ORF307 – Optimization

13. Duality

Bartolomeo Stellato — Spring 2024

Ed Forum

- For phase 1 vs phase 2, I understand it finds an extreme point that is not necessarily the optimal but I do not understand how it gets this point by setting $x = 0$ and $y = b$.
- how does the simplex method's approach to handling degeneracy and cycling impact its efficiency and reliability in practical applications, such as logistics or resource allocation? Are there examples where alternative methods might be more effective due to these issues?

Complexity

Complexity of a single simplex iteration

1. Compute the reduced costs $\bar{c}$
   - Solve $A_B^T p = c_B$
   - $\bar{c} = c - A^T p$
2. If $\bar{c} \geq 0$, x **optimal. break**
3. Choose j such that $\bar{c}_j < 0$
4. Compute search direction d with $d_j = 1$ and $A_B d_B = -A_j$
5. If $d_B \geq 0$, the problem is **unbounded** and the optimal value is $-\infty$. **break**
6. Compute step length $\theta^\star = \min_{\{i \in B | d_i < 0\}} \left(-\frac{x_i}{d_i} \right)$
7. Define y such that $y = x + \theta^\star d$
8. Get new basis $\bar{B}$ (i exits and j enters)

Bottleneck

Two linear systems

Linear system solutions

Very similar linear systems

$$A_B^T p = c_B$$
$$A_B d_B = -A_j$$

→

LU factorization

$(2/3)n^3$ flops

$$A_B = PLU$$

→

Easy linear systems

$4n^2$ flops

$$U^T L^T P^T p = c_B$$
$$PLU d_B = -A_j$$

Factorization is expensive

Do we need to recompute it at every iteration?

Basis update

Index update

- j enters (x_j becomes $\theta^\star$)
- $i = B(\ell)$ exists (x_i becomes 0)

$\longrightarrow$

Basis matrix change

$$A_{\bar{B}} = A_B + (A_i - A_j)e_\ell^T$$

Example

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$B = \{4, 1, 6\} \quad \rightarrow \quad \bar{B} = \{4, 1, 2\}$$

- 2 enters
- $6 = B(3)$ exists

$$A_{\bar{B}} = \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}}_{A_B} + \underbrace{\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}}_{A_2 e_3^T} - \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A_6 e_3^T} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

Smarter linear system solution

Basis matrix change

$$A_{\bar{B}} = A_B + \overbrace{(A_i - A_j)}^{v} e_\ell^T$$

$\longrightarrow$

Matrix inversion lemma
(from homework 2)

$$(A_B + v e_\ell^T)^{-1} = \left(I - \frac{1}{1 + e_\ell^T A_B^{-1} v} A_B^{-1} v e_\ell^T\right) A_B^{-1}$$

Solve $A_{\bar{B}} d_{\bar{B}} = -A_j$

1. Solve $A_B z^1 = e_\ell$ ($2n^2$ flops)
2. Solve $A_B z^2 = -A_j$ ($2n^2$ flops)
3. Solve $d_{\bar{B}} = z^2 - \frac{v^T z^2}{1+v^T z^1} z^1$

Remarks

- Same complexity for $A_B^T p = c_B$ ($4n^2$ flops)
- k-th next iteration ($4kn^2$ flops, derive as exercise...)
- Once in a while (e.g., $k = 100$), better to refactor A_B

Complexity of a single simplex iteration

1. Compute the reduced costs $\bar{c}$
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8. Get new basis $\bar{B}$ (i exits and j enters)

Bottleneck
Two linear systems

→

Matrix inversion lemma trick
$\approx n^2$ per iteration
(very cheap)

How many iterations do we need?

Complexity of the simplex method

Example of worst-case behavior

Innocent-looking problem

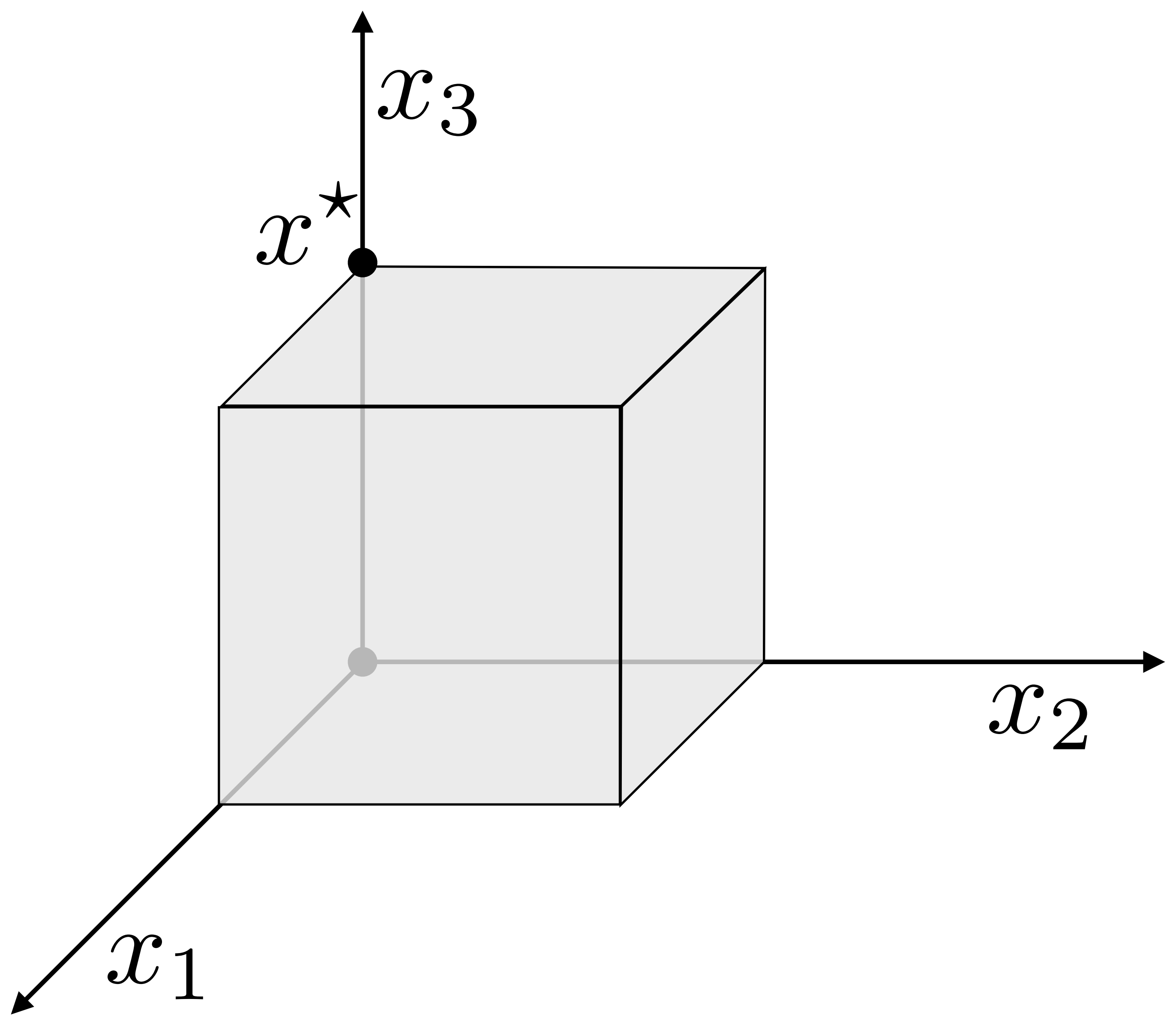
minimize $\quad -x_n$

subject to $\quad 0 \leq x \leq 1$

2^n **vertices**

$2^n/2$ vertices: cost = 1

$2^n/2$ vertices: cost = 0



Perturb unit cube

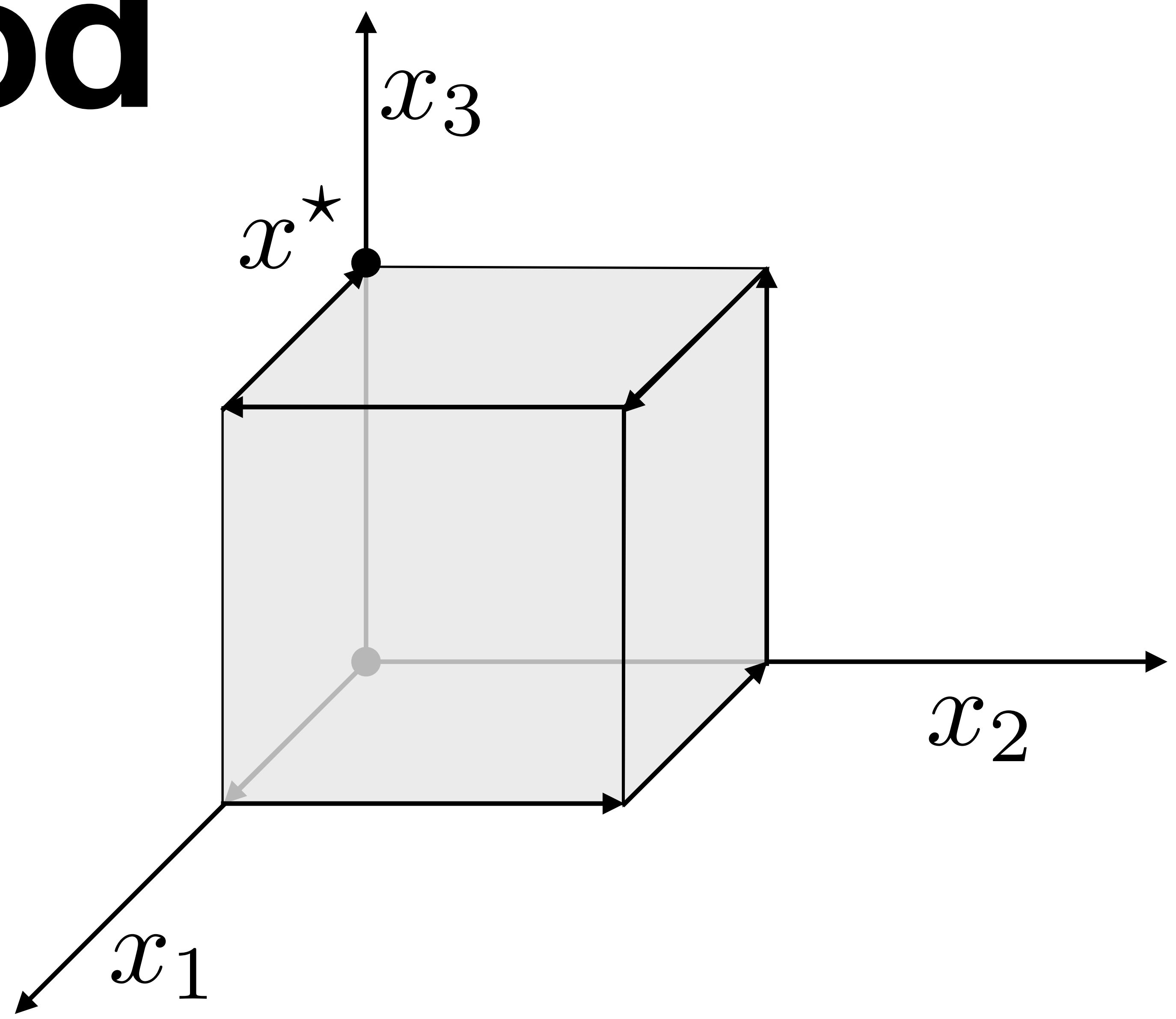
minimize $\quad -x_n$

subject to $\quad \epsilon \leq x_1 \leq 1$

$$\epsilon x_{i-1} \leq x_i \leq 1 - \epsilon x_{i-1}, \quad i = 2, \ldots, n$$

Complexity of the simplex method

Example of worst-case behavior



minimize $\quad -x_n$

subject to $\quad \epsilon \leq x_1 \leq 1$

$$\epsilon x_{i-1} \leq x_i \leq 1 - \epsilon x_{i-1}, \quad i = 2, \dots, n$$

Theorem

- The vertices can be ordered so that each one is adjacent to and has a **lower cost than the previous one**
- There exists a pivoting rule under which the simplex method terminates after $2^n - 1$ **iterations**

Remark

- A **different pivot rule** would have converged in one iteration.
- We have a bad example for every pivot rule.

Complexity of the simplex method

We do **not know any polynomial version of the simplex method**, no matter which pivoting rule we pick. → Still **open research question!**

Worst-case

There are problem instances where the simplex method will run an **exponential number of iterations** in terms of the dimensions, e.g. 2^n

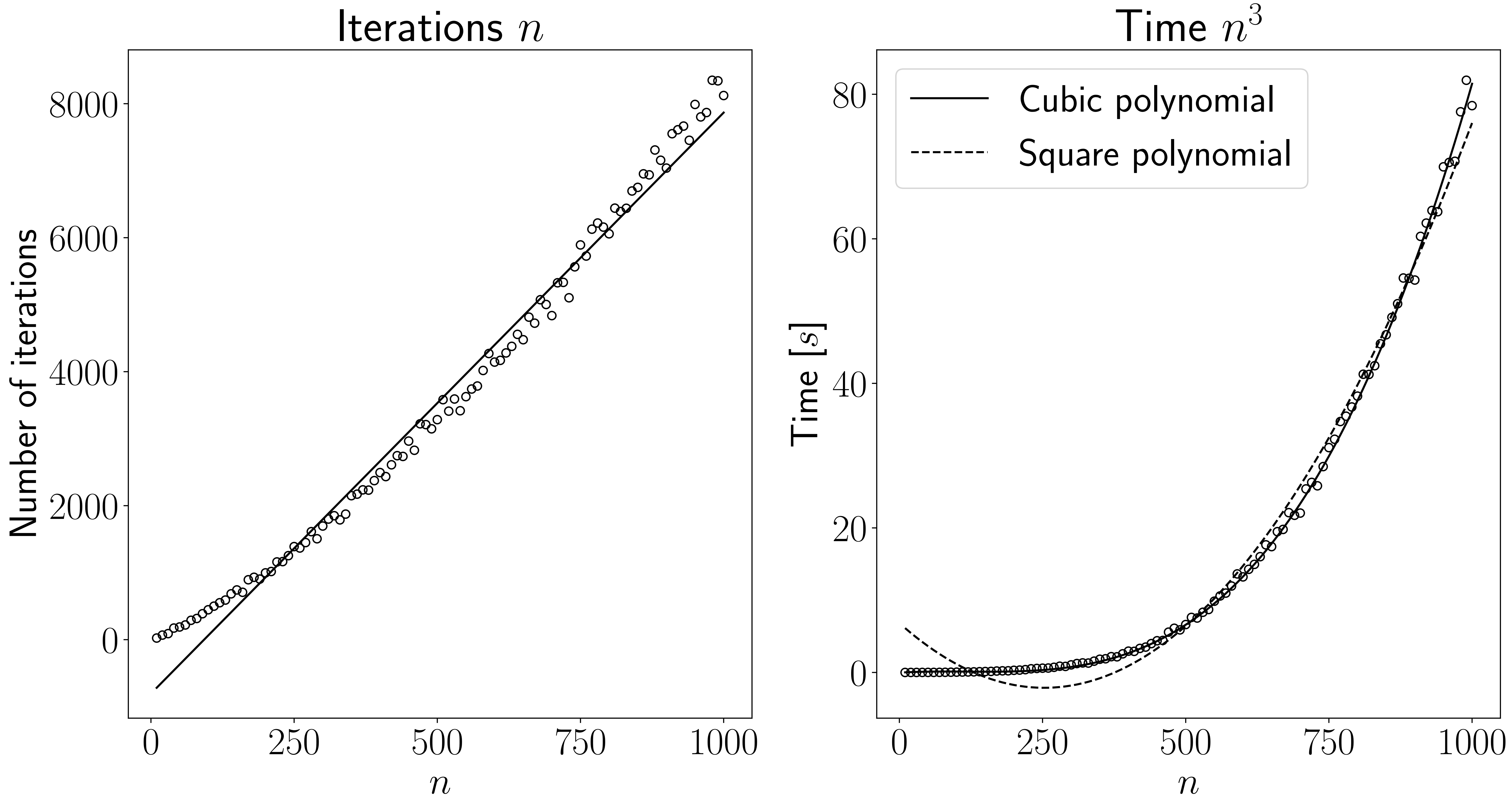
Good news: average-case
Practical performance is very good. On average, it stops in n iterations.

Average simplex complexity

Random LPs

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

n variables
$3n$ constraints



Recap

Linear optimization formulations

Standard form LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Inequality form LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

Today's agenda

Duality

- Obtaining lower bounds
- The dual problem
- Weak and strong duality

Obtaining lower bounds

Obtaining lower bounds

A simple example

$$
\begin{array}{ll}
\text{minimize} & x_1 + 3x_2 \\
\text{subject to} & x_1 + 3x_2 \geq 2
\end{array}
$$

What is a **lower bound** on the optimal cost?

A lower bound is 2 because $x_1 + 3x_2 \geq 2$

Obtaining lower bounds

Another example

$$\begin{aligned} \text{minimize} \quad & x_1 + 3x_2 \\ \text{subject to} \quad & x_1 + x_2 \geq 2 \\ & x_2 \geq 1 \end{aligned}$$

What is a **lower bound** on the optimal cost?

Let's sum the constraints

$$\begin{aligned} & 1 \cdot (x_1 + x_2 \geq 2) \\ + & 2 \cdot (x_2 \geq 1) \\ = & x_1 + 3x_2 \geq 4 \end{aligned}$$

A lower bound is 4

Obtaining lower bounds

A more interesting example

$$
\begin{aligned}
\text{minimize} \quad & x_1 + 3x_2 \\
\text{subject to} \quad & x_1 + x_2 \geq 2 \\
& x_2 \geq 1 \\
& x_1 - x_2 \geq 3
\end{aligned}
$$

How can we obtain a lower bound?

Add constraints

$$
\begin{aligned}
& y_1 \cdot (x_1 + x_2 \geq 2) \\
+ & y_2 \cdot (x_2 \geq 1) \\
+ & y_3 \cdot (x_1 - x_2 \geq 3) \\
\Rightarrow & (y_1 + y_3)x_1 + (y_1 + y_2 - y_3)x_2 \geq 2y_1 + y_2 + 3y_3
\end{aligned}
$$

Bound

Match cost coefficients

$$
\begin{aligned}
y_1 + y_3 &= 1 \\
y_1 + y_2 - y_3 &= 3 \\
y_1, y_2, y_3 &\geq 0
\end{aligned}
$$

Many options

$y = (1, 2, 0) \Rightarrow$ Bound 4

$y = (0, 4, 1) \Rightarrow$ Bound 7

How can we get the **best one**?

Obtaining lower bounds

A more interesting example — Best lower bound

We can obtain the **best lower bound** by solving the following problem

$$\begin{array}{ll} \text{maximize} & 2y_1 + y_2 + 3y_3 \\ \text{subject to} & y_1 + y_3 = 1 \\ & y_1 + y_2 - y_3 = 3 \\ & y_1, y_2, y_3 \geq 0 \end{array}$$

This linear optimization problem is called the **dual problem**

The dual problem

Lagrange multipliers

Consider the LP in standard form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Relax the constraint

$$g(y) = \begin{array}{ll} \underset{x}{\text{minimize}} & c^T x + y^T (Ax - b) \\ \text{subject to} & x \geq 0 \end{array}$$

Lower bound

$$g(y) \leq c^T x^\star + y^T (Ax^\star - b) = c^T x^\star$$

Best lower bound

$$\underset{y}{\text{maximize}} \quad g(y)$$

The dual

Dual function

$$
\begin{aligned}
g(y) &= \underset{x \geq 0}{\text{minimize}} \left(c^T x + y^T (Ax - b)\right) \\
&= -b^T y + \underset{x \geq 0}{\text{minimize}} \left(c + A^T y\right)^T x
\end{aligned}
$$

$$
g(y) = \begin{cases} -b^T y & \text{if } c + A^T y \geq 0 \\ -\infty & \text{otherwise} \end{cases}
$$

Dual problem (find the best bound)

$$
\underset{y}{\text{maximize}} \quad g(y) \quad = \quad \begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c \geq 0 \end{array}
$$

Primal and dual problems

Primal problem	**Dual problem**
minimize $\quad c^T x$	maximize $\quad -b^T y$
subject to $\quad Ax = b$	subject to $\quad A^T y + c \geq 0$
$\quad x \geq 0$	

Primal variable $x \in \mathbf{R}^n$

Dual variable $y \in \mathbf{R}^m$

The dual problem carries **useful information** for the primal problem

Duality is useful also to **solve** optimization problems

Dual of inequality form LP

What if you find an LP with inequalities?

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \le b \end{array}$$

1. We could first transform it to standard form
2. We can compute the dual function (same procedure as before)

Relax the constraint

$$g(y) = \underset{x}{\text{minimize}} \quad c^T x + y^T (Ax - b)$$

Lower bound

$$g(y) \le c^T x^\star + y^T (Ax^\star - b) \le c^T x^\star$$

we must have $y \ge 0$

Dual of LP with inequalities

Derivation

Dual function

$$g(y) = \underset{x}{\text{minimize}} \left(c^T x + y^T (Ax - b)\right)$$
$$= -b^T y + \underset{x}{\text{minimize}} \left(c + A^T y\right)^T x$$

$$g(y) = \begin{cases} -b^T y & \text{if } c + A^T y = 0 \\ -\infty & \text{otherwise} \end{cases} \quad \text{(and } y \geq 0\text{)}$$

Dual problem (find the best bound)

$$\underset{y}{\text{maximize}} \quad g(y) \quad = \quad \begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c = 0 \\ & y \geq 0 \end{array}$$

General forms

Standard form LP

Primal

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Dual

$$\begin{aligned} \text{maximize} \quad & -b^T y \\ \text{subject to} \quad & A^T y + c \geq 0 \end{aligned}$$

Inequality form LP

Primal

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax \leq b \end{aligned}$$

Dual

$$\begin{aligned} \text{maximize} \quad & -b^T y \\ \text{subject to} \quad & A^T y + c = 0 \\ & y \geq 0 \end{aligned}$$

LP with inequalities and equalities

Primal

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax \leq b \\ & Cx = d \end{aligned}$$

Dual

$$\begin{aligned} \text{maximize} \quad & -b^T y - d^T z \\ \text{subject to} \quad & A^T y + C^T z + c = 0 \\ & y \geq 0 \end{aligned}$$

Example from before

minimize	$x_1 + 3x_2$
subject to	$x_1 + x_2 \geq 2$
	$x_2 \geq 1$
	$x_1 - x_2 \geq 3$

→

Inequality form LP

minimize	$c^T x$
subject to	$Ax \leq b$

$$c = (1, 3)$$

$$A = \begin{bmatrix} -1 & -1 \\ 0 & -1 \\ -1 & 1 \end{bmatrix}$$

$$b = (-2, -1, -3)$$

Dual

maximize	$-b^T y$
subject to	$A^T y + c = 0$
	$y \geq 0$

→

maximize	$2y_1 + y_2 + 3y_3$
subject to	$-y_1 - y_3 = -1$
	$-y_1 - y_2 + y_3 = -3$
	$y_1, y_2, y_3 \geq 0$

To memorize

Ways to get the dual

- Derive dual function directly
- Transform the problem in inequality form LP and dualize

Sanity-checks and signs convention

- Consider constraints as $Ax - b \leq 0$ or $Ax - b = 0$ (not ≥ 0)
- Each dual variable is associated to a primal constraint
- y free for primal equalities and $y \geq 0$ for primal inequalities

Dual of the dual

Theorem

If we transform the primal into its dual and then transform the dual to its dual, we obtain a problem equivalent to the original problem. In other words, the **dual of the dual is the primal**.

Exercise

Derive dual and dualize again

Primal

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax \leq b \\
 & Cx = d
\end{array}
$$

Dual

$$
\begin{array}{ll}
\text{maximize} & -b^T y - d^T z \\
\text{subject to} & A^T y + C^T z + c = 0 \\
 & y \geq 0
\end{array}
$$

Theorem

If we **transform a linear optimization problem to another form** (inequality form, standard form, inequality and equality form), **the dual of the two problems will be equivalent**.

Weak and strong duality

Optimal objective values

Primal

minimize $c^T x$

subject to $Ax \leq b$

Dual

maximize $-b^T y$

subject to $A^T y + c = 0$

$y \geq 0$

$p^\star$ is the primal optimal value

$d^\star$ is the dual optimal value

Primal infeasible: $p^\star = +\infty$
Primal unbounded: $p^\star = -\infty$

Dual infeasible: $d^\star = -\infty$
Dual unbounded: $d^\star = +\infty$

Weak duality

Theorem

If x, y satisfy:

- x is a feasible solution to the primal problem
- y is a feasible solution to the dual problem

$\longrightarrow$ $-b^T y \leq c^T x$

Proof

We know that $Ax \leq b$, $A^T y + c = 0$ and $y \geq 0$. Therefore,

$$0 \leq y^T(b - Ax) = b^T y - y^T A x = c^T x + b^T y \qquad \blacksquare$$

Remark

- Any dual feasible y gives a **lower bound** on the primal optimal value
- Any primal feasible x gives an **upper bound** on the dual optimal value
- $c^T x + b^T y$ is the **duality gap**

Weak duality

Corollaries

Unboundedness vs feasibility

- Primal unbounded ($p^\star = -\infty$) $\Rightarrow$ dual infeasible ($d^\star = -\infty$)
- Dual unbounded ($d^\star = +\infty$) $\Rightarrow$ primal infeasible ($p^\star = +\infty$)

Optimality condition

If x, y satisfy:

- x is a feasible solution to the primal problem
- y is a feasible solution to the dual problem
- The duality gap is zero, *i.e.*, $c^T x + b^T y = 0$

Then x and y are **optimal solutions** to the primal and dual problem respectively

Strong duality

Theorem
If a linear optimization problem has an optimal solution, so does its dual, and the optimal value of primal and dual are equal

$$d^\star = p^\star$$

Strong duality

Constructive proof

Given a primal optimal solution $x^\star$ we will construct a dual optimal solution $y^\star$

Apply simplex to problem in **standard form**

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax = b \\
 & x \geq 0
\end{array}
\longrightarrow
$$

- optimal basis B
- optimal solution $x^\star$ with $A_B x_B^\star = b$
- reduced costs $\bar{c} = c - A^T A_B^{-T} c_B \geq 0$

Define $y^\star$ such that $y^\star = -A_B^{-T} c_B$. Therefore, $A^T y^\star + c \geq 0$ ($y^\star$ dual feasible).

$$-b^T y^\star = -b^T(-A_B^{-T} c_B) = c_B^T(A_B^{-1} b) = c_B^T x_B^\star = c^T x^\star$$

By weak duality theorem corollary, $y^\star$ is an optimal solution of the dual.
Therefore, $d^\star = p^\star$. ■

Exception to strong duality

Primal

minimize $\quad x$

subject to $\quad 0 \cdot x \leq -1$

Optimal value is $p^\star = +\infty$

Dual

maximize $\quad y$

subject to $\quad 0 \cdot y + 1 = 0$

$\quad y \geq 0$

Optimal value is $d^\star = -\infty$

Both **primal** and **dual infeasible**

Relationship between primal and dual

	$p^\star = +\infty$	$p^\star$ finite	$p^\star = -\infty$
$d^\star = +\infty$	primal inf. dual unb.		
$d^\star$ finite		optimal values equal	
$d^\star = -\infty$	exception		primal unb. dual inf

- Upper-right excluded by **weak duality**
- $(1, 1)$ and $(3, 3)$ proven by **weak duality**
- $(3, 1)$ and $(2, 2)$ proven by **strong duality**

Example

Production problem

maximize $x_1 + 2x_2$ ⟵ Profits

subject to
$$x_1 \leq 100$$
$$2x_2 \leq 200$$
$$x_1 + x_2 \leq 150$$
⟵ Resources
$$x_1, x_2 \geq 0$$

$$c = (-1, -2)$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$b = (100, 200, 150, 0, 0)$$

Dualize

1. Transform in inequality form

minimize $c^T x$

subject to $Ax \leq b$

2. Derive dual

maximize $-b^T y$

subject to $A^T y + c = 0$

$y \geq 0$

Production problem

Dualized

$$\begin{aligned} &\text{maximize} && -b^T y \\ &\text{subject to} && A^T y + c = 0 \\ & && y \geq 0 \end{aligned}$$

$$c = (-1, -2)$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$b = (100, 200, 150, 0, 0)$$

Fill-in data

$$\begin{aligned} &\text{minimize} && 100y_1 + 200y_2 + 150y_3 \\ &\text{subject to} && y_1 + y_3 \cancel{- y_4} = 1 \\ & && 2y_2 + y_3 \cancel{- y_5} = 2 \\ & && y_1, y_2, y_3, y_4, y_5 \geq 0 \end{aligned}$$

$$\longrightarrow$$

Eliminate variables

$$\begin{aligned} &\text{minimize} && 100y_1 + 200y_2 + 150y_3 \\ &\text{subject to} && y_1 + y_3 \geq 1 \\ & && 2y_2 + y_3 \geq 2 \\ & && y_1, y_2, y_3 \geq 0 \end{aligned}$$

Production problem

The dual

$$
\begin{aligned}
\text{minimize} \quad & 100y_1 + 200y_2 + 150y_3 \\
\text{subject to} \quad & y_1 + y_3 \geq 1 \\
& 2y_2 + y_3 \geq 2 \\
& y_1, y_2, y_3 \geq 0
\end{aligned}
$$

Interpretation

- **Sell all your resources** at a fair (minimum) price
- Selling must be **more convenient than producing**:
  - Product 1 (price 1, needs $1\times$ resource 1 and 3): $y_1 + y_3 \geq 1$
  - Product 2 (price 2, needs $2\times$ resource 2 and $1\times$ resource 3): $2y_2 + y_3 \geq 2$

Linear optimization duality

Today, we learned to:

- **Dualize** linear optimization problems
- **Prove** weak and strong duality conditions
- **Interpret** simple dual optimization problems

References

- Bertsimas and Tsitsiklis: Introduction to Linear Optimization
  - Chapter 4: Duality theory
- R. Vanderbei: Linear Programming — Foundations and Extensions
  - Chapter 5: Duality theory

Next lecture

More on duality:

- Game theory
- Complementary slackness
- Farkas lemma