem carries **useful information** for the primal problem

Duality is useful also to **solve** optimization problems

# Dual of inequality form LP

What if you find an LP with inequalities?

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax \leq b \end{aligned}$$

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Relax the constraint

$$g(y) = \underset{x}{\text{minimize}} \quad c^T x + y^T (Ax - b)$$

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Relax the constraint

$$g(y) = \underset{x}{\text{minimize}} \quad c^T x + y^T (Ax - b)$$

Lower bound

$$g(y) \leq c^T x^\star + y^T (Ax^\star - b) \leq c^T x^\star$$

we must have $y \geq 0$

# Dual of LP with inequalities

## Derivation

**Dual function**

$$g(y) = \underset{x}{\text{minimize}} \left(c^T x + y^T (Ax - b)\right)$$

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$$\underset{y}{\text{maximize}} \quad g(y) \quad = \quad \begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c = 0 \\ & y \geq 0 \end{array}$$

# General forms

## Standard form LP

**Primal**

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

**Dual**

$$\begin{aligned} \text{maximize} \quad & -b^T y \\ \text{subject to} \quad & A^T y + c \geq 0 \end{aligned}$$

## Inequality form LP

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## LP with inequalities and equalities

**Primal**

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax \leq b \\ & Cx = d \end{aligned}$$

**Dual**

$$\begin{aligned} \text{maximize} \quad & -b^T y - d^T z \\ \text{subject to} \quad & A^T y + C^T z + c = 0 \\ & y \geq 0 \end{aligned}$$

# Example from before

minimize $x_1 + 3x_2$

subject to $x_1 + x_2 \geq 2$

$x_2 \geq 1$

$x_1 - x_2 \geq 3$

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**Inequality form LP**

minimize $c^T x$

subject to $Ax \leq b$

$$c = (1, 3)$$

$$A = \begin{bmatrix} -1 & -1 \\ 0 & -1 \\ -1 & 1 \end{bmatrix}$$

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**Dual**

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$y \geq 0$

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$$b = (-2, -1, -3)$$

**Dual**

maximize $-b^T y$

subject to $A^T y + c = 0$

$y \geq 0$

maximize $2y_1 + y_2 + 3y_3$

subject to $-y_1 - y_3 = -1$

$-y_1 - y_2 + y_3 = -3$

$y_1, y_2, y_3 \geq 0$

# To memorize

**Ways to get the dual**

- Derive dual function directly
- Transform the problem in inequality form LP and dualize

**Sanity-checks and signs convention**

- Consider constraints as $Ax - b \leq 0$ or $Ax - b = 0$ (not $\geq 0$)
- Each dual variable is associated to a primal constraint
- $y$ free for primal equalities and $y \geq 0$ for primal inequalities

# Dual of the dual

**Theorem**

If we transform the primal into its dual and then transform the dual to its dual, we obtain a problem equivalent to the original problem. In other words, the **dual of the dual is the primal**.

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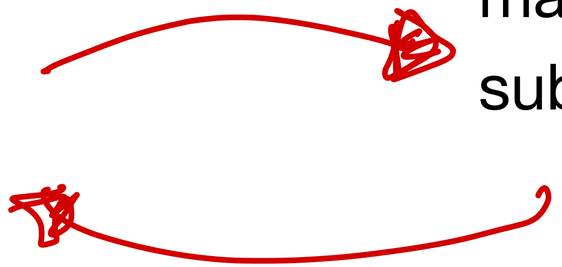
If we transform the primal into its dual and then transform the dual to its dual, we obtain a problem equivalent to the original problem. In other words, the **dual of the dual is the primal**.

**Exercise**

Derive dual and dualize again

| **Primal** | | | **Dual** | |
|---|---|---|---|---|
| minimize | $c^T x$ | | maximize | $-b^T y - d^T z$ |
| subject to | $Ax \leq b$ | | subject to | $A^T y + C^T z + c = 0$ |
| | $Cx = d$ | | | $y \geq 0$ |



# Dual of the dual

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If we transform the primal into its dual and then transform the dual to its dual, we obtain a problem equivalent to the original problem. In other words, the **dual of the dual is the primal**.

**Exercise**

Derive dual and dualize again

**Primal**

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax \leq b \\ & Cx = d \end{aligned}$$

**Dual**

$$\begin{aligned} \text{maximize} \quad & -b^T y - d^T z \\ \text{subject to} \quad & A^T y + C^T z + c = 0 \\ & y \geq 0 \end{aligned}$$

**Theorem**

If we **transform a linear optimization problem to another form** (inequality form, standard form, inequality and equality form), **the dual of the two problems will be equivalent**.

# Weak and strong duality

# Optimal objective values

**Primal**

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

**Dual**

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c = 0 \\ & y \geq 0 \end{array}$$

$p^\star$ is the primal optimal value

$d^\star$ is the dual optimal value

Primal infeasible: $p^\star = +\infty$
Primal unbounded: $p^\star = -\infty$

Dual infeasible: $d^\star = -\infty$
Dual unbounded: $d^\star = +\infty$

# Weak duality

**Theorem**

If $x, y$ satisfy:

- $x$ is a feasible solution to the primal problem
- $y$ is a feasible solution to the dual problem

$\longrightarrow \quad -b^T y \leq c^T x$

# Weak duality

**Theorem**

If $x, y$ satisfy:

- $x$ is a feasible solution to the primal problem
- $y$ is a feasible solution to the dual problem

$\longrightarrow \quad -b^T y \leq c^T x$

**Proof**

We know that $Ax \leq b$, $A^T y + c = 0$ and $y \geq 0$. Therefore,

$$0 \leq y^T(b - Ax) = b^T y - y^T Ax = c^T x + b^T y$$

# Weak duality

**Theorem**

If $x, y$ satisfy:

- $x$ is a feasible solution to the primal problem
- $y$ is a feasible solution to the dual problem

$$\longrightarrow \quad -b^T y \leq c^T x$$

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We know that $Ax \leq b$, $A^T y + c = 0$ and $y \geq 0$. Therefore,

$$0 \leq y^T(b - Ax) = b^T y - y^T Ax = c^T x + b^T y$$

■

**Remark**

- Any dual feasible $y$ gives a **lower bound** on the primal optimal value
- Any primal feasible $x$ gives an **upper bound** on the dual optimal value
- $c^T x + b^T y$ is the **duality gap**

# Weak duality

## Corollaries

**Unboundedness vs feasibility**

- Primal unbounded ($p^\star = -\infty$) $\Rightarrow$ dual infeasible ($d^\star = -\infty$)
- Dual unbounded ($d^\star = +\infty$) $\Rightarrow$ primal infeasible ($p^\star = +\infty$)

# Weak duality

## Corollaries

**Unboundedness vs feasibility**

- Primal unbounded ($p^\star = -\infty$) $\Rightarrow$ dual infeasible ($d^\star = -\infty$)
- Dual unbounded ($d^\star = +\infty$) $\Rightarrow$ primal infeasible ($p^\star = +\infty$)

**Optimality condition**

If $x, y$ satisfy:

- $x$ is a feasible solution to the primal problem
- $y$ is a feasible solution to the dual problem
- The duality gap is zero, *i.e.*, $c^T x + b^T y = 0$

Then $x$ and $y$ are **optimal solutions** to the primal and dual problem respectively

# Strong duality

**Theorem**
If a linear optimization problem has an optimal solution, so does its dual, and the optimal value of primal and dual are equal

$$d^\star = p^\star$$

# Strong duality

## Constructive proof

Given a primal optimal solution $x^\star$ we will construct a dual optimal solution $y^\star$

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Given a primal optimal solution $x^\star$ we will construct a dual optimal solution $y^\star$

Apply simplex to problem in **standard form**

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

# Strong duality

## Constructive proof

Given a primal optimal solution $x^\star$ we will construct a dual optimal solution $y^\star$

Apply simplex to problem in **standard form**

minimize $c^T x$

subject to $Ax = b$

$x \geq 0$

$\longrightarrow$

- optimal basis $B$
- optimal solution $x^\star$ with $A_B x_B^\star = b$
- reduced costs $\bar{c} = c - A^T A_B^{-T} c_B \geq 0$

$A_B^T y^* = -c_B \Rightarrow y = -p$

# Strong duality

## Constructive proof

Given a primal optimal solution $x^\star$ we will construct a dual optimal solution $y^\star$

Apply simplex to problem in **standard form**

$$\begin{array}{lll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \longrightarrow \begin{array}{l} \bullet \text{ optimal basis } B \\ \bullet \text{ optimal solution } x^\star \text{ with } A_B x_B^\star = b \\ \bullet \text{ reduced costs } \bar{c} = c - A^T A_B^{-T} c_B \geq 0 \end{array}$$

Define $y^\star$ such that $y^\star = -A_B^{-T} c_B$. Therefore, $A^T y^\star + c \geq 0$ ($y^\star$ dual feasible).

# Strong duality

## Constructive proof

Given a primal optimal solution $x^\star$ we will construct a dual optimal solution $y^\star$

Apply simplex to problem in **standard form**

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned} \longrightarrow$$

- optimal basis $B$
- optimal solution $x^\star$ with $A_B x_B^\star = b$
- reduced costs $\bar{c} = c - A^T A_B^{-T} c_B \geq 0$

$x_B^* = A_B^{-1} b$

Define $y^\star$ such that $y^\star = -A_B^{-T} c_B$. Therefore, $A^T y^\star + c \geq 0$ ($y^\star$ dual feasible).

$$-b^T y^\star = -b^T(-A_B^{-T} c_B) = c_B^T (A_B^{-1} b) = c_B^T x_B^\star = c^T x^\star$$

# Strong duality

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Given a primal optimal solution $x^\star$ we will construct a dual optimal solution $y^\star$

Apply simplex to problem in **standard form**

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$$-b^T y^\star = -b^T(-A_B^{-T} c_B) = c_B^T (A_B^{-1} b) = c_B^T x_B^\star = c^T x^\star$$

By weak duality theorem corollary, $y^\star$ is an optimal solution of the dual.
Therefore, $d^\star = p^\star$.

# Exception to strong duality

**Primal**

$$\begin{array}{ll} \text{minimize} & x \\ \text{subject to} & 0 \cdot x \leq -1 \end{array}$$

Optimal value is $p^\star = +\infty$

**Dual**

$$\begin{array}{ll} \text{maximize} & y \\ \text{subject to} & 0 \cdot y + 1 = 0 \\ & y \geq 0 \end{array}$$

Optimal value is $d^\star = -\infty$

# Exception to strong duality

**Primal**

$$\begin{array}{ll} \text{minimize} & x \\ \text{subject to} & 0 \cdot x \leq -1 \end{array}$$

Optimal value is $p^\star = +\infty$

**Dual**

$$\begin{array}{ll} \text{maximize} & y \\ \text{subject to} & 0 \cdot y + 1 = 0 \\ & y \geq 0 \end{array}$$

Optimal value is $d^\star = -\infty$

Both **primal** and **dual infeasible**

# Relationship between primal and dual

| | $p^\star = +\infty$ | $p^\star$ finite | $p^\star = -\infty$ |
|---|---|---|---|
| $d^\star = +\infty$ | primal inf.<br>dual unb. | | |
| $d^\star$ finite | | optimal values<br>equal | |
| $d^\star = -\infty$ | exception | | primal unb.<br>dual inf |

- Upper-right excluded by **weak duality**
- $(1, 1)$ and $(3, 3)$ proven by **weak duality**
- $(3, 1)$ and $(2, 2)$ proven by **strong duality**

# Example

# Production problem

maximize $x_1 + 2x_2$

subject to $x_1 \leq 100$

$2x_2 \leq 200$

$x_1 + x_2 \leq 150$

$x_1, x_2 \geq 0$

# Production problem

maximize $x_1 + 2x_2$ ⟵ Profits

subject to $x_1 \leq 100$

$2x_2 \leq 200$

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# Production problem

maximize $x_1 + 2x_2$ ⟵ Profits

subject to

$$x_1 \leq 100$$
$$2x_2 \leq 200 \quad \longleftarrow \text{Resources}$$
$$x_1 + x_2 \leq 150$$
$$x_1, x_2 \geq 0$$

# Production problem

maximize $x_1 + 2x_2$ ⟵ Profits

subject to
$$x_1 \leq 100$$
$$2x_2 \leq 200$$
$$x_1 + x_2 \leq 150$$
$$x_1, x_2 \geq 0$$

⟵ Resources

**Dualize**

1. Transform in inequality form

minimize $c^T x$

subject to $Ax \leq b$

$$c = (-1, -2)$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$b = (100, 200, 150, 0, 0)$$

# Production problem

maximize $x_1 + 2x_2$ ← Profits

subject to
$$x_1 \leq 100$$
$$2x_2 \leq 200$$ ← Resources
$$x_1 + x_2 \leq 150$$
$$x_1, x_2 \geq 0$$

$$c = (-1, -2)$$

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$$b = (100, 200, 150, 0, 0)$$

**Dualize**

1. Transform in inequality form

minimize $c^T x$
subject to $Ax \leq b$

2. Derive dual

maximize $-b^T y$
subject to $A^T y + c = 0$
$y \geq 0$

# Production problem

## Dualized

$$\begin{aligned} \text{maximize} \quad & -b^T y \\ \text{subject to} \quad & A^T y + c = 0 \\ & y \geq 0 \end{aligned}$$

$$c = (-1, -2)$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$b = (100, 200, 150, 0, 0)$$

# Production problem

## Dualized

$$
\begin{aligned}
\text{maximize} \quad & -b^T y \\
\text{subject to} \quad & A^T y + c = 0 \\
& y \geq 0
\end{aligned}
$$

$$c = (-1, -2)$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$b = (100, 200, 150, 0, 0)$$

## Fill-in data

$$
\begin{aligned}
\text{minimize} \quad & 100y_1 + 200y_2 + 150y_3 \\
\text{subject to} \quad & y_1 + y_3 - y_4 = 1 \\
& 2y_2 + y_3 - y_5 = 2 \\
& y_1, y_2, y_3, y_4, y_5 \geq 0
\end{aligned}
$$

# Production problem

## Dualized

$$\begin{aligned} \text{maximize} \quad & -b^T y \\ \text{subject to} \quad & A^T y + c = 0 \\ & y \geq 0 \end{aligned}$$

$$c = (-1, -2)$$

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### Fill-in data

$$\begin{aligned} \text{minimize} \quad & 100y_1 + 200y_2 + 150y_3 \\ \text{subject to} \quad & y_1 + y_3 - y_4 = 1 \\ & 2y_2 + y_3 - y_5 = 2 \\ & y_1, y_2, y_3, y_4, y_5 \geq 0 \end{aligned}$$

### Eliminate variables

$$\begin{aligned} \text{minimize} \quad & 100y_1 + 200y_2 + 150y_3 \\ \text{subject to} \quad & y_1 + y_3 \geq 1 \\ & 2y_2 + y_3 \geq 2 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

# Production problem

## The dual

$$
\begin{aligned}
\text{minimize} \quad & 100y_1 + 200y_2 + 150y_3 \\
\text{subject to} \quad & y_1 + y_3 \geq 1 \\
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& y_1, y_2, y_3 \geq 0
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# Production problem

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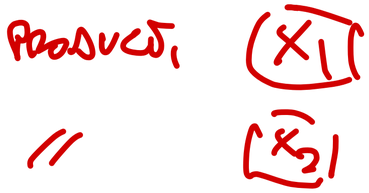
& y_1, y_2, y_3 \geq 0
\end{aligned}
$$




### Interpretation

- **Sell all your resources** at a fair (minimum) price
- Selling must be **more convenient than producing**:
  - Product $1$ (price $1$, needs $1\times$ resource $1$ and $3$): $y_1 + y_3 \geq 1$
  - Product $2$ (price $2$, needs $2\times$ resource $2$ and $1\times$ resource $3$): $2y_2 + y_3 \geq 2$

# Linear optimization duality

Today, we learned to:

- **Dualize** linear optimization problems
- **Prove** weak and strong duality conditions
- **Interpret** simple dual optimization problems

# References


- Bertsimas and Tsitsiklis: Introduction to Linear Optimization
  - Chapter 4: Duality theory
- R. Vanderbei: Linear Programming — Foundations and Extensions
  - Chapter 5: Duality theory

# Next lecture

More on duality:

- Game theory
- Complementary slackness
- Farkas lemma