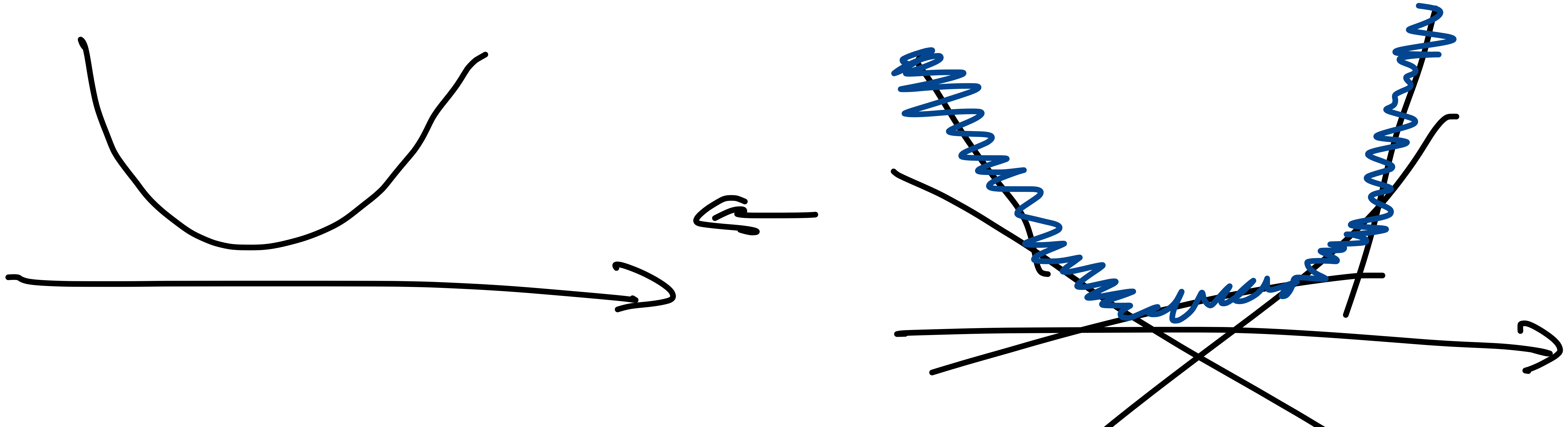
ORF307 – Optimization

9. Geometry and polyhedra

Bartolomeo Stellato— Spring 2024

Ed Forum



- when doing convex piecewise linear minimization, how do we know how many pieces to split the curve into?

- We also discussed how to turn a vector norm problem into a LP problem, which I don't fully understand and will need to review.

$$\min \left[\max_{i \in 1,\ldots,N} a_i^T x + b_i \right] \leftarrow \qquad \min \; t \quad \text{s.t. } a_i^T x + b_i \leq t \quad i = 1, \ldots, N$$

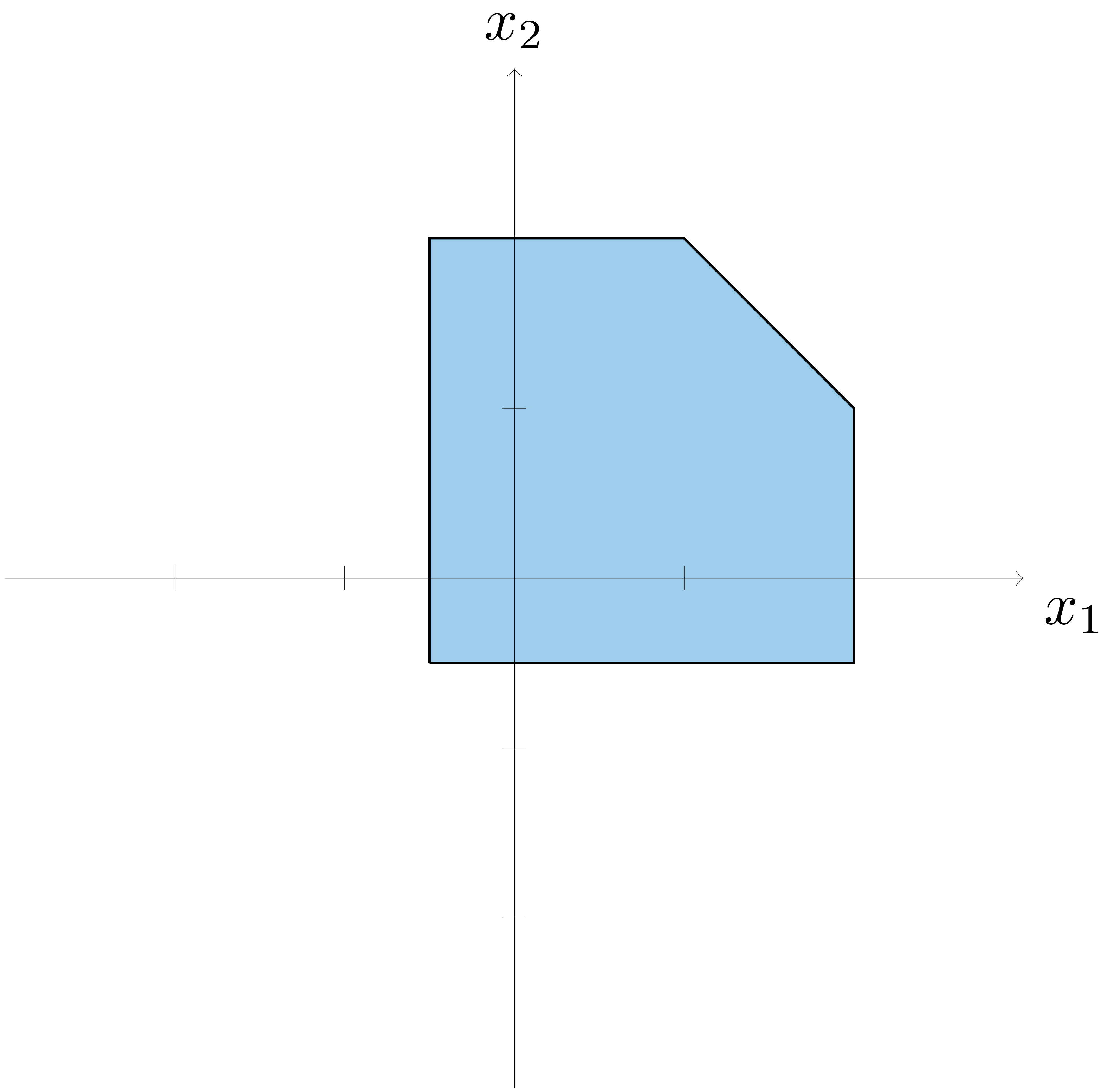
Today's lecture

Geometry and polyhedra

- Simple example
- Polyhedra
- Corners: extreme points, vertices, basic feasible solutions
- Constructing basic solutions
- Existence and optimality of extreme points

A simple example

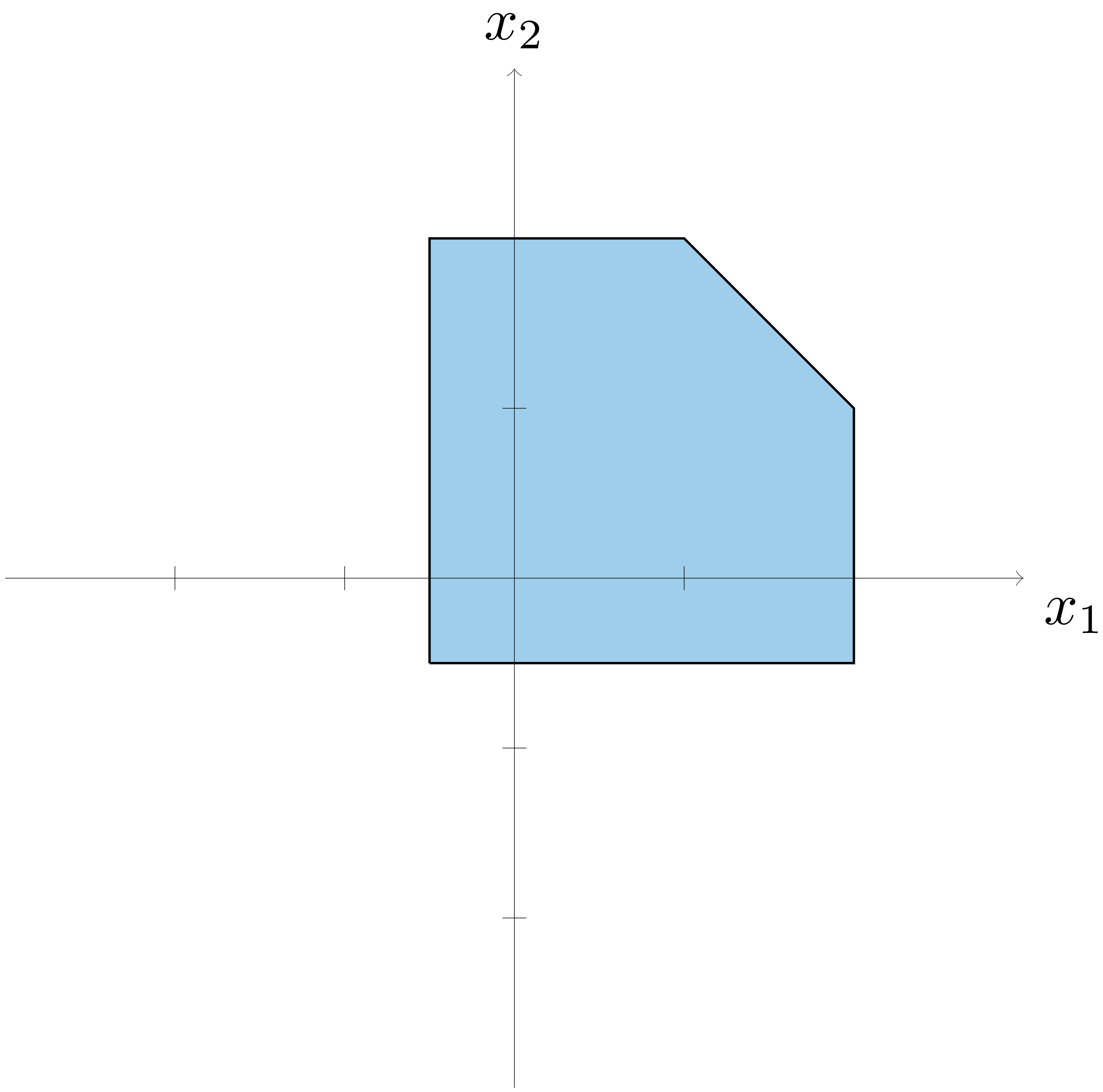
$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & -1/2 \leq x_1 \leq 2 \\ & -1/2 \leq x_2 \leq 2 \\ & x_1 + x_2 \leq 2 \end{array}$$



A simple example

minimize	$c^T x$
subject to	$-1/2 \leq x_1 \leq 2$
	$-1/2 \leq x_2 \leq 2$
	$x_1 + x_2 \leq 3$

What kind of optimal solutions do we get?



A simple example

minimize	$c^T x$
subject to	$-1/2 \le x_1 \le 2$
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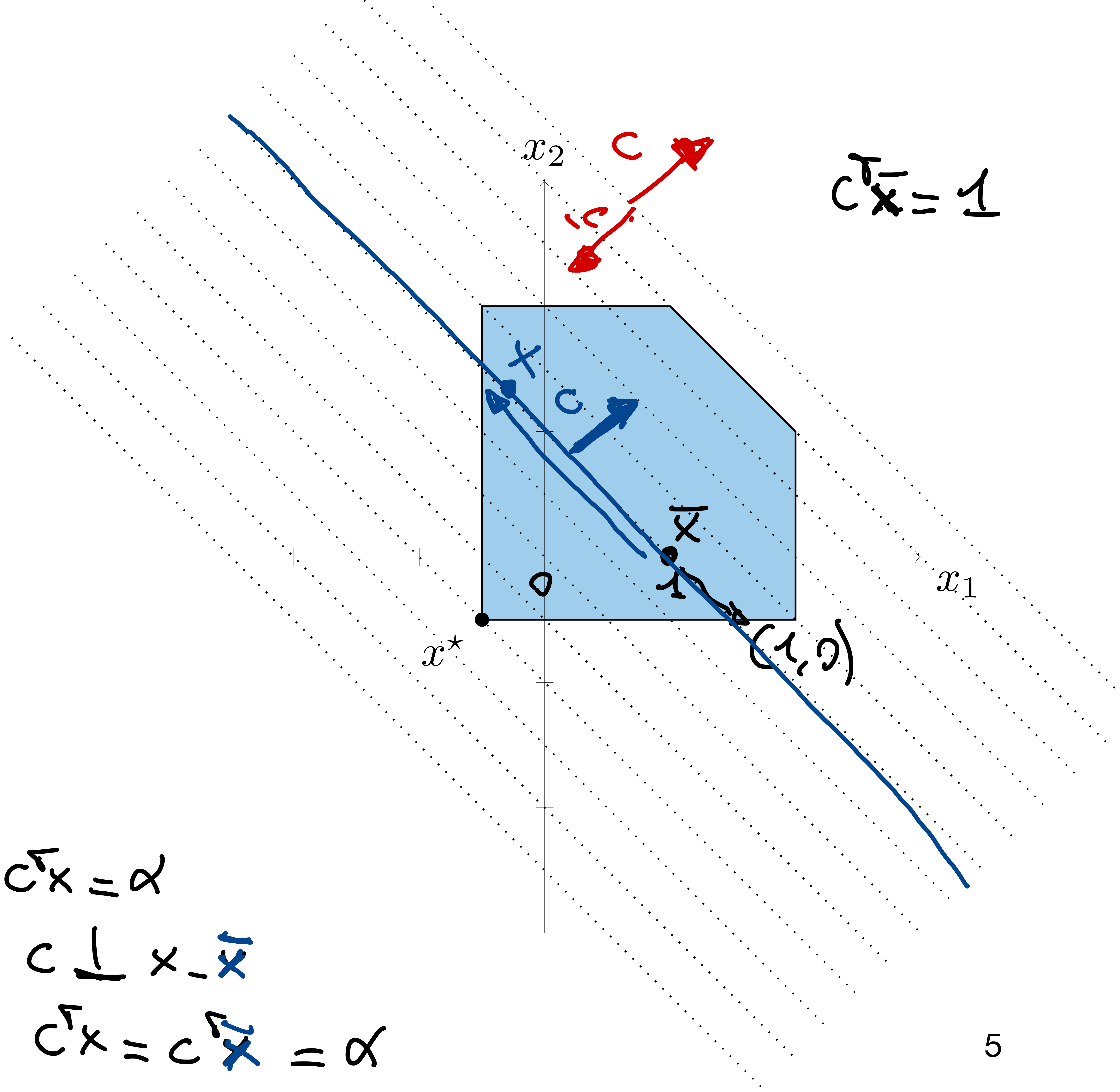
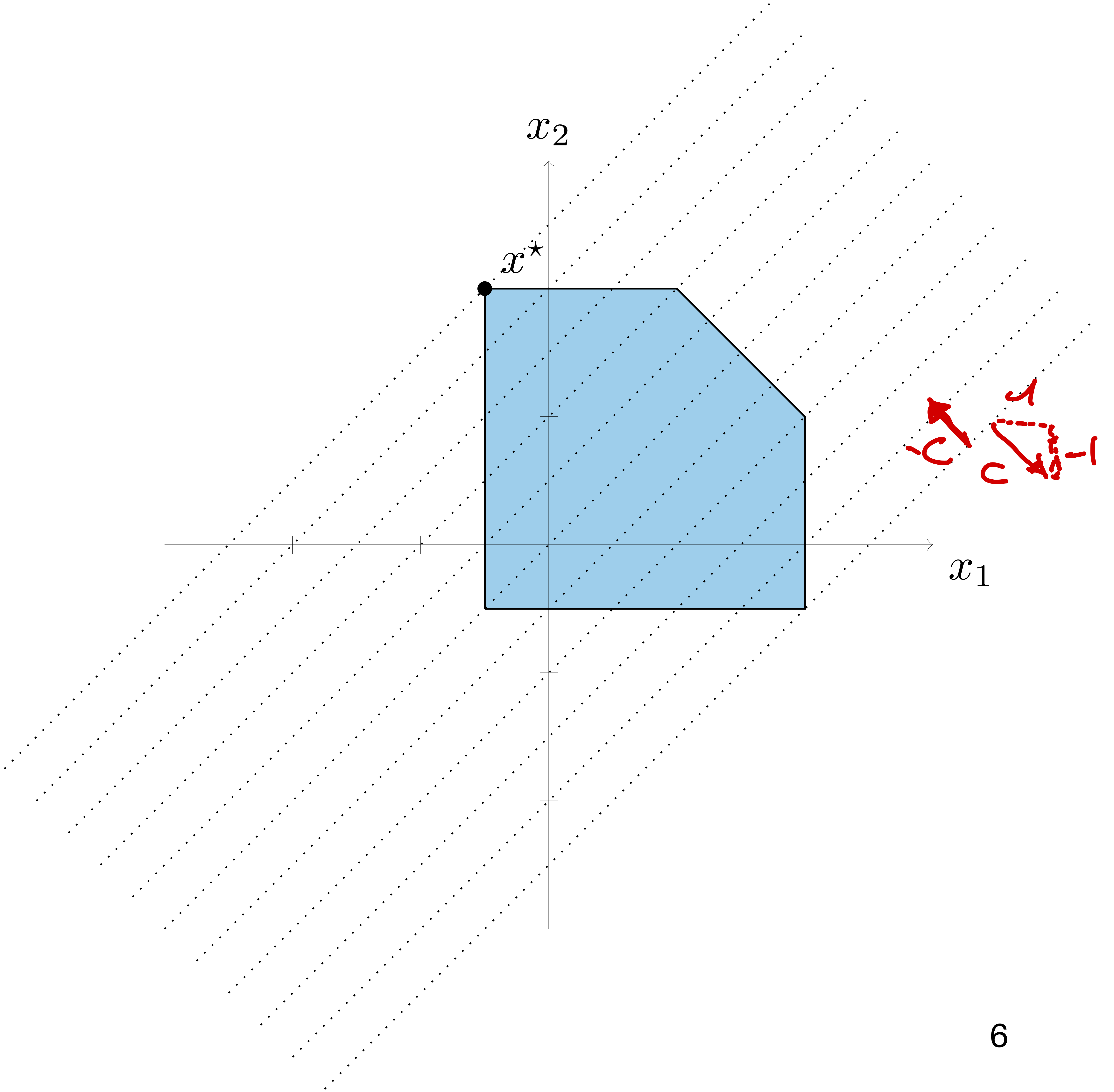
Suppose $c = (1, 1)$



A simple example

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & -1/2 \leq x_1 \leq 2 \\ & -1/2 \leq x_2 \leq 2 \\ & x_1 + x_2 \leq 3 \end{array}$$

Suppose $c = (1, -1)$



A simple example

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & -1/2 \le x_1 \le 2 \\ & -1/2 \le x_2 \le 2 \\ & x_1 + x_2 \le 3 \end{array}$$

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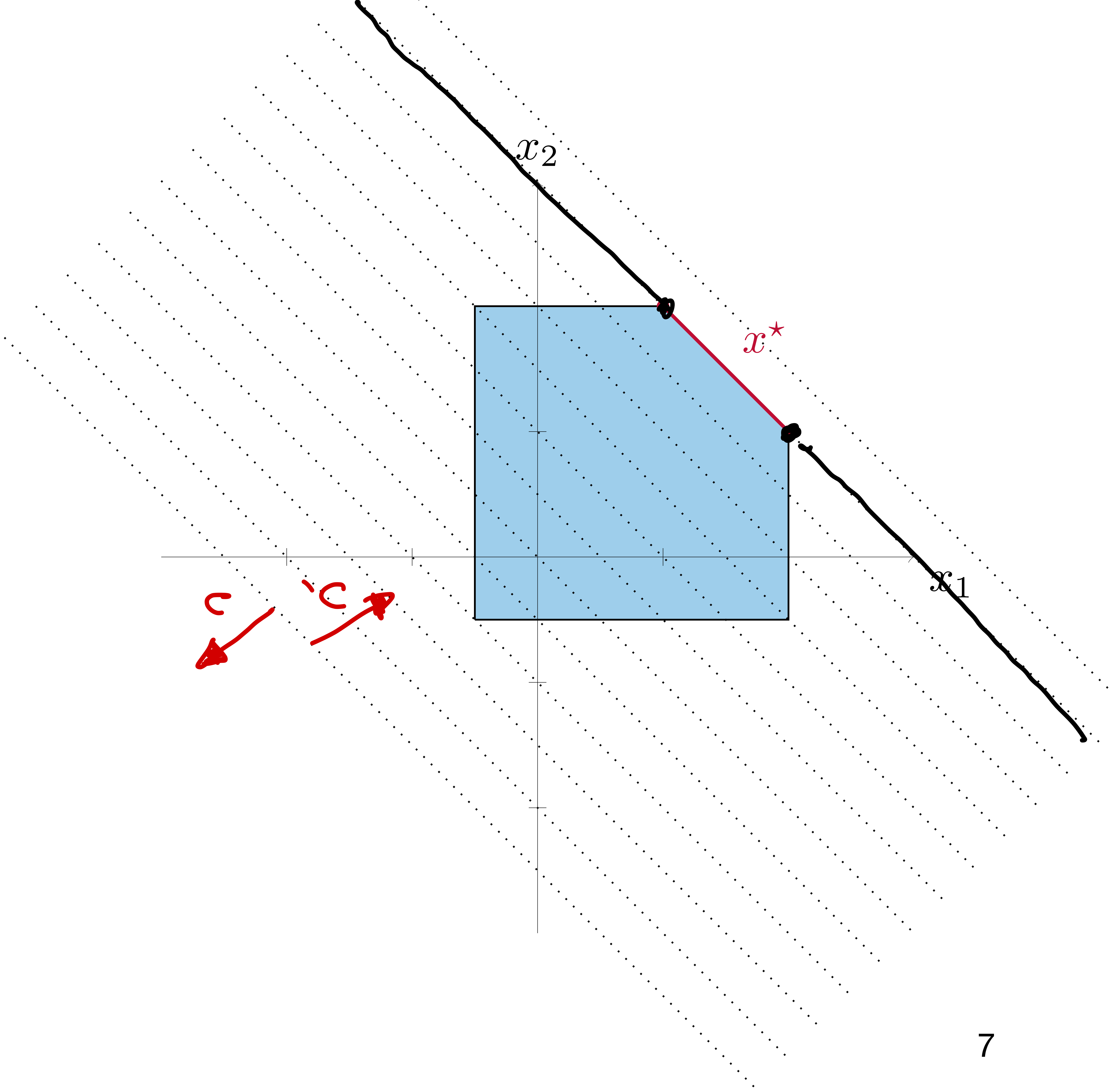


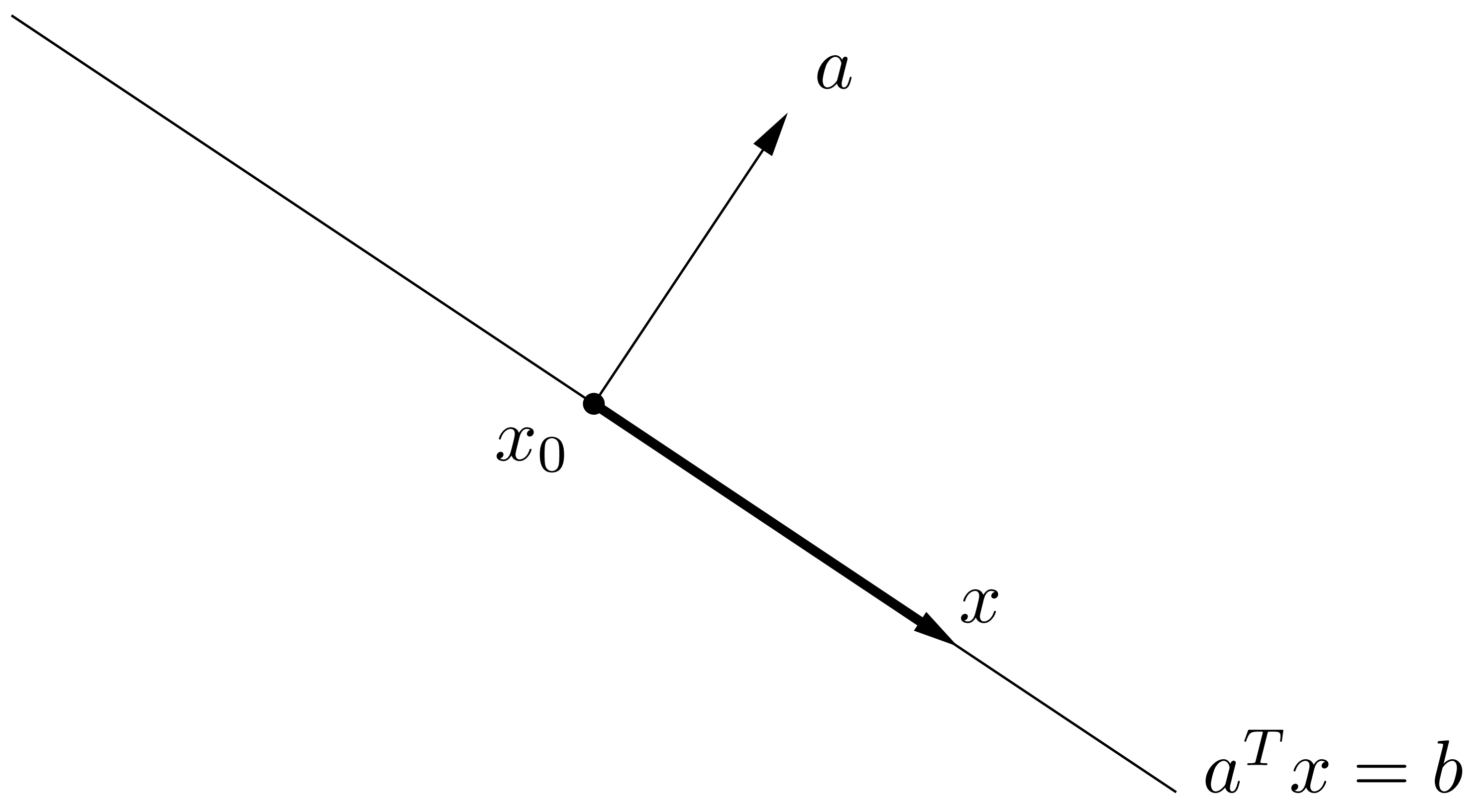
Polyhedra and linear algebra

Hyperplanes and halfspaces

Definitions

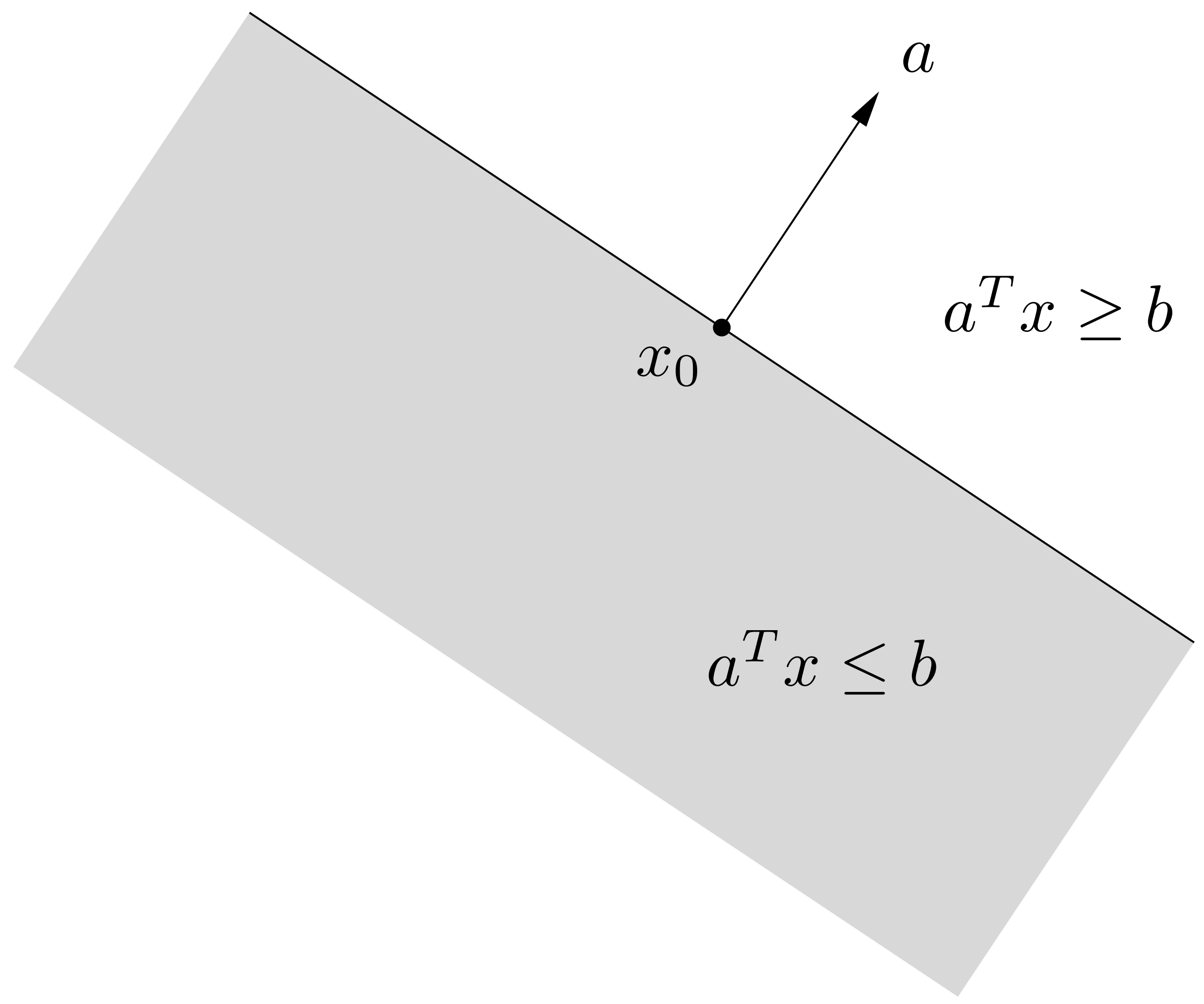
Hyperplane

$$\{x \mid a^T x = b\}$$



Halfspace

$$\{x \mid a^T x \leq b\}$$



Hyperplanes and halfspaces

Definitions

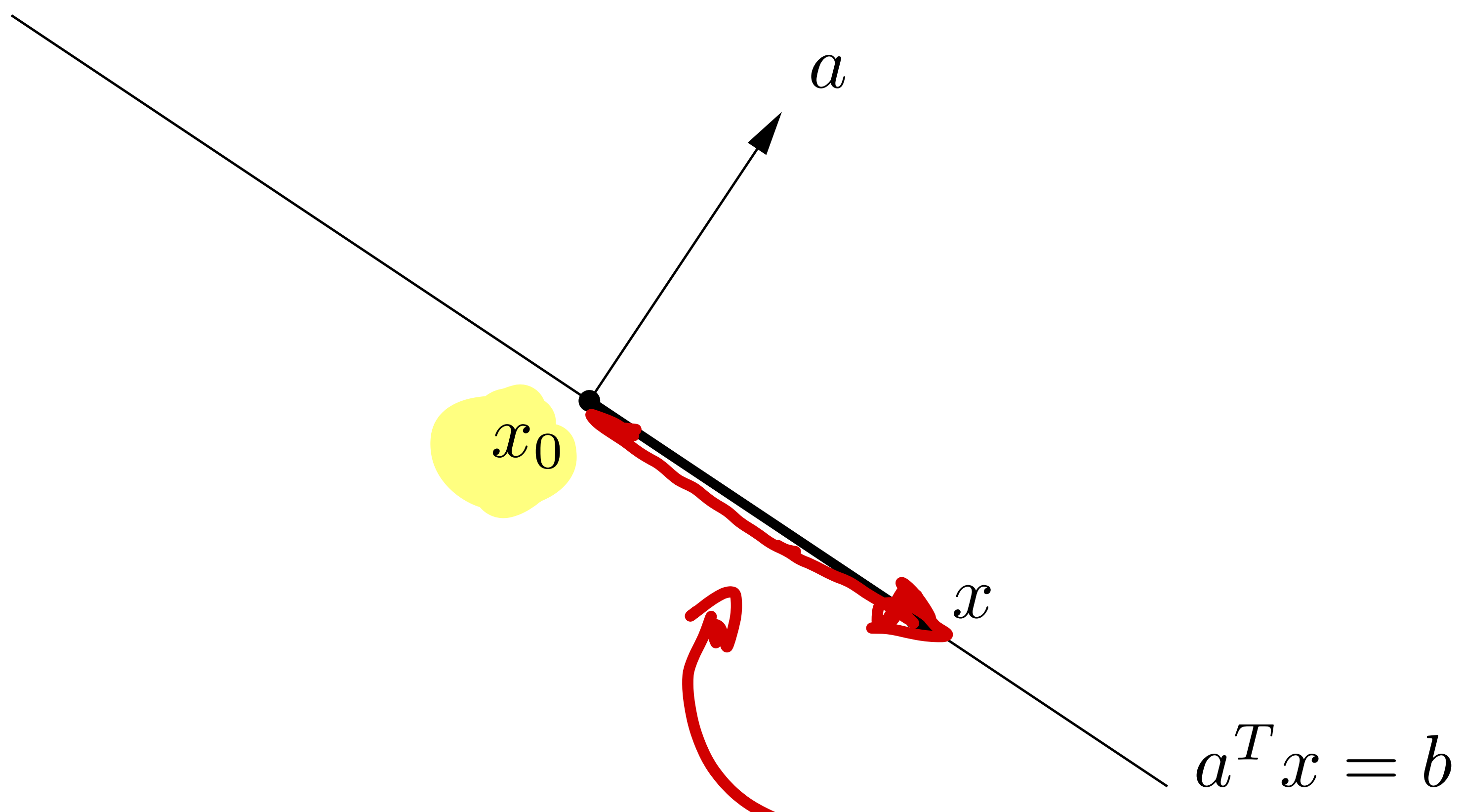
$c^T x = \alpha$

$c \perp x - x_0$

$c^T x = c^T x_0 = \alpha$

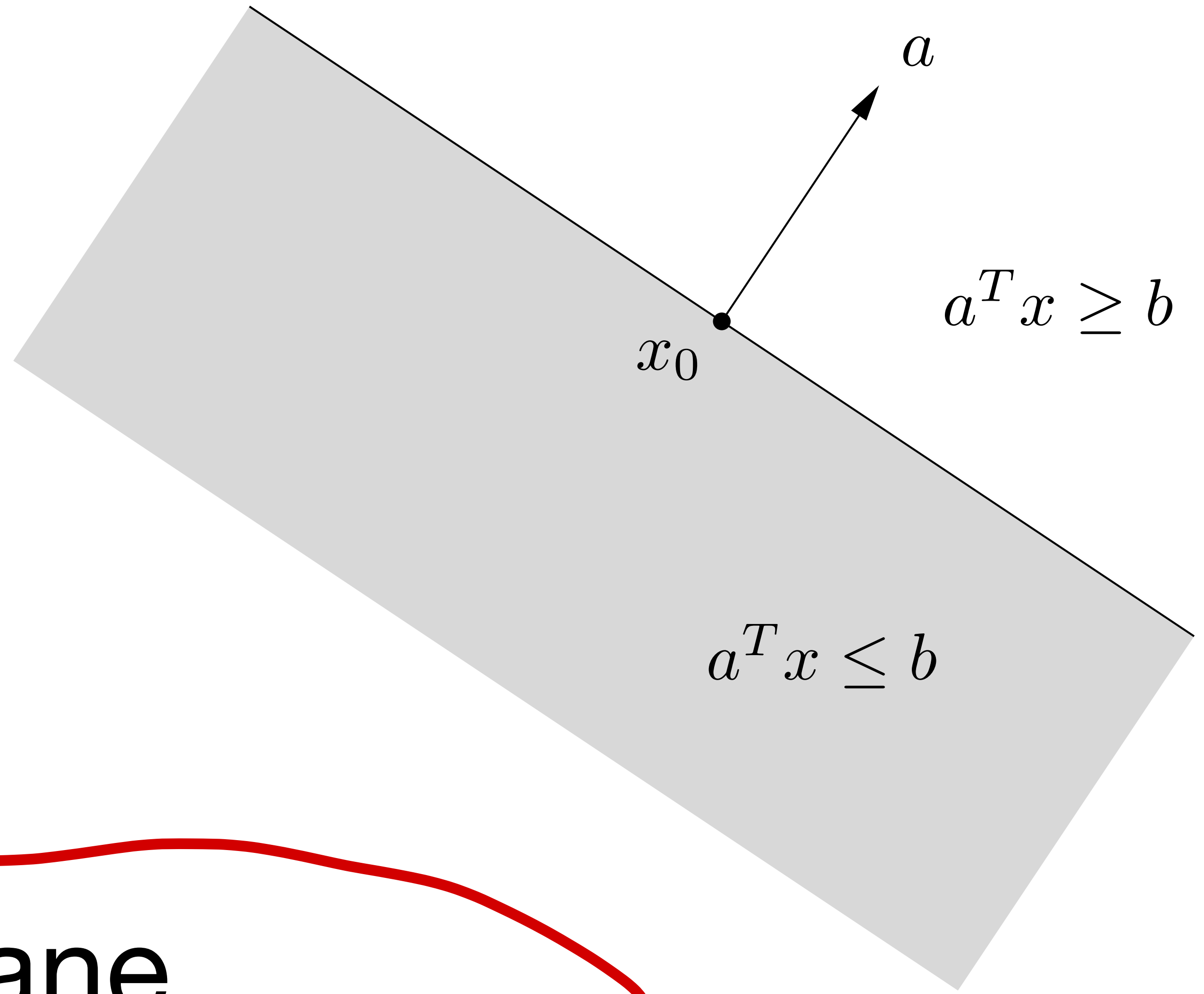
Hyperplane

$\{x \mid a^T x = b\}$



Halfspace

$\{x \mid a^T x \leq b\}$

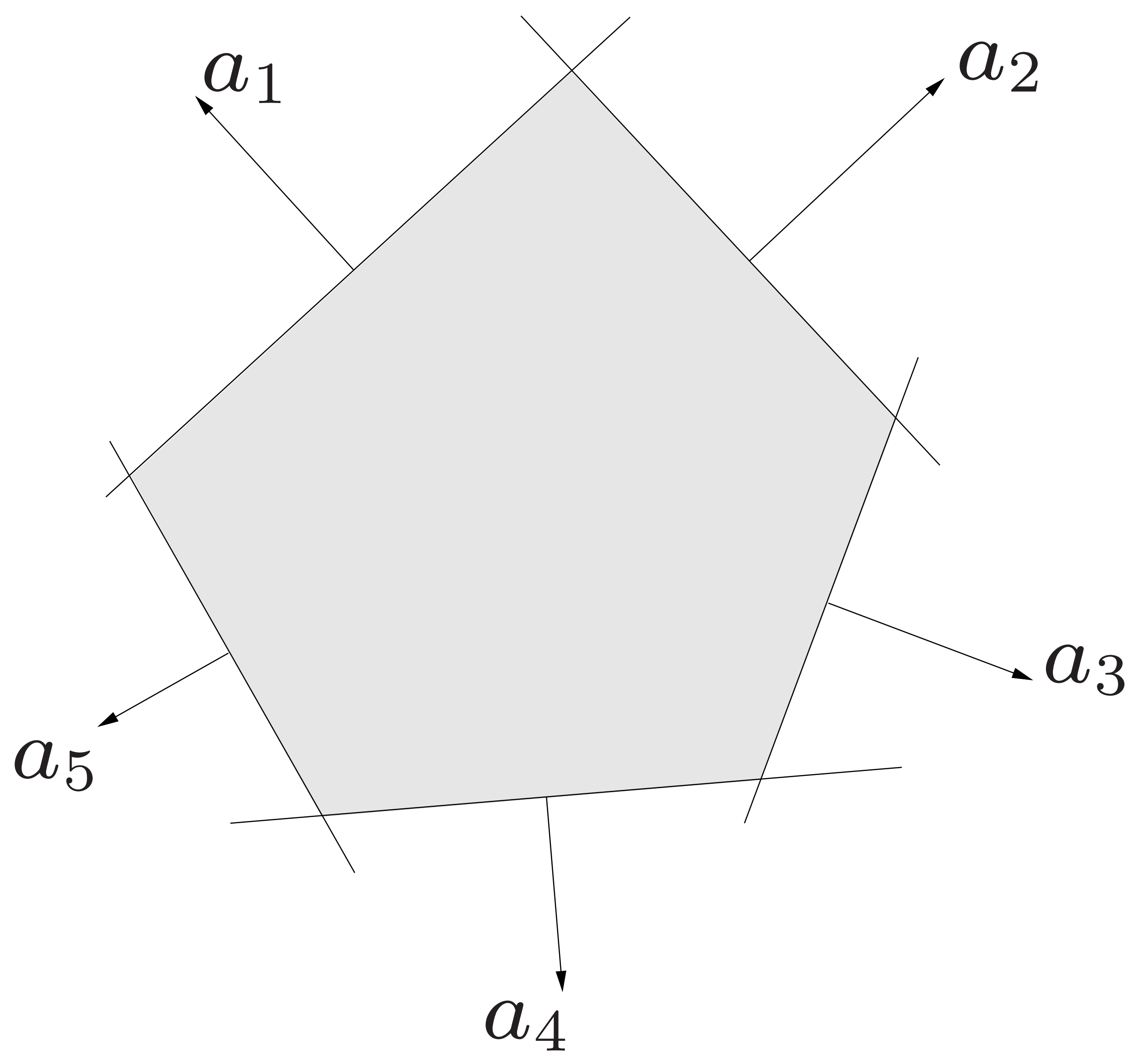


- x_0 is a specific point in the hyperplane
- For any x in the hyperplane defined by $a^T x = b$, $x - x_0 \perp a$
- The halfspace determined by $a^T x \leq b$ extends in the direction of $-a$

Polyhedron

Definition

$$P = \{x \mid a_i^T x \leq b_i, \quad i = 1, \ldots, m\} = \{x \mid Ax \leq b\}$$



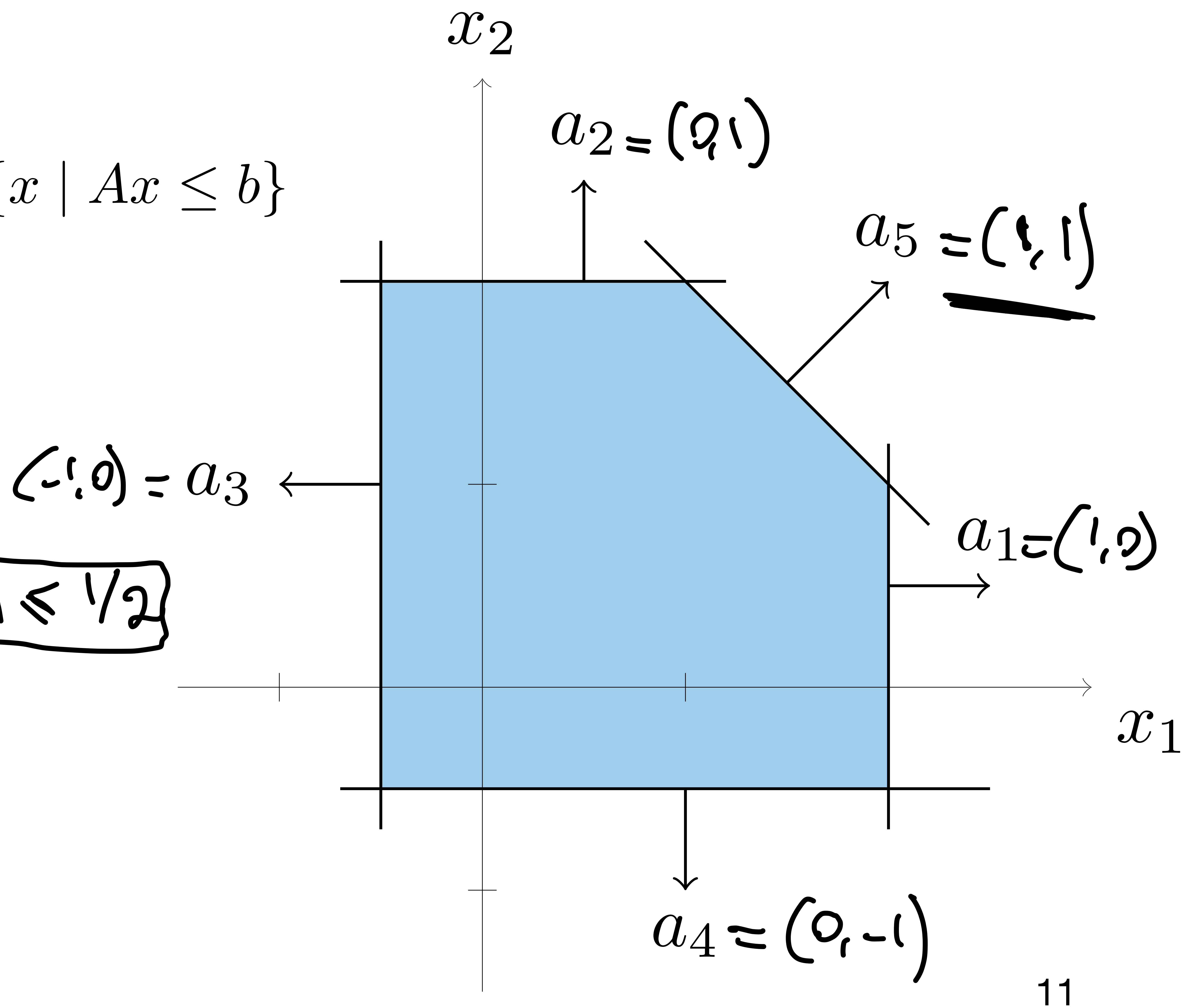
- Intersection of finite number of halfspaces
- Can include equalities

Polyhedron

Example

$$P = \{x \mid a_i^T x \le b_i, \quad i = 1, \dots, m\} = \{x \mid Ax \le b\}$$

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & x_1 \le 2 \\
& x_2 \le 2 \\
& x_1 \ge -1/2 \quad \rightarrow \boxed{-x_1 \le 1/2} \\
& x_2 \ge -1/2 \\
& x_1 + x_2 \le 3
\end{array}
$$



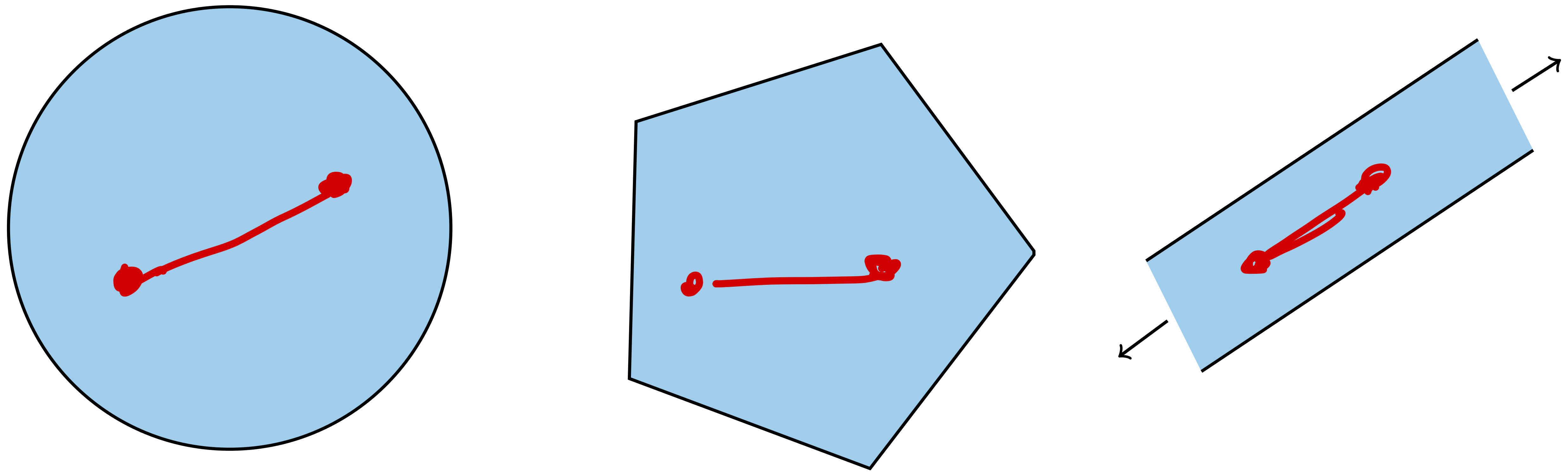
$a_5^T x = 1 \cdot x_1 + 1 \cdot x_2$

Convex set

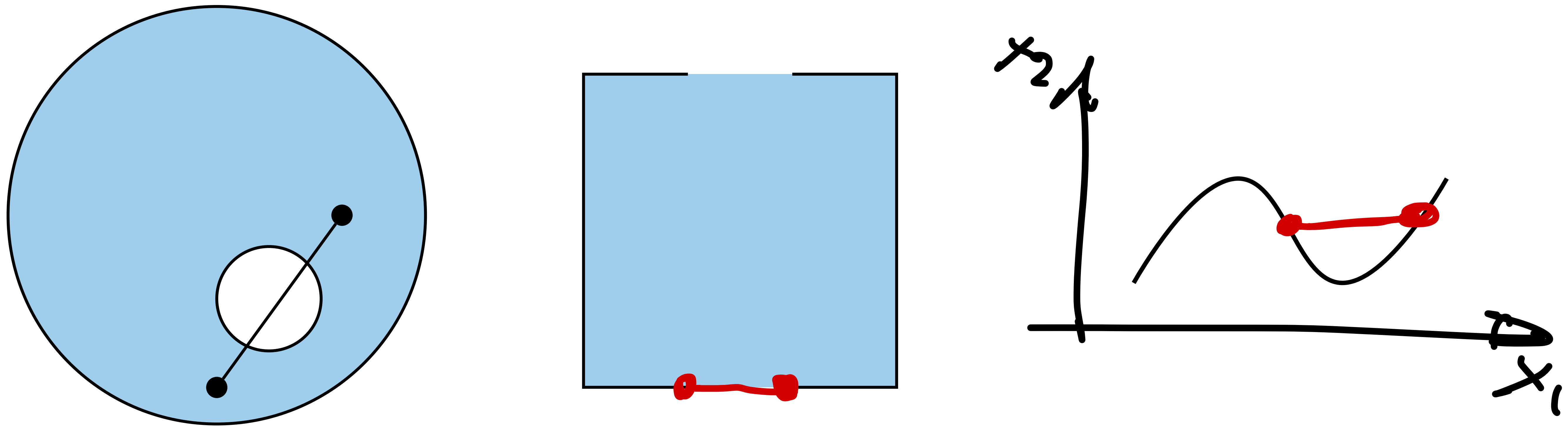
Definition

For any $x, y \in C$ and any $\alpha \in [0, 1]$ $\qquad \alpha x + (1-\alpha) y \in C$

Convex



Nonconvex



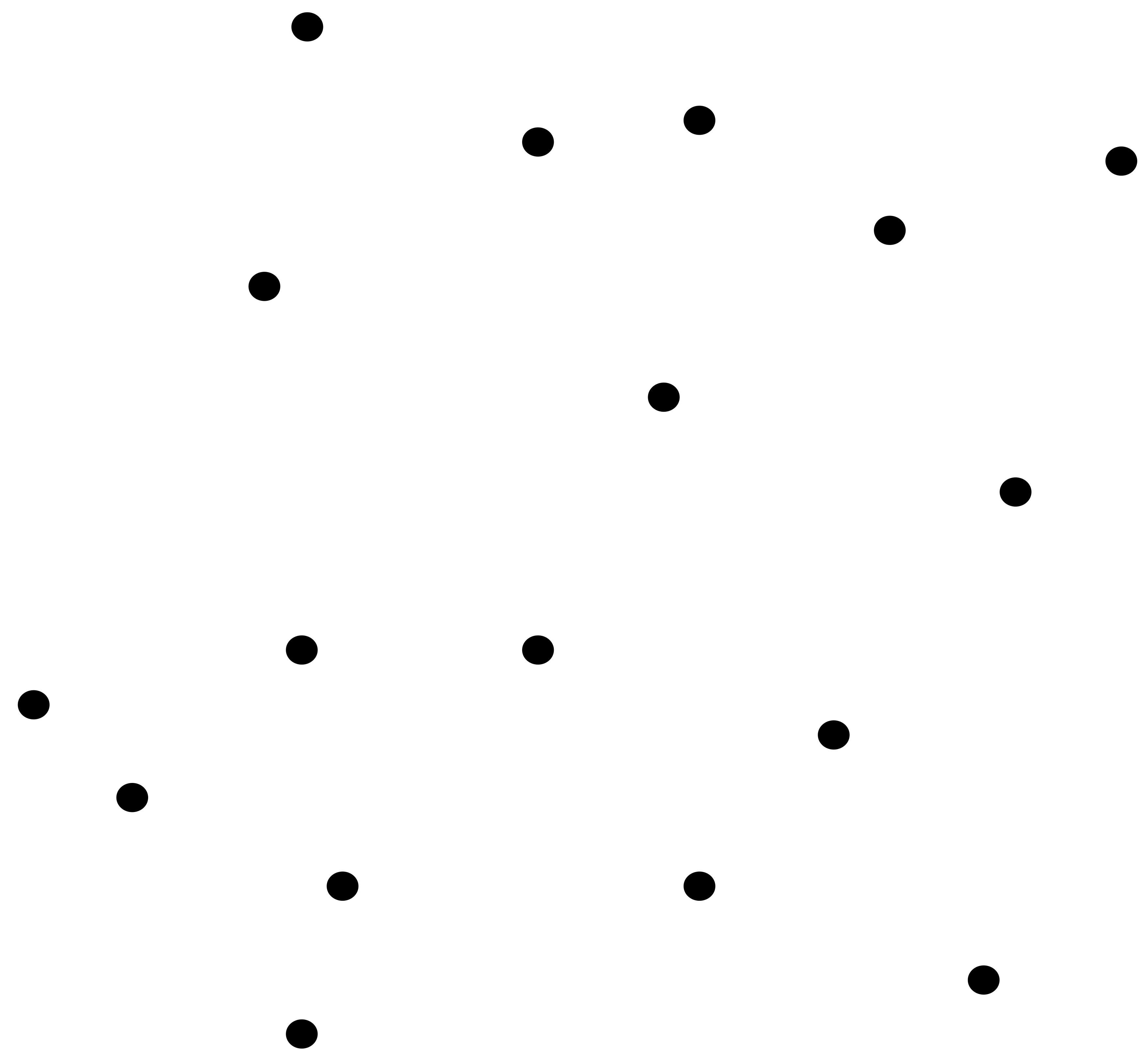
Examples

- $\mathbf{R}^n$
- Hyperplanes
- Halfspaces
- Polyhedra

Convex combinations

Ingredients :

- A collection of points $C = \{x_1, \ldots, x_k\}$
- A collection of non-negative weights α_i
- The weights α_i sum to 1

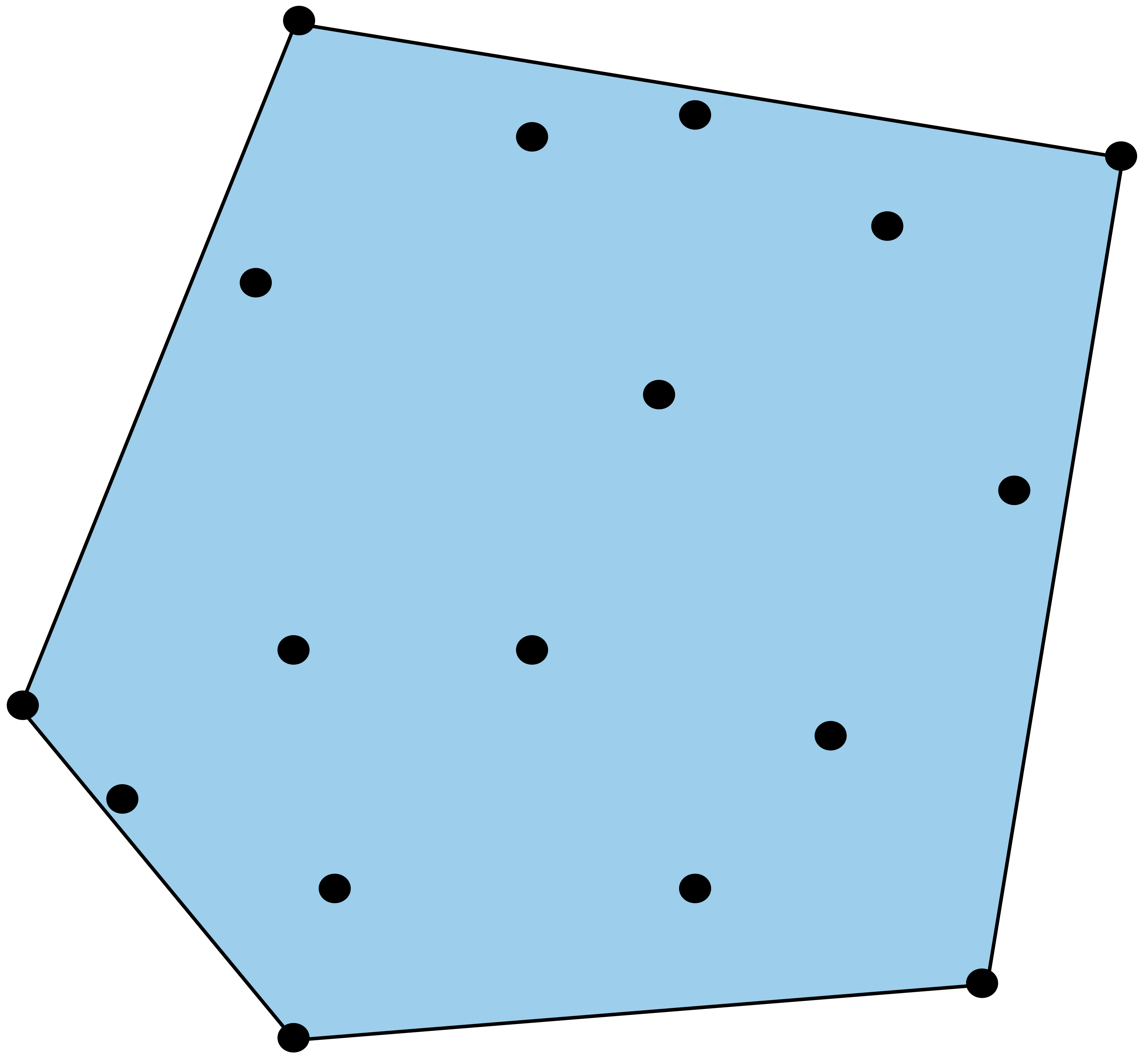


The vector $v = \alpha_1 x_1 + \cdots + \alpha_k x_k$ is a **convex combination** of the points.

Convex hull

The **convex hull** is the set of all possible convex combinations of the points.

$$\mathbf{conv}\, C = \left\{ \sum_{i=1}^{n} \alpha_i x_i \mid \alpha_i \geq 0,\ i = 1, \ldots, n,\ \mathbf{1}^T \alpha = 1 \right\}$$

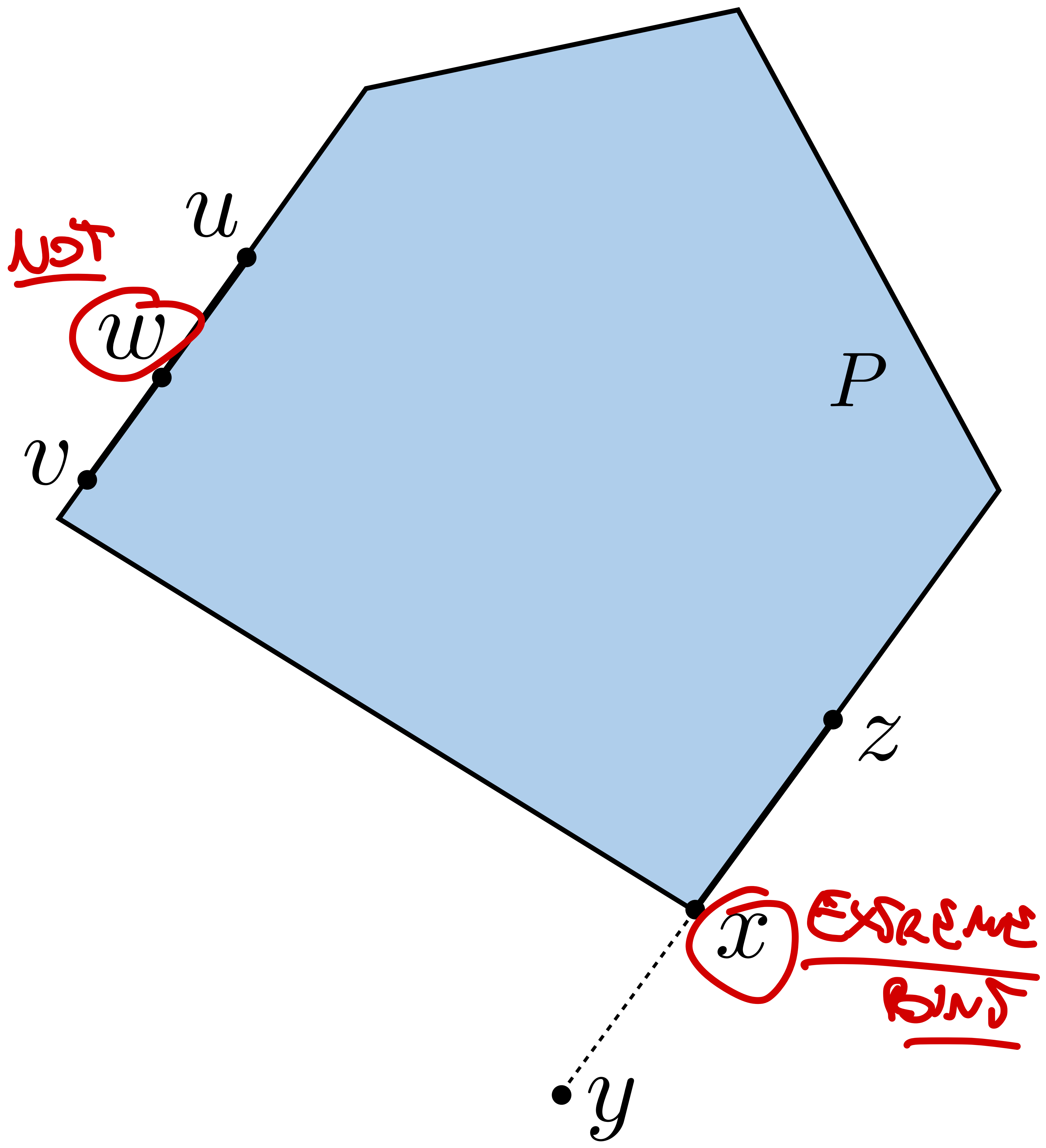


Corners

Extreme points

Definition:

An **extreme point** of a set is one not on a straight line between any other points in the set.

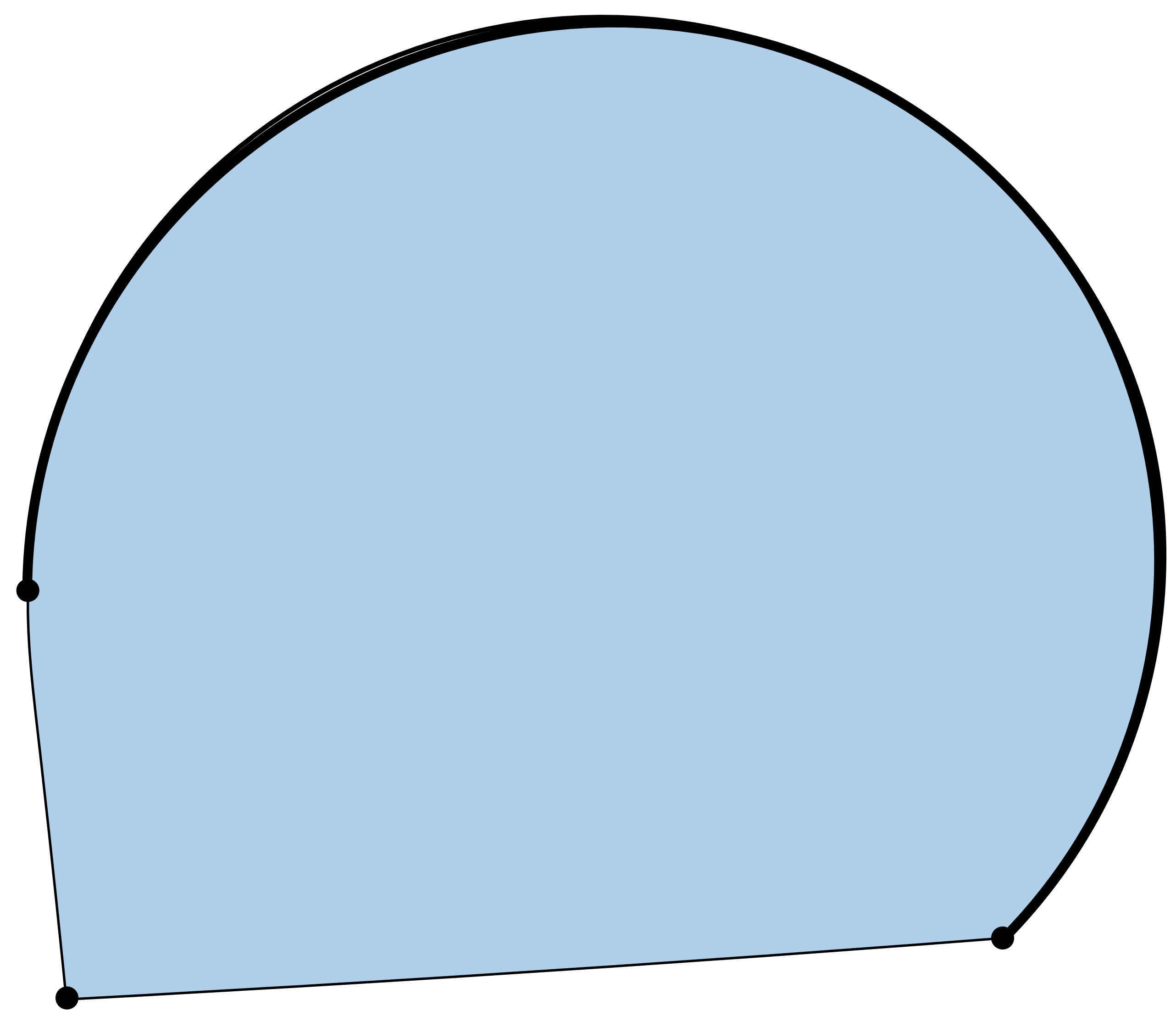


More formal definition:

The point $x \in P$ is an **extreme point** of P if

$\nexists y, z \in P \ (y \neq x, z \neq x)$ and $\alpha \in [0, 1]$ such that $x = \alpha y + (1 - \alpha) z$

Extreme points



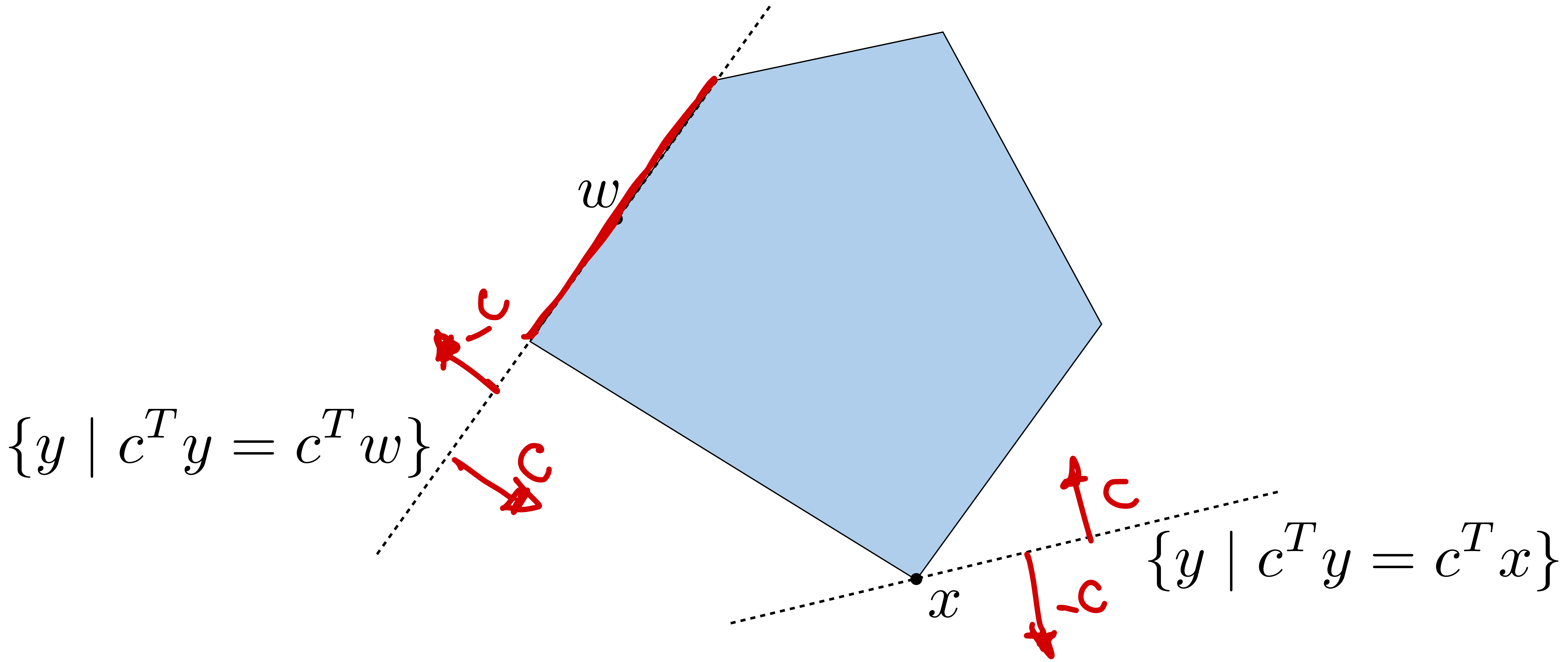
- General convex sets can have an infinite number of extreme points
- **Polyhedra** are convex sets with a finite number of extreme points

Vertices

$c^T x < c^T y \quad \forall y \in P, \; y \neq x$

The point $x \in P$ is a **vertex** if $\exists c$ such that x is the unique optimum of

$$\begin{array}{ll} \text{minimize} & c^T y \\ \text{subject to} & y \in P \end{array}$$



Basic feasible solution

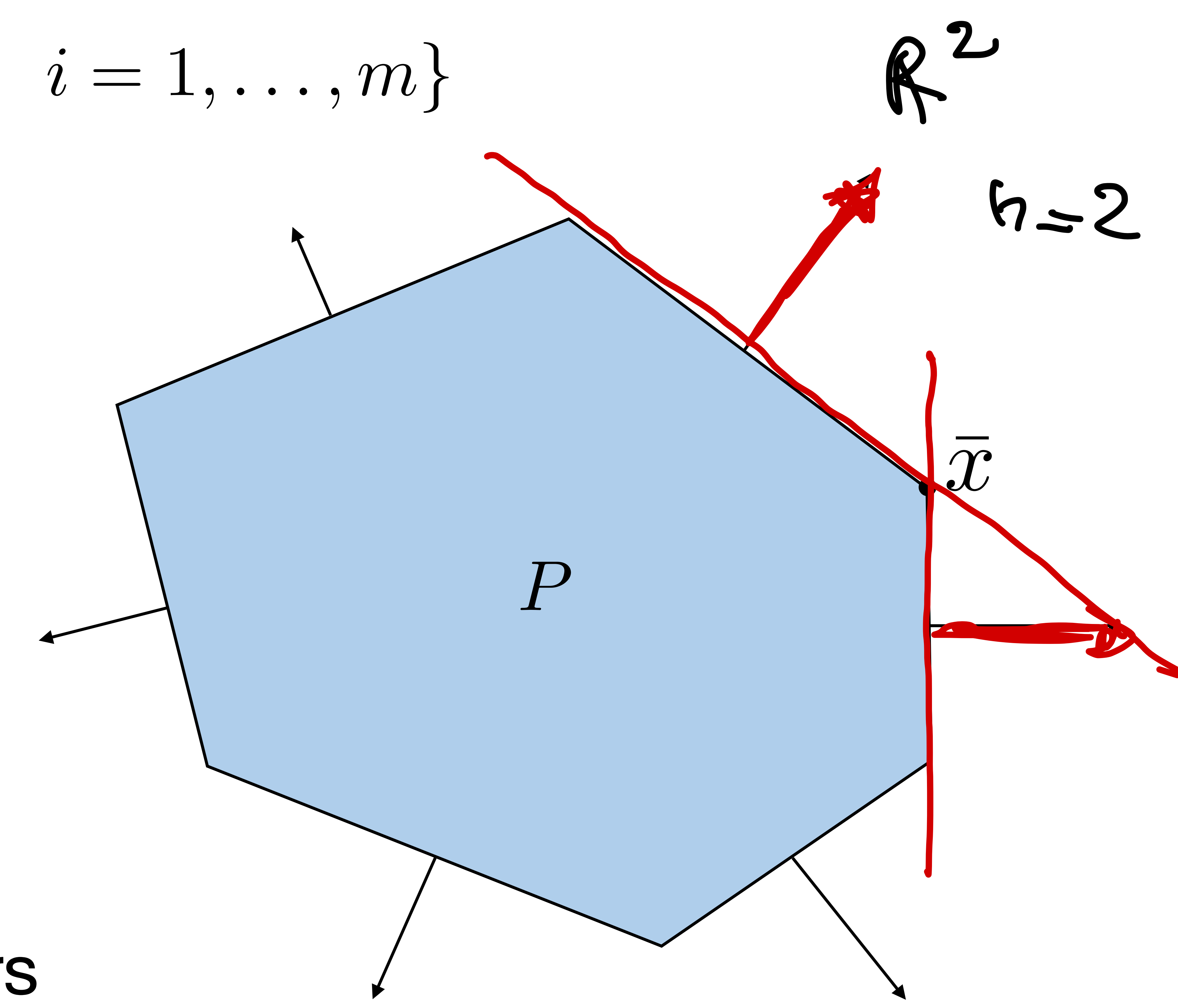
Assume we have a polytope $P = \{x \mid a_i^T x \leq b_i, \quad i = 1, \dots, m\}$

Active constraints at $\bar{x}$

$$\mathcal{I}(\bar{x}) = \{i \in \{1, \dots, m\} \mid a_i^T \bar{x} = b_i\}$$

Basic feasible solution $\bar{x} \in P$

$\{a_i \mid i \in \mathcal{I}(\bar{x})\}$ has n linearly independent vectors



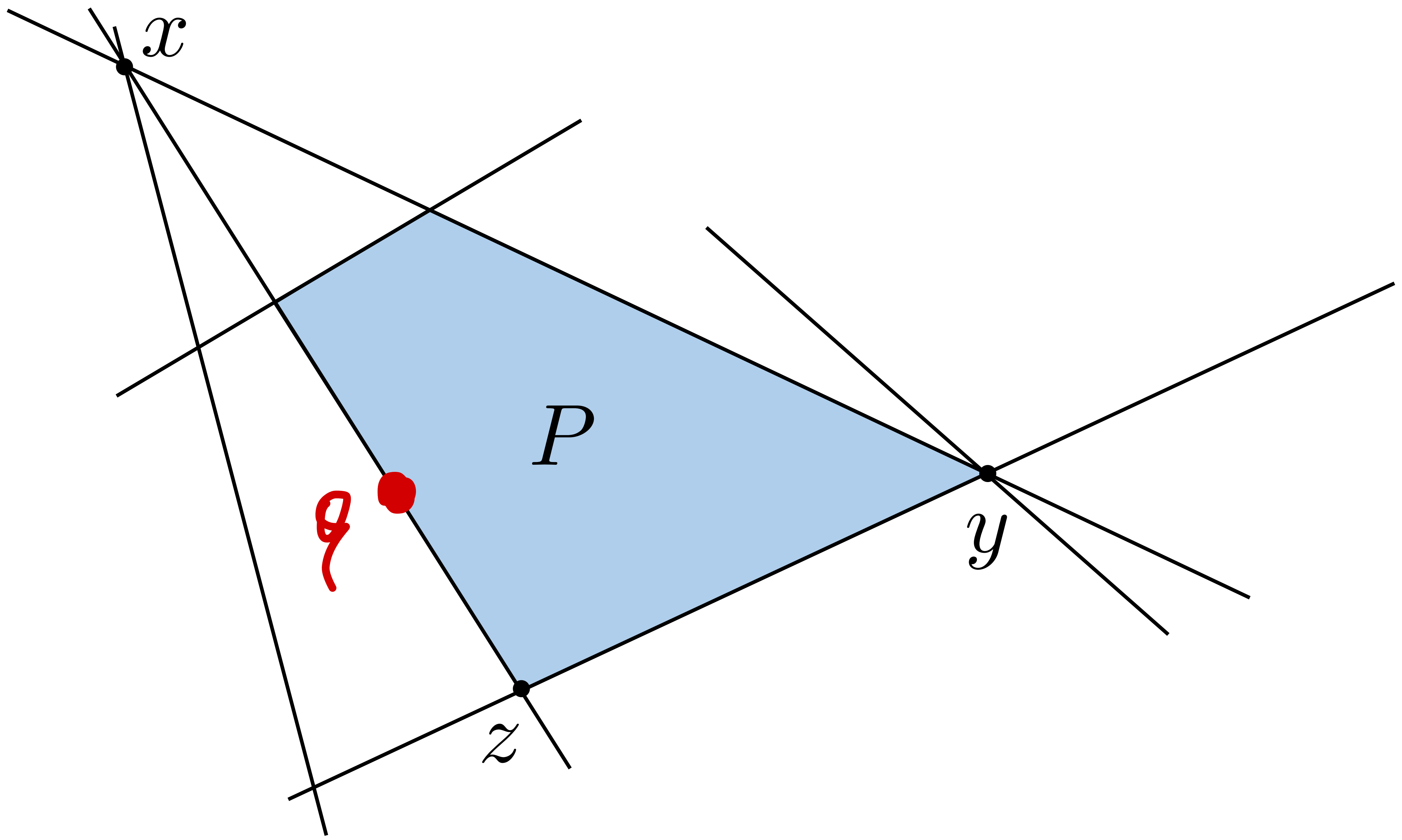
Degenerate basic feasible solutions

A solution $\bar{x}$ is **degenerate** if $|\mathcal{I}(\bar{x})| > n$

$\mathbb{R}^2$

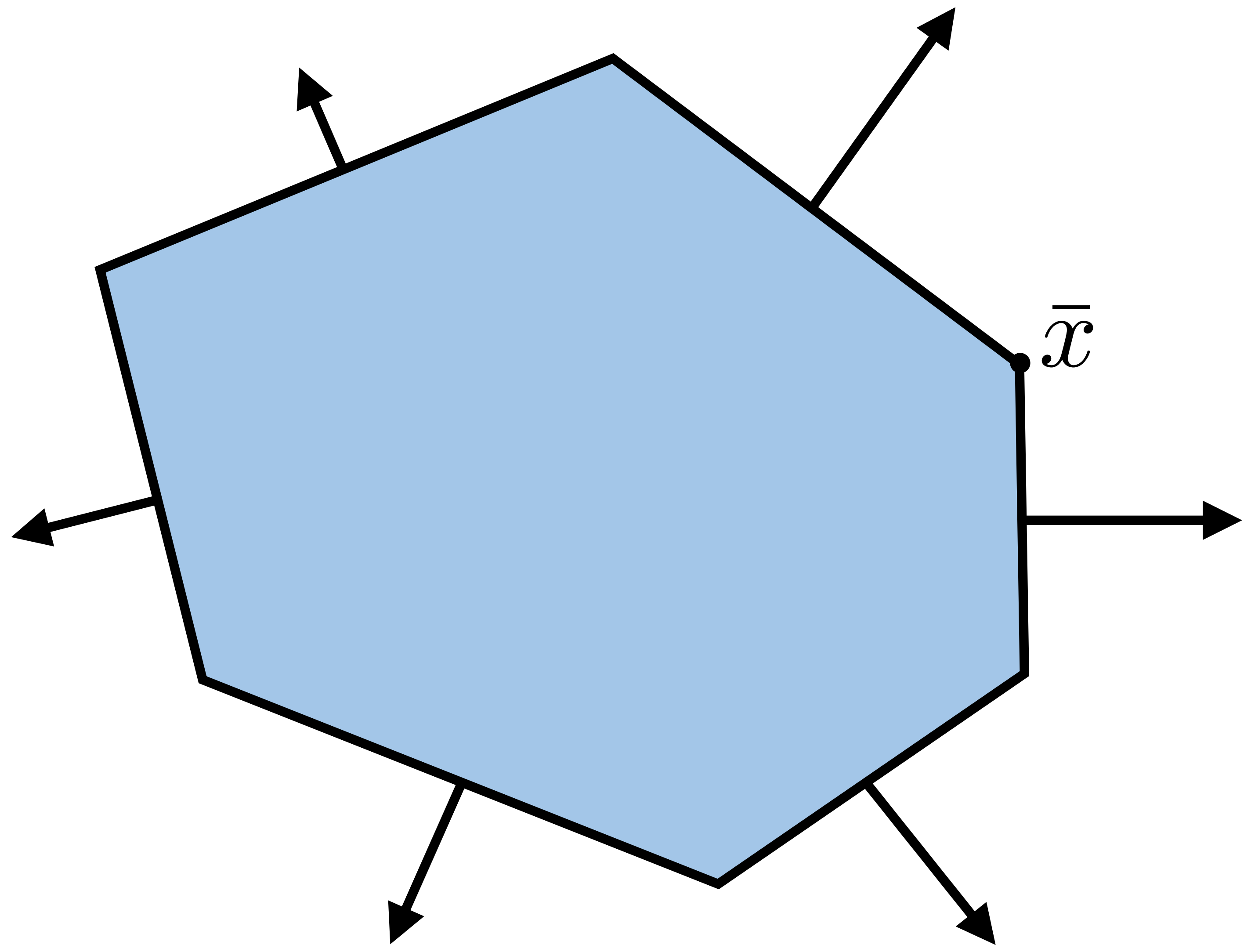
True or False?

	Basic	Feasible	Degenerate
x	Y	NO	Y
y	Y	Y	Y
z	Y	Y	N
q	N	Y	N



An Equivalence Theorem

Given a nonempty polyhedron $P = \{x \mid Ax \leq b\}$



x is a **vertex** $\iff$ x is an **extreme point** $\iff$ x is a **basic feasible solution**

Equivalent theorem proof

Vertex —> Extreme point

If x is a vertex, $\exists c$ such that $c^T x < c^T y, \quad \forall y \in P, y \neq x$

Let's assume x is not an extreme point:

$\exists y, z \neq x$ such that $x = \lambda y + (1 - \lambda) z$ $\lambda \in [0,1]$

Since x is a vertex, $c^T x < c^T y$ and $c^T x < c^T z$

Equivalent theorem proof

Vertex —> Extreme point

If x is a vertex, $\exists c$ such that $c^T x < c^T y, \quad \forall y \in P, y \neq x$

Let's assume x is not an extreme point:

$\exists y, z \neq x$ such that $x = \lambda y + (1 - \lambda) z$

Since x is a vertex, $c^T x < c^T y$ and $c^T x < c^T z$

Therefore, $c^T x = \lambda c^T y + (1 - \lambda) c^T z > \lambda c^T x + (1 - \lambda) c^T x = c^T x$

$\Longrightarrow$ **contradiction** ∎

Equivalent theorem proof

Extreme point —> Basic feasible solution (proof by **contraposition**)

Suppose $x \in P$ is **not basic feasible solution**

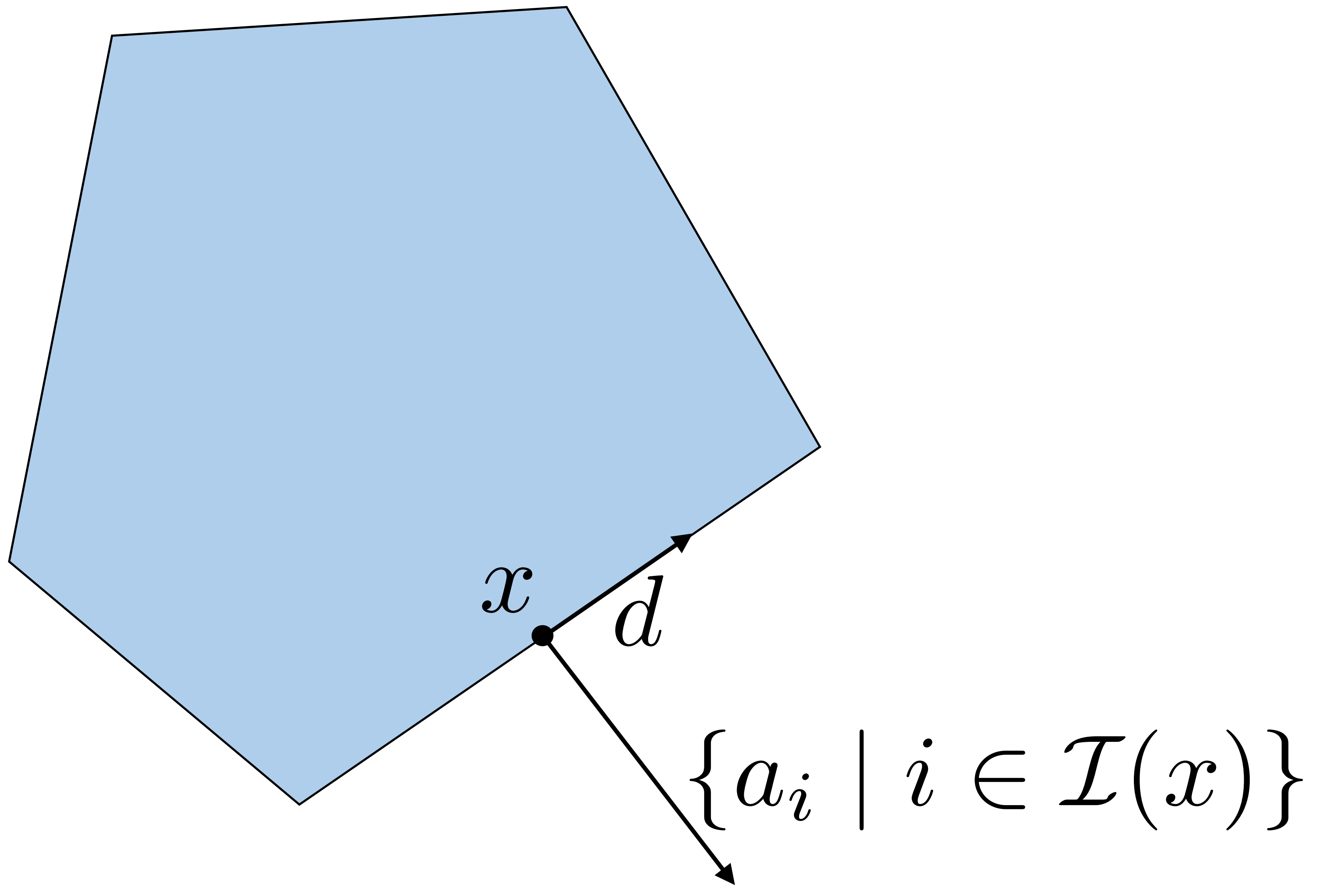
Equivalent theorem proof

Extreme point —> Basic feasible solution (proof by **contraposition**)

Suppose $x \in P$ is **not basic feasible solution**

$\{a_i \mid i \in \mathcal{I}(x)\}$ does not span $\mathbf{R}^n$

$\exists d \in \mathbf{R}^n$ perpendicular to all of them: $a_i^T d = 0, \quad \forall i \in \mathcal{I}(x)$



Equivalent theorem proof

Extreme point —> Basic feasible solution (proof by **contraposition**)

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Equivalent theorem proof

Extreme point —> Basic feasible solution (proof by **contraposition**)

Suppose $x \in P$ is **not basic feasible solution**

$\{a_i \mid i \in \mathcal{I}(x)\}$ does not span $\mathbf{R}^n$

$\exists d \in \mathbf{R}^n$ perpendicular to all of them: $a_i^T d = 0, \quad \forall i \in \mathcal{I}(x)$

Let $\epsilon > 0$ and define $y = x + \epsilon d$ and $z = x - \epsilon d$ $\{a_i^T x = b_i, \quad a_i^T d = 0\}$

For $i \in \mathcal{I}(x)$ we have $a_i^T y = b_i$ and $a_i^T z = b_i$

For $i \notin \mathcal{I}(x)$ we have $a_i^T x < b_i \quad \Rightarrow \quad a_i^T(x + \epsilon d) < b_i$ and $a_i^T(x - \epsilon d) < b_i$

Equivalent theorem proof

Extreme point —> Basic feasible solution (proof by **contraposition**)

Suppose $x \in P$ is **not basic feasible solution**

$\{a_i \mid i \in \mathcal{I}(x)\}$ does not span $\mathbf{R}^n$

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For $i \in \mathcal{I}(x)$ we have $a_i^T y = b_i$ and $a_i^T z = b_i$

For $i \notin \mathcal{I}(x)$ we have $a_i^T x < b_i \quad \Rightarrow \quad a_i^T(x + \epsilon d) < b_i$ and $a_i^T(x - \epsilon d) < b_i$

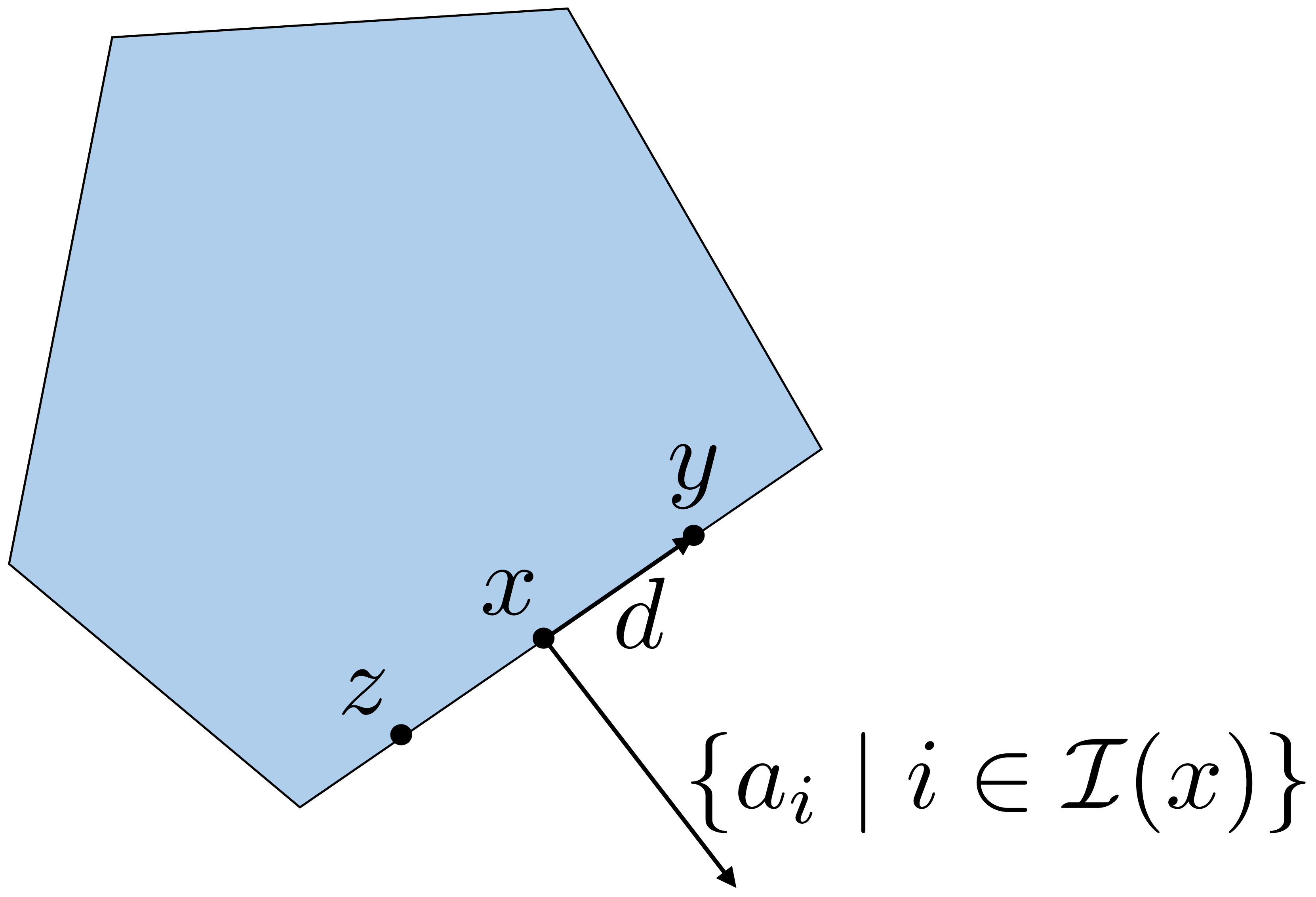
Hence, $y, z \in P$ and $x = \lambda y + (1 - \lambda) z$ with $\lambda = 0.5$.

$\implies x$ is **not an extreme point** ■

Equivalent theorem proof

Extreme point —> Basic feasible solution (proof by **contraposition**)

Suppose $x \in P$ is **not basic feasible solution**



Hence, $y, z \in P$ and $x = \lambda y + (1 - \lambda) z$ with $\lambda = 0.5$.

$\implies x$ is **not an extreme point** ∎

Equivalence theorem proof

Basic feasible solution —> Vertex

Left as exercise

Hint

Define $c = \sum_{i \in \mathcal{I}(x)} a_i$

Constructing basic solutions

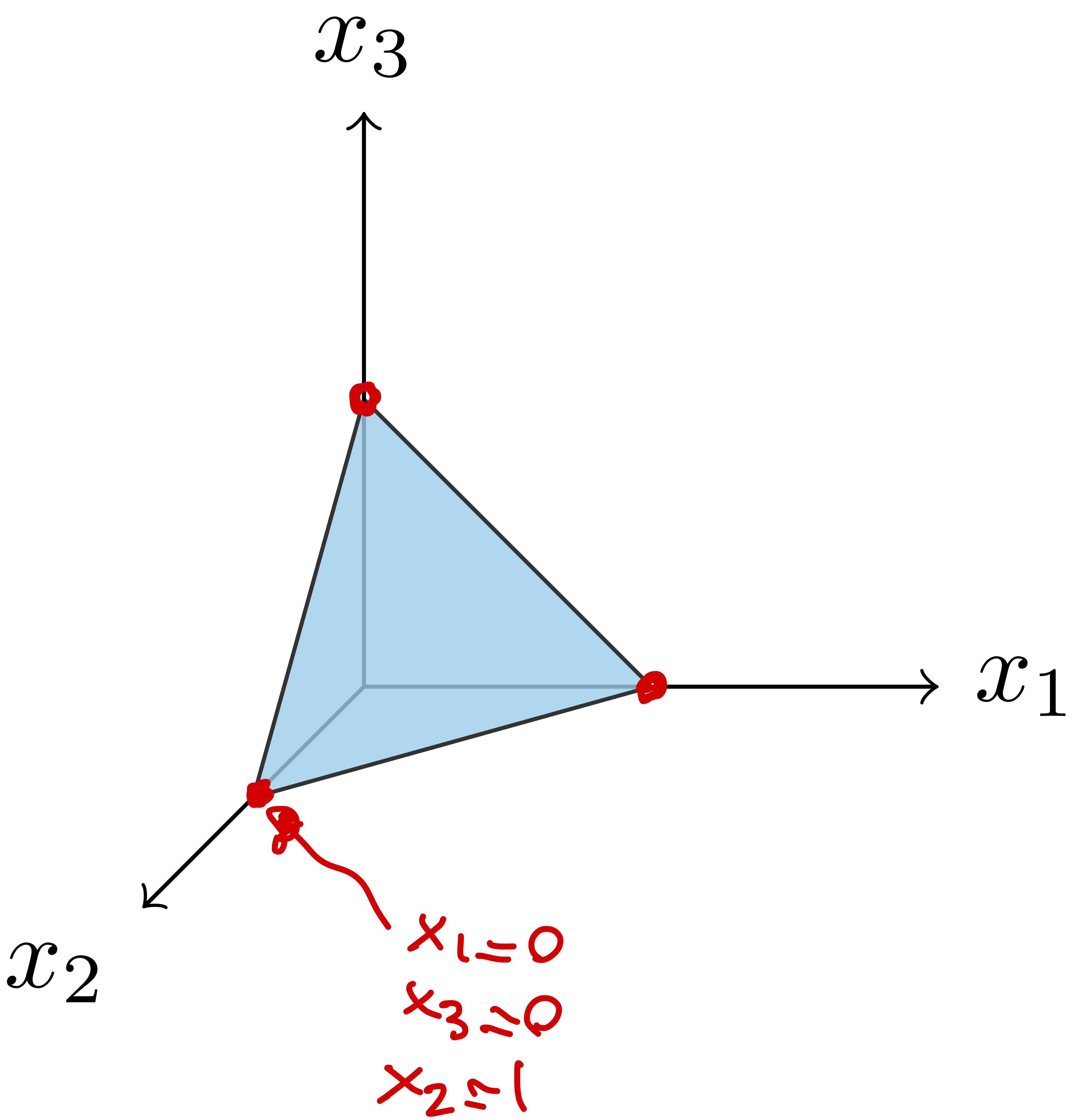
3D example

One equality $(m = 1, n = 3)$

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & x_1 + x_2 + x_3 = 1 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Basic feasible solution $\bar{x}$ has n linearly independent active constraints.

$n - m = 2$ inequalities have to be tight: $x_i = 0$

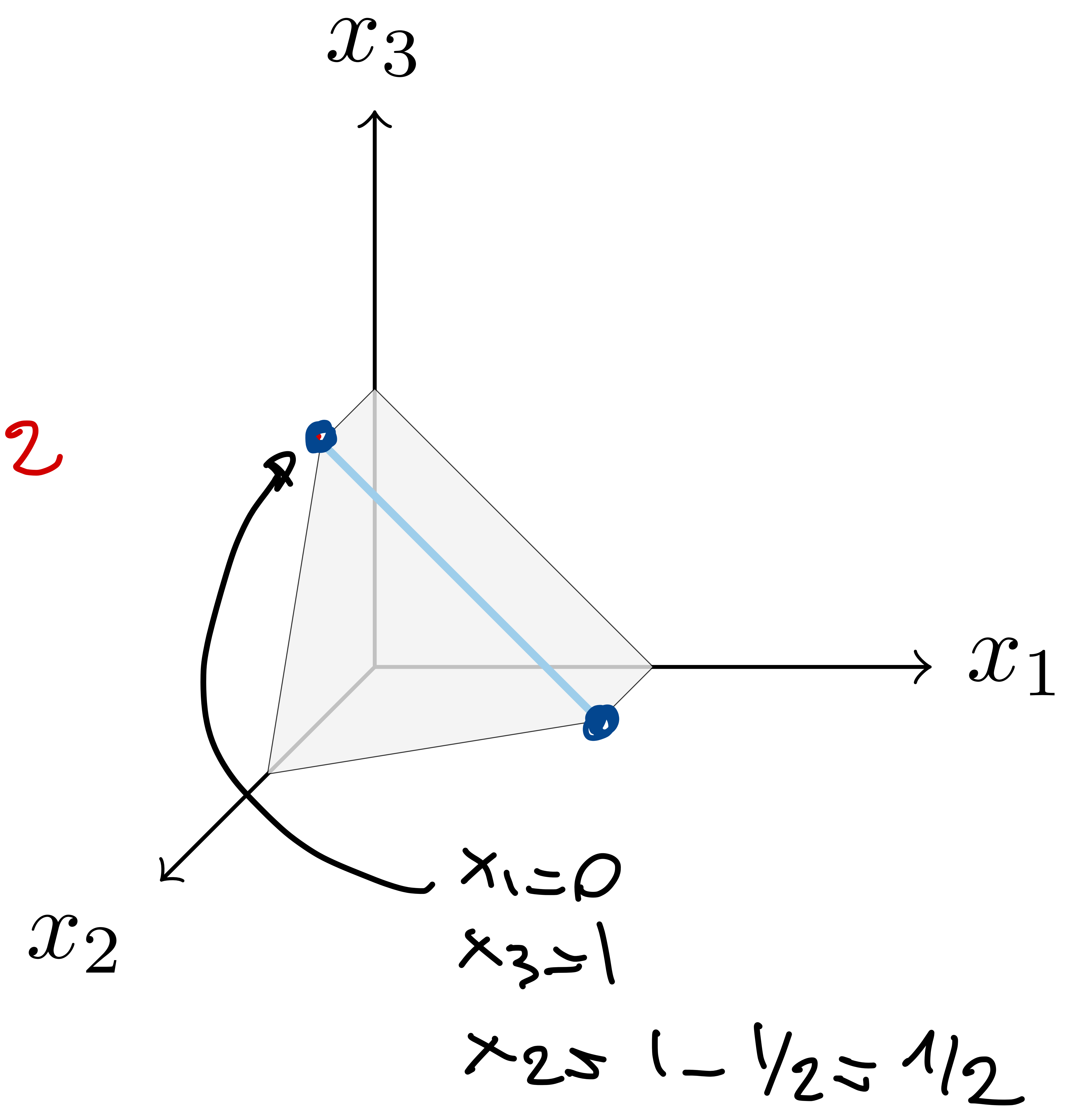


3D example

Two equalities $(m = 2, n = 3)$

minimize	$c^T x$
subject to	$x_1 + x_3 = 1$
	$(1/2)x_1 + x_2 + (1/2)x_3 = 1$
	$x_1, x_2, x_3 \geq 0$

m = 2



Basic feasible solution $\bar{x}$ has n linearly independent active constraints.

↓

$n - m = 1$ inequalities have to be tight: $x_i = 0$

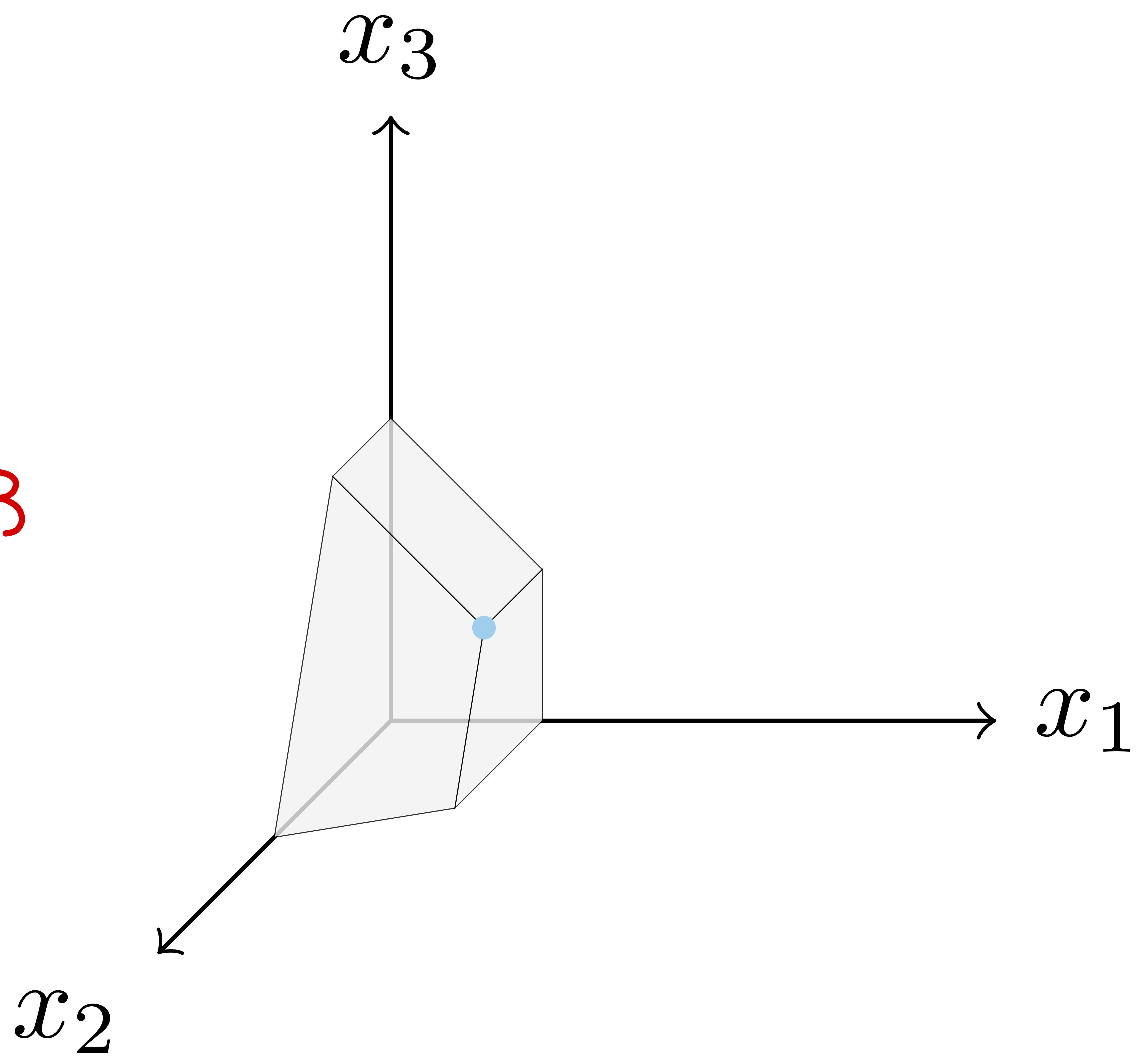
3D example

Three equalities $(m = 3, n = 3)$

minimize $c^T x$

subject to
$$x_1 + x_3 = 1$$
$$(1/2)x_1 + x_2 + (1/2)x_3 = 1$$
$$2x_1 = 1$$
$$x_1, x_2, x_3 \geq 0$$

m=3



Basic feasible solution $\bar{x}$ has n linearly independent active constraints.

↓

$n - m = 0$ inequalities have to be tight: $x_i = 0$

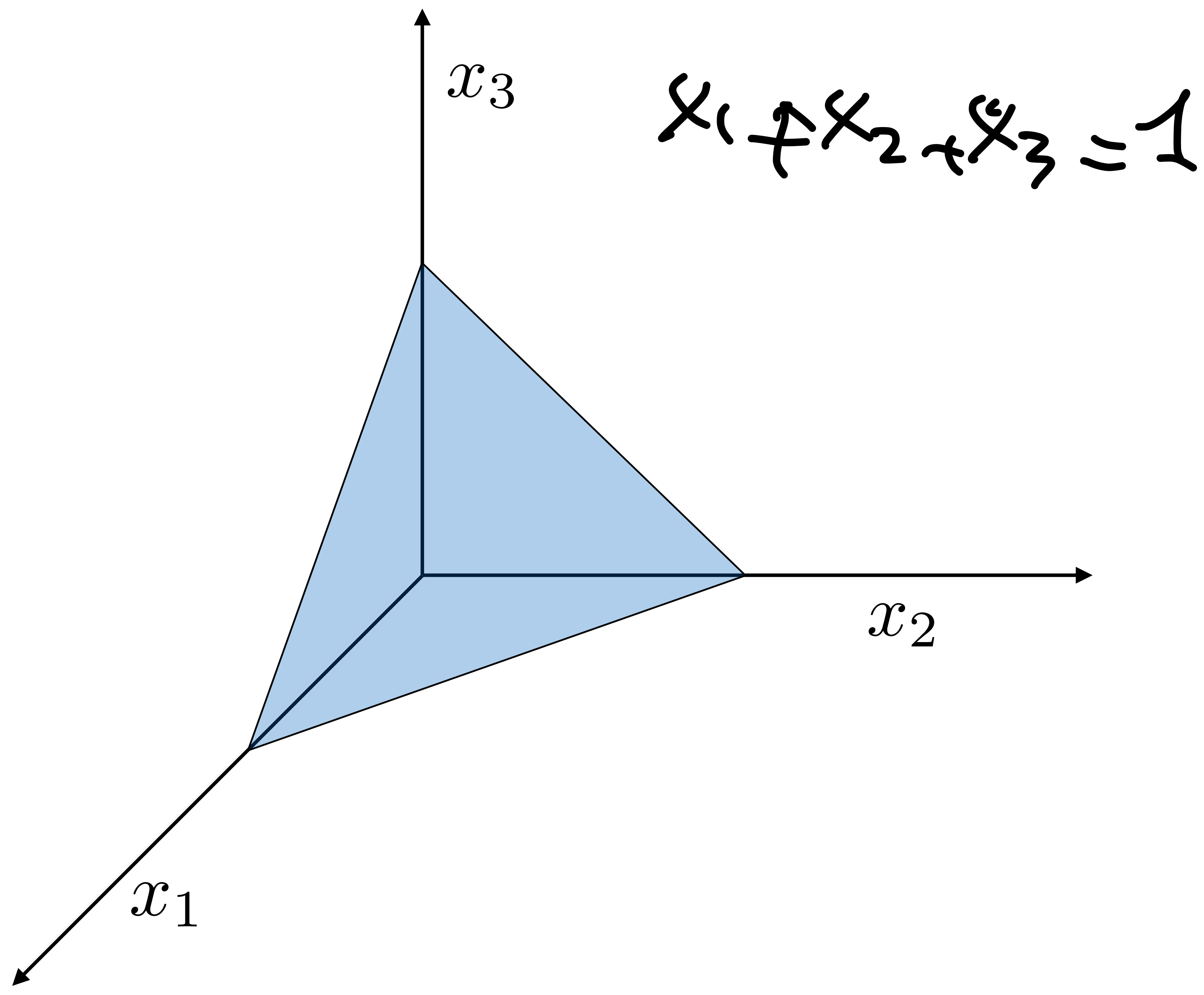
Standard form polyhedra

Standard form LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Standard form polyhedron

$$P = \{x \mid Ax = b,\ x \geq 0\}$$



$n = 3, m = 1$

Assumption

$A \in \mathbf{R}^{m \times n}$ has full row rank $m \leq n$

Interpretation

P is an $(n-m)$-dimensional surface

Constructing a basic solution

Two equalities $(m = 2, n = 3)$

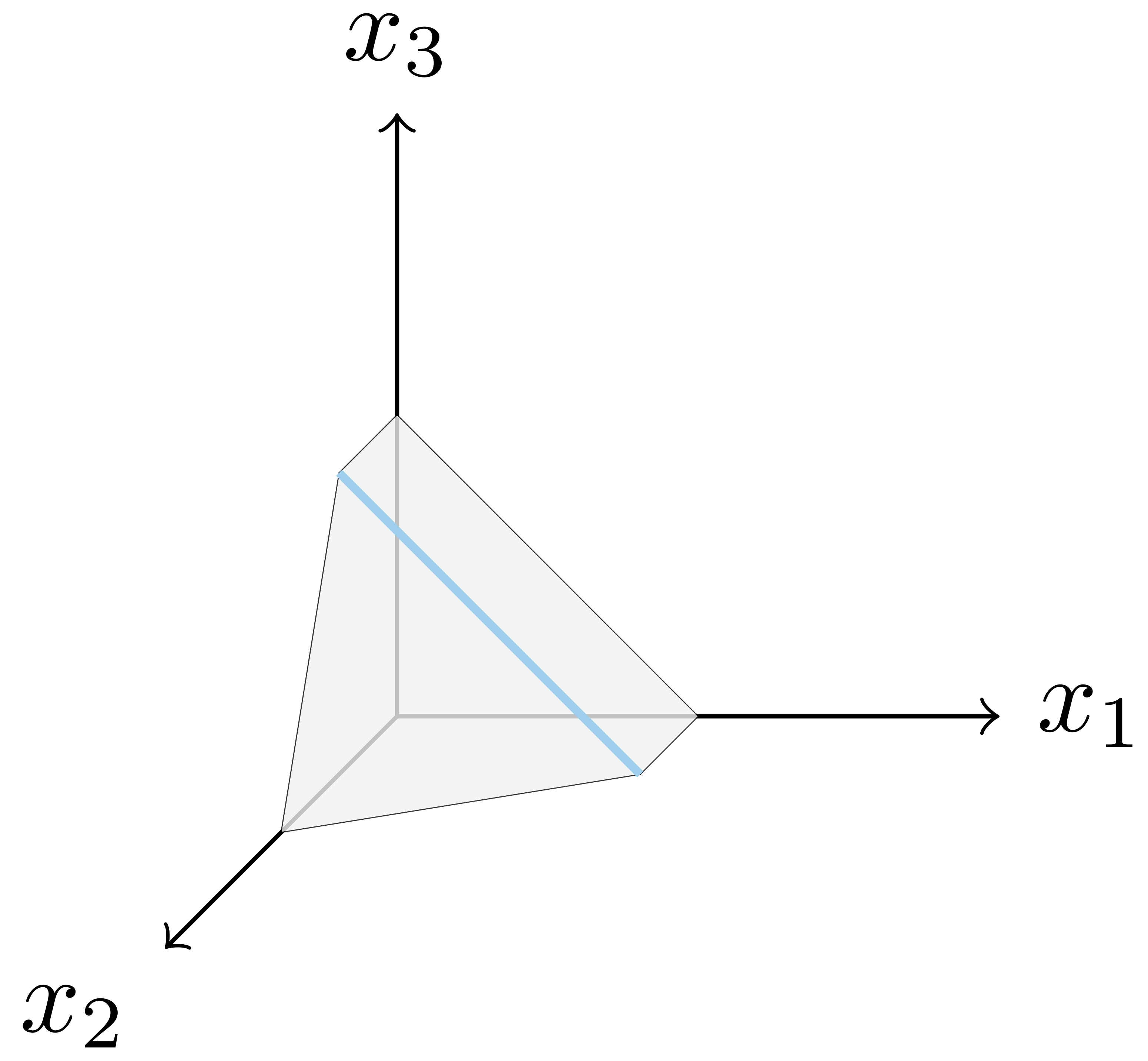
minimize $c^T x$

subject to $x_1 + x_3 = 1$

$(1/2)x_1 + x_2 + (1/2)x_3 = 1$

$x_1, x_2, x_3 \geq 0$

$n - m = 1$ inequalities have to be tight: $x_i = 0$



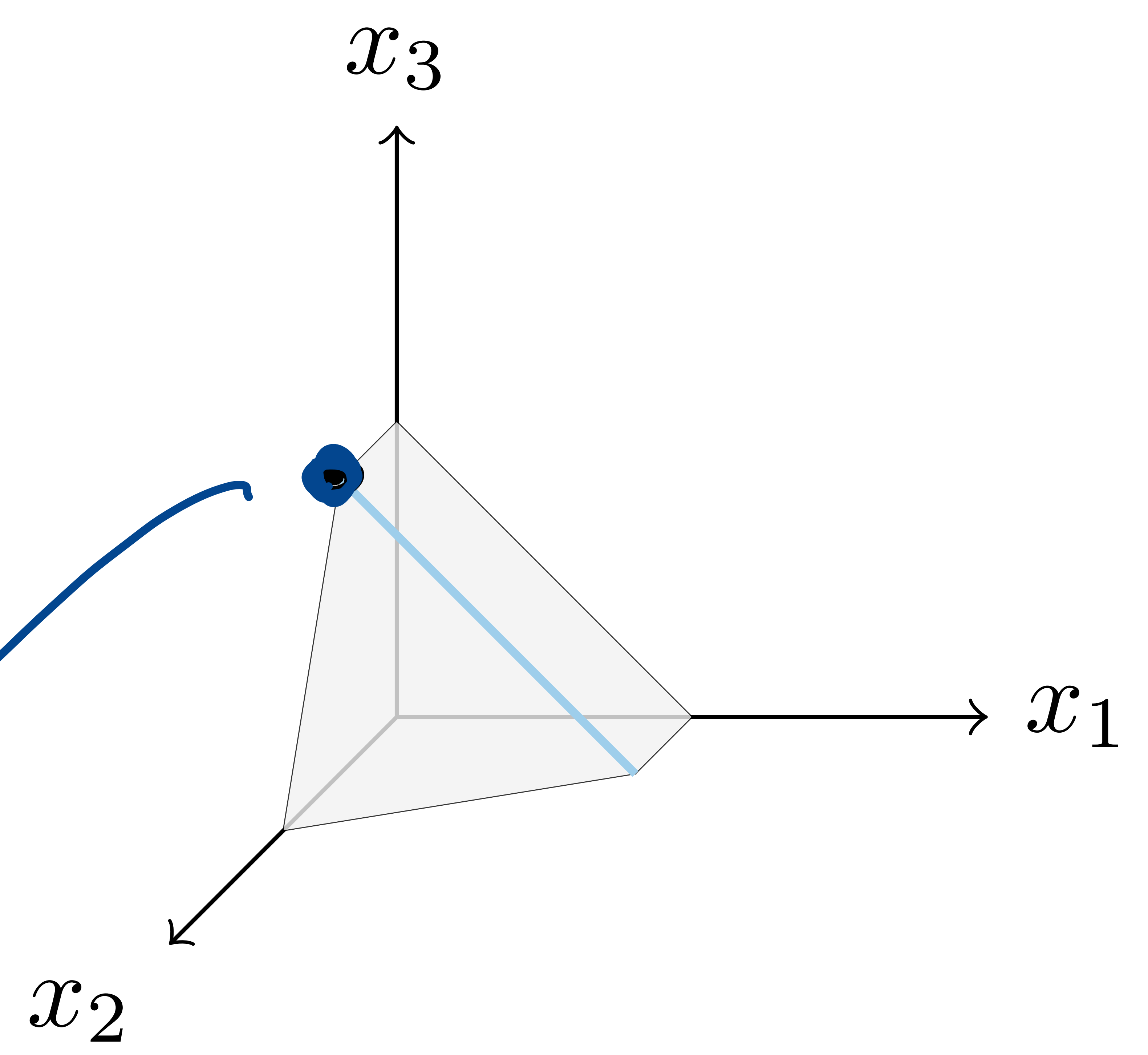
Constructing a basic solution

Two equalities $(m = 2, n = 3)$

minimize $c^T x$

subject to
$$x_1 + x_3 = 1$$
$$(1/2)x_1 + x_2 + (1/2)x_3 = 1$$
$$x_1, x_2, x_3 \geq 0$$

Ax=b

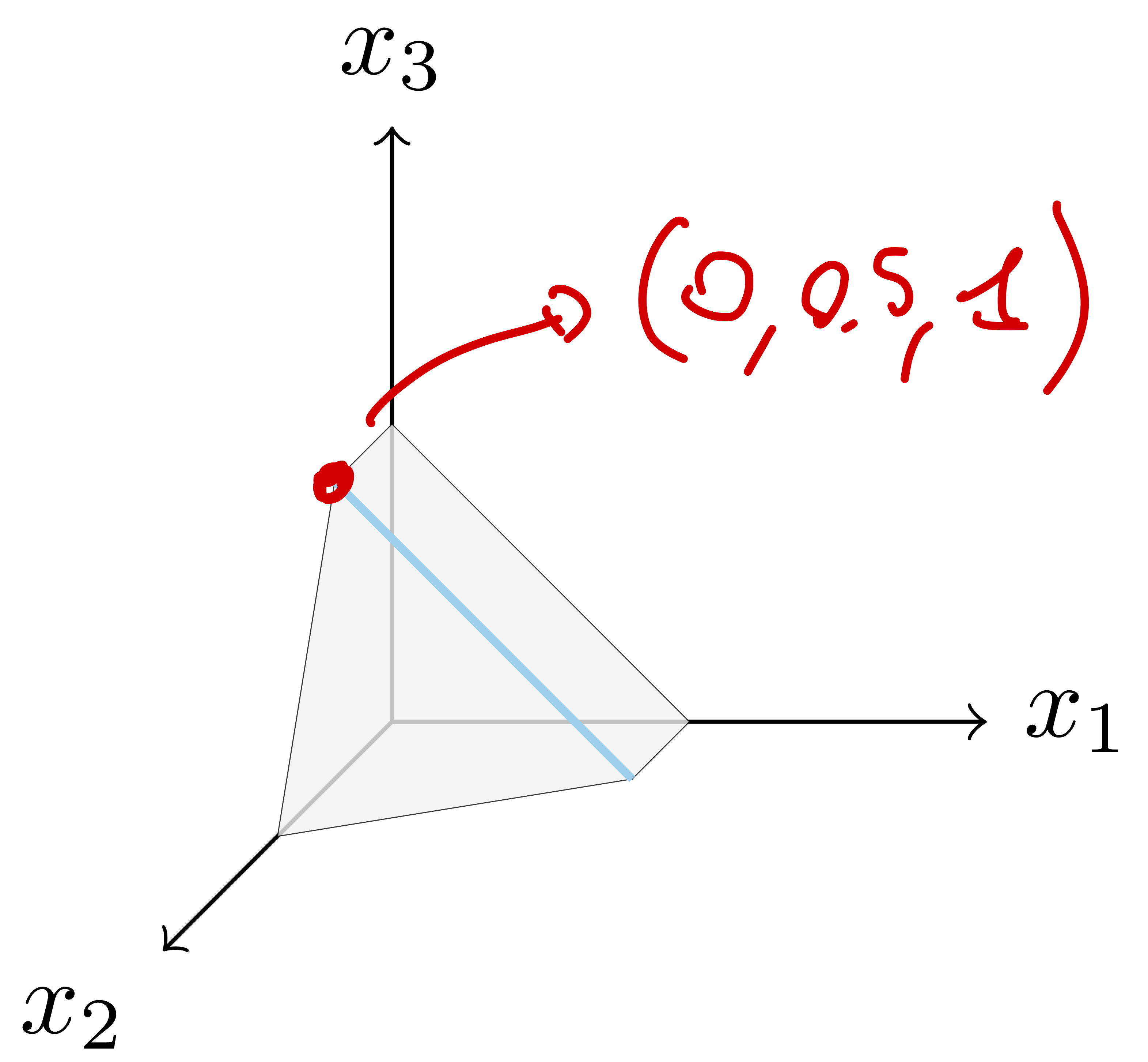


$n - m = 1$ inequalities have to be tight: $x_i = 0$

Set $x_1 = 0$ and solve

$$\begin{bmatrix} 1 & 0 & 1 \\ 1/2 & 1 & 1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 \\ 1 & 1/2 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Constructing a basic solution



Two equalities $(m = 2, n = 3)$

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x_1 + x_3 = 1 \\ & (1/2)x_1 + x_2 + (1/2)x_3 = 1 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

$n - m = 1$ inequalities have to be tight: $x_i = 0$

Set $x_1 = 0$ and solve

$$\begin{bmatrix} 1 & 0 & 1 \\ 1/2 & 1 & 1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 \\ 1 & 1/2 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \longrightarrow (x_2, x_3) = (0.5, 1)$$

Basic solutions

Standard form polyhedra

$P = \{x \mid Ax = b,\ x \geq 0\}$ with $A \in \mathbf{R}^{m \times n}$ has full row rank $m \leq n$

Basic solutions

Standard form polyhedra

$P = \{x \mid Ax = b,\ x \geq 0\}$ with $A \in \mathbf{R}^{m \times n}$ has full row rank $m \leq n$

x is a **basic solution** if and only if

- $Ax = b$
- There exist indices $B(1), \dots, B(m)$ such that
  - columns $A_{B(1)}, \dots, A_{B(m)}$ are linearly independent
  - $x_i = 0$ for $i \neq B(1), \dots, B(m)$

Basic solutions

Standard form polyhedra

$P = \{x \mid Ax = b,\ x \geq 0\}$ with $A \in \mathbf{R}^{m \times n}$ has full row rank $m \leq n$

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  - columns $A_{B(1)}, \dots, A_{B(m)}$ are linearly independent
  - $x_i = 0$ for $i \neq B(1), \dots, B(m)$

x is a **basic feasible solution** if x is a **basic solution** and $x \geq 0$

Constructing basic solution

1. Choose any m independent columns of A: $A_{B(1)}, \ldots, A_{B(m)}$
2. Let $x_i = 0$ for all $i \neq B(1), \ldots, B(m)$
3. Solve $Ax = b$ for the remaining $x_{B(1)}, \ldots, x_{B(m)}$

Constructing basic solution

1. Choose any m independent columns of A: $A_{B(1)}, \ldots, A_{B(m)}$
2. Let $x_i = 0$ for all $i \neq B(1), \ldots, B(m)$
3. Solve $Ax = b$ for the remaining $x_{B(1)}, \ldots, x_{B(m)}$

Basis matrix

Basis columns

Basic variables

$$A_B = \begin{bmatrix} | & | & & | \\ A_{B(1)} & A_{B(2)} & \ldots & A_{B(m)} \\ | & | & & | \end{bmatrix}, \quad x_B = \begin{bmatrix} x_{B(1)} \\ \vdots \\ x_{B(m)} \end{bmatrix} \longrightarrow$$

Solve $A_B x_B = b$

Constructing basic solution

1. Choose any m independent columns of A: $A_{B(1)}, \ldots, A_{B(m)}$
2. Let $x_i = 0$ for all $i \neq B(1), \ldots, B(m)$
3. Solve $Ax = b$ for the remaining $x_{B(1)}, \ldots, x_{B(m)}$

Basis matrix — Basis columns — Basic variables

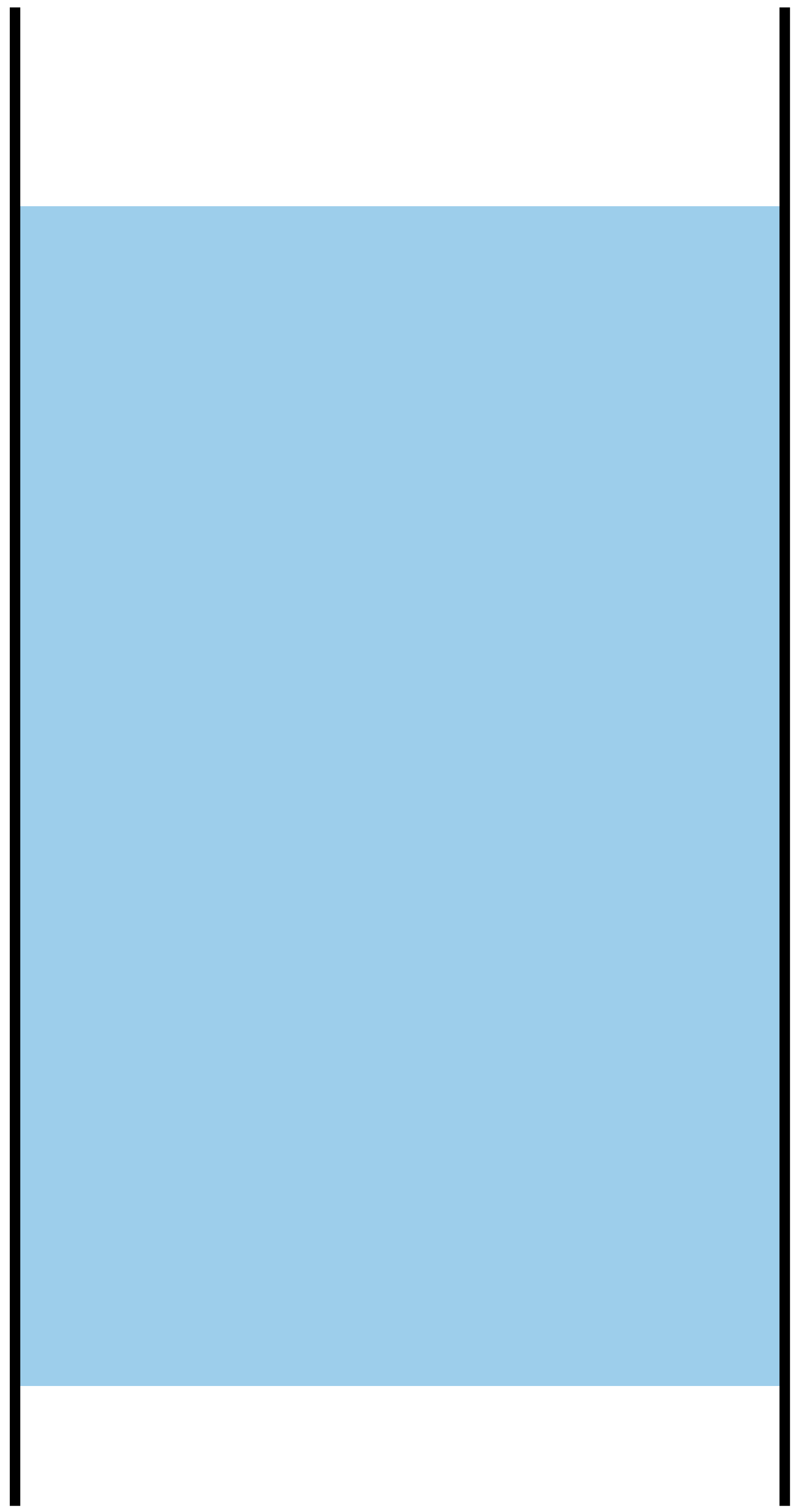
$$A_B = \begin{bmatrix} | & | & & | \\ A_{B(1)} & A_{B(2)} & \ldots & A_{B(m)} \\ | & | & & | \end{bmatrix}, \quad x_B = \begin{bmatrix} x_{B(1)} \\ \vdots \\ x_{B(m)} \end{bmatrix} \longrightarrow \text{Solve } A_B x_B = b$$

If $x_B \geq 0$, then x is a **basic feasible solution**

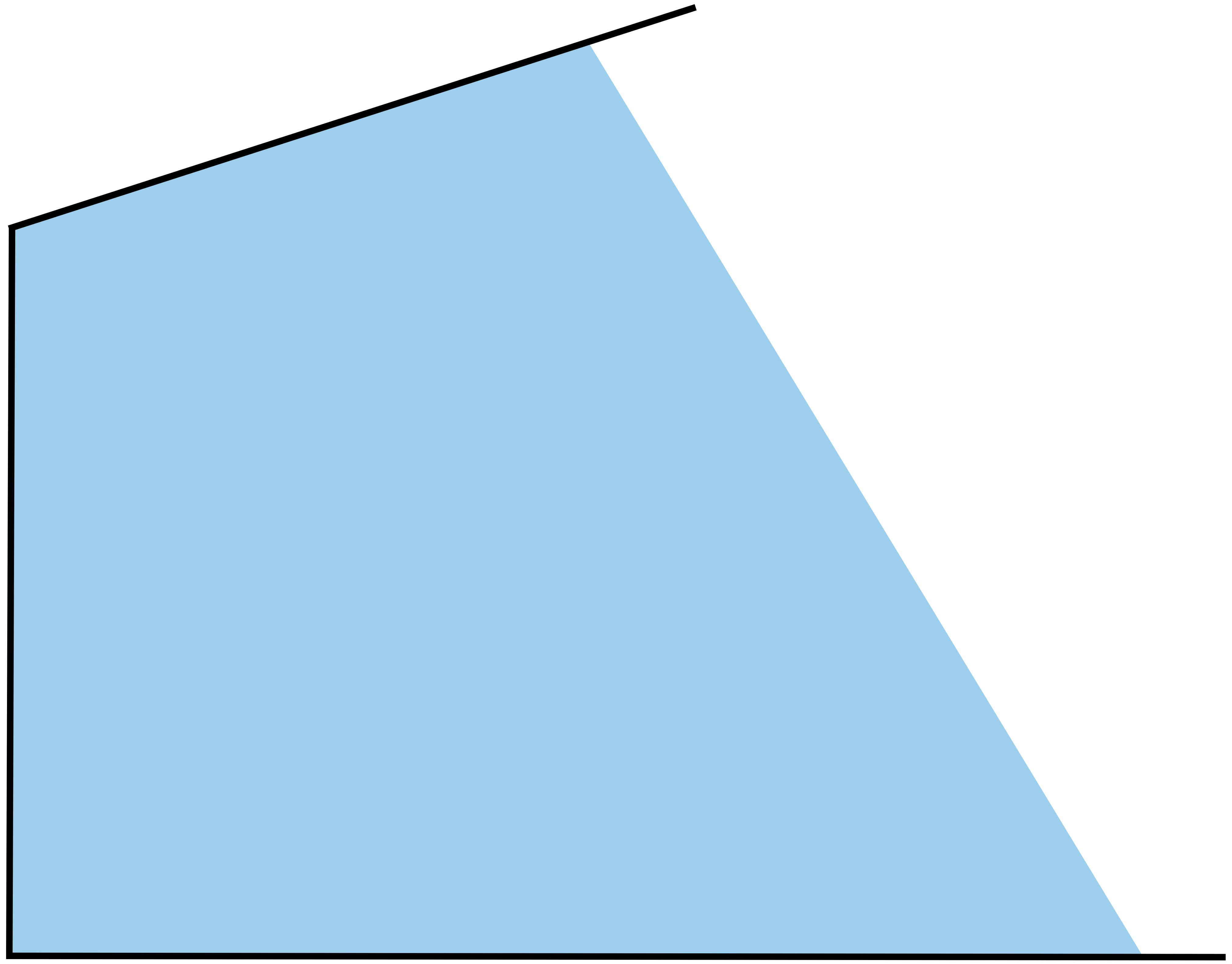
Existence and optimality of extreme points

Existence of extreme points

Example



No extreme points



Extreme points

Existence of extreme points

Characterization

A polyhedron P **contains a line** if

$\exists x \in P$ and a nonzero vector d such that $x + \lambda d \in P, \forall \lambda \in \mathbf{R}$.

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Given a polyhedron $P = \{x \mid a_i^T x \leq b_i, \quad i = 1, \ldots, m\}$, the following are **equivalent**

- P does not contain a line
- P has at least one extreme point
- n of the a_i vectors are linearly independent

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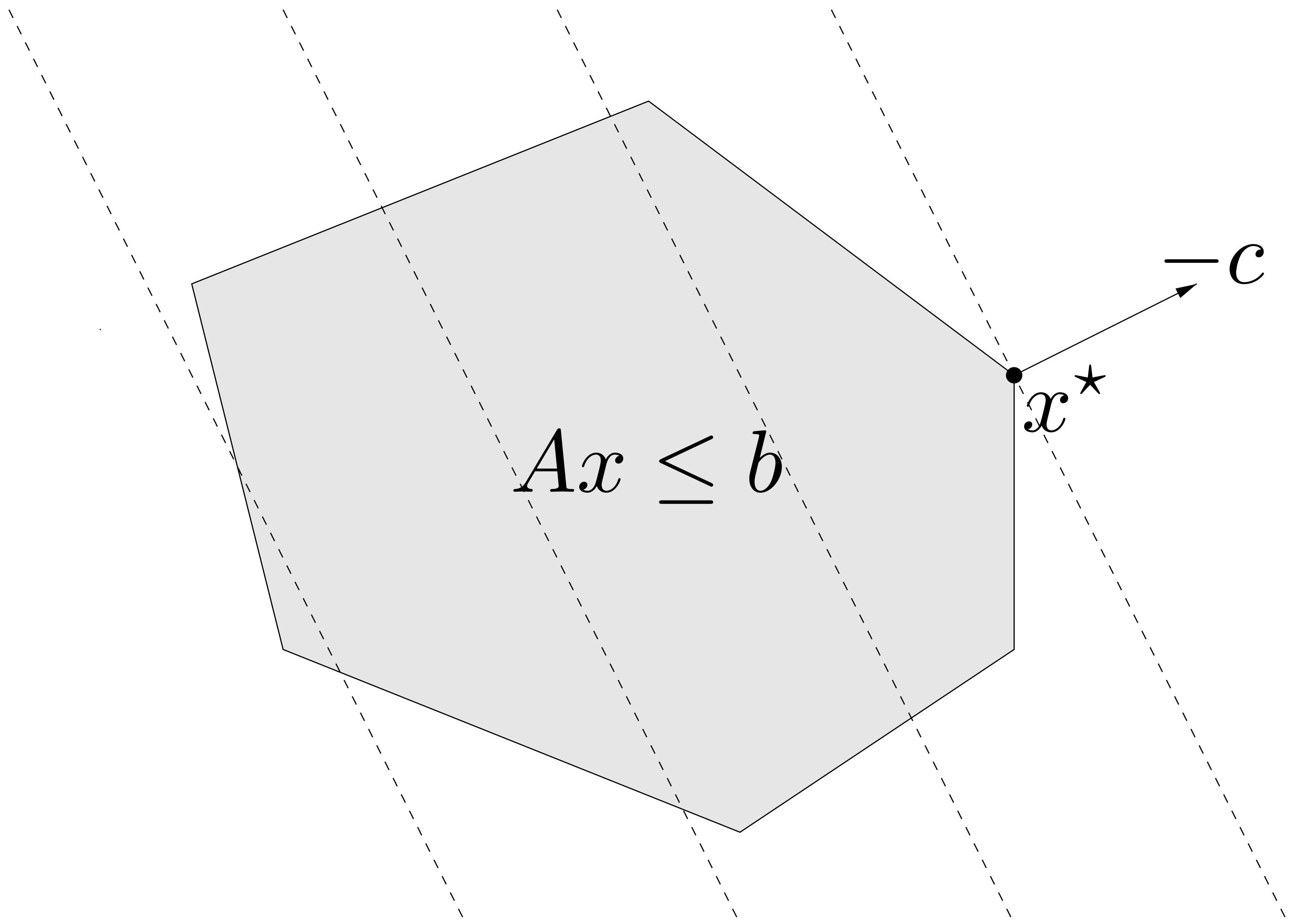
Corollary

Every nonempty **bounded polyhedron** has **at least one basic feasible solution**

Optimality of extreme points



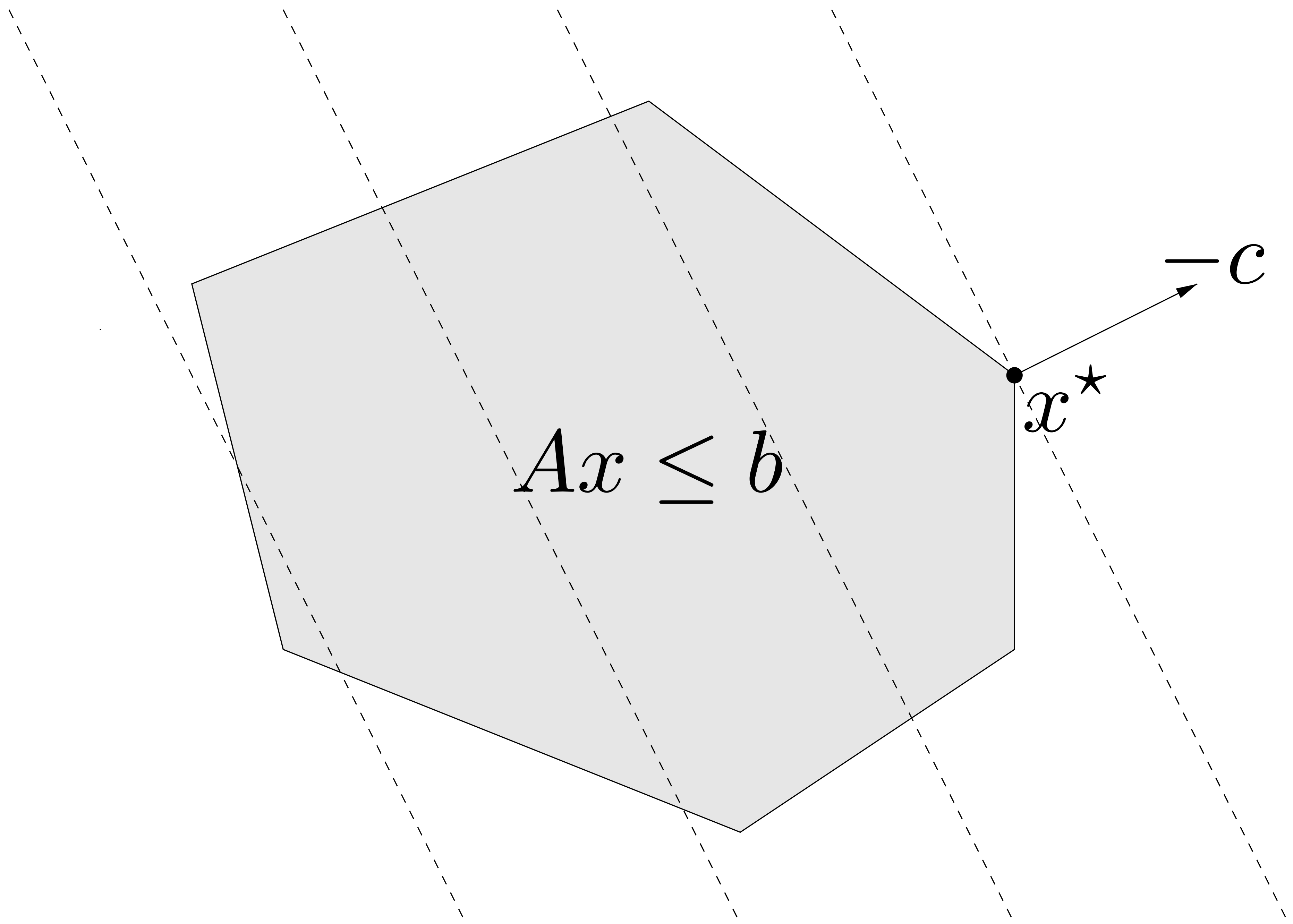
$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

If

- P has at least one extreme point
- There exists an optimal solution $x^\star$

Then, there exists an optimal solution that is an **extreme point** of P.

Optimality of extreme points



minimize $c^T x$
subject to $Ax \leq b$

If

- P has at least one extreme point
- There exists an optimal solution $x^\star$

Then, there exists an optimal solution that is an **extreme point** of P.

Solution method: restrict search to **extreme points.**

How to search among basic feasible solutions?

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Idea

List all the basic feasible solutions, compare objective values and pick the best one.

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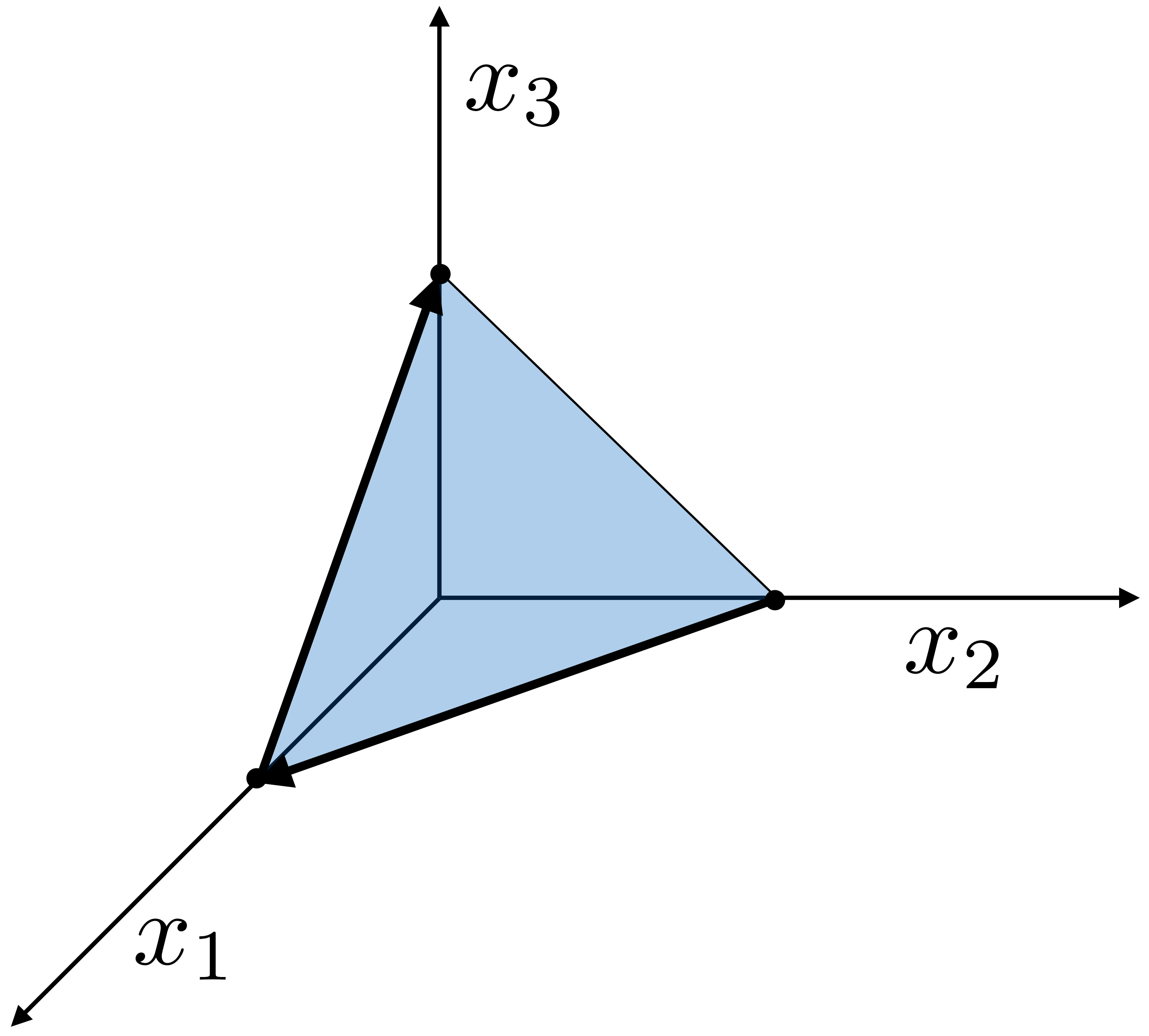
List all the basic feasible solutions, compare objective values and pick the best one.

Intractable!

If $n = 1000$ and $m = 100$, we have 10^{143} combinations!

Conceptual algorithm

- Start at corner
- Visit neighboring corner that improves the objective



Geometry of linear optimization

Today, we learned to:

- **Apply geometric and algebraic properties** of polyhedra to characterize the "corners" of the feasible region.
- **Construct basic feasible solutions** by solving a linear system.
- **Recognize existence and optimality** of extreme points.

References

- Bertsimas and Tsitsiklis: Introduction to Linear Programming
    - Chapter 2.1—2.6 : geometry of linear programming

Next topics

More applications

The simplex method