ORF307 – Optimization

8. Piecewise linear optimization

Bartolomeo Stellato — Spring 2024

$a^T x \geq b \rightarrow (-a)^T x \leq (-b)$

Ed Forum

- In some examples we have >= and in others <= constraints. Is that allowed?
- How does software recognize which problem is an LP and how is it converted to LPs?
- How the one-norm specifically enhances robustness to outliers in linear optimization?

Recap

Standard form
Definition

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

- Minimization
- Equality constraints
- Nonnegative variables

- Matrix notation for **theory**
- Standard form for **algorithms**

Standard form

Transformation tricks

Change objective

If “maximize”, use $-c$ instead of c and change to “minimize”.

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Eliminate inequality constraints

If $Ax \leq b$, define s and write $Ax + s = b,\ s \geq 0$.

If $Ax \geq b$, define s and write $Ax - s = b,\ s \geq 0$.

s are the **slack variables**

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Change variable signs

If $x_i \leq 0$, define $y_i = -x_i$.

Standard form

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s are the **slack variables**

Change variable signs

If $x_i \leq 0$, define $y_i = -x_i$.

Eliminate "free" variables

If x_i unconstrained, define $x_i = x_i^+ - x_i^-$, with $x_i^+ \geq 0$ and $x_i^- \geq 0$.

Standard form

Transformation example

$x_2 = x_2^+ - x_2^-$

$$
\begin{array}{llcl}
\text{minimize} & 2x_1 & + & 4x_2 \\
\text{subject to} & x_1 & + & x_2 \geq 3 \\
 & 3x_1 & + & 2x_2 = 14 \\
 & x_1 & & \geq 0
\end{array}
$$

$$\downarrow$$

$$
\begin{array}{lccccccl}
\text{minimize} & 2x_1 & + & 4x_2^+ & - & 4x_2^- & & \\
\text{subject to} & x_1 & + & x_2^+ & - & x_2^- & - & x_3 = 3 \\
 & 3x_1 & + & 2x_2^+ & - & 2x_2^- & & = 14 \\
 & x_1, & & x_2^+, & & x_2^-, & & x_3 \geq 0.
\end{array}
$$

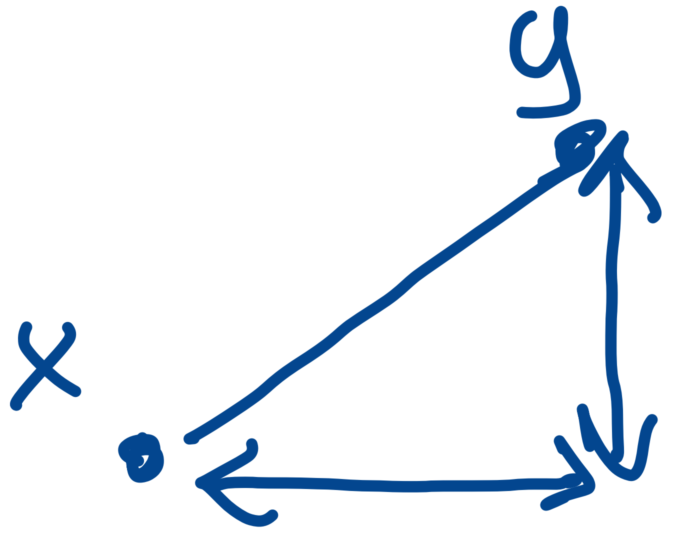
Today's lecture

Piecewise linear optimization

- Vector norms
- Piecewise linear optimization
- Turning vector norm problems into LPs
- Sparse signal recovery
- Support vector machines

Vector norms

Vector norms



$\|y-x\| = |y_1 - x_1| + |y_2 - x_2|$

Euclidean norm

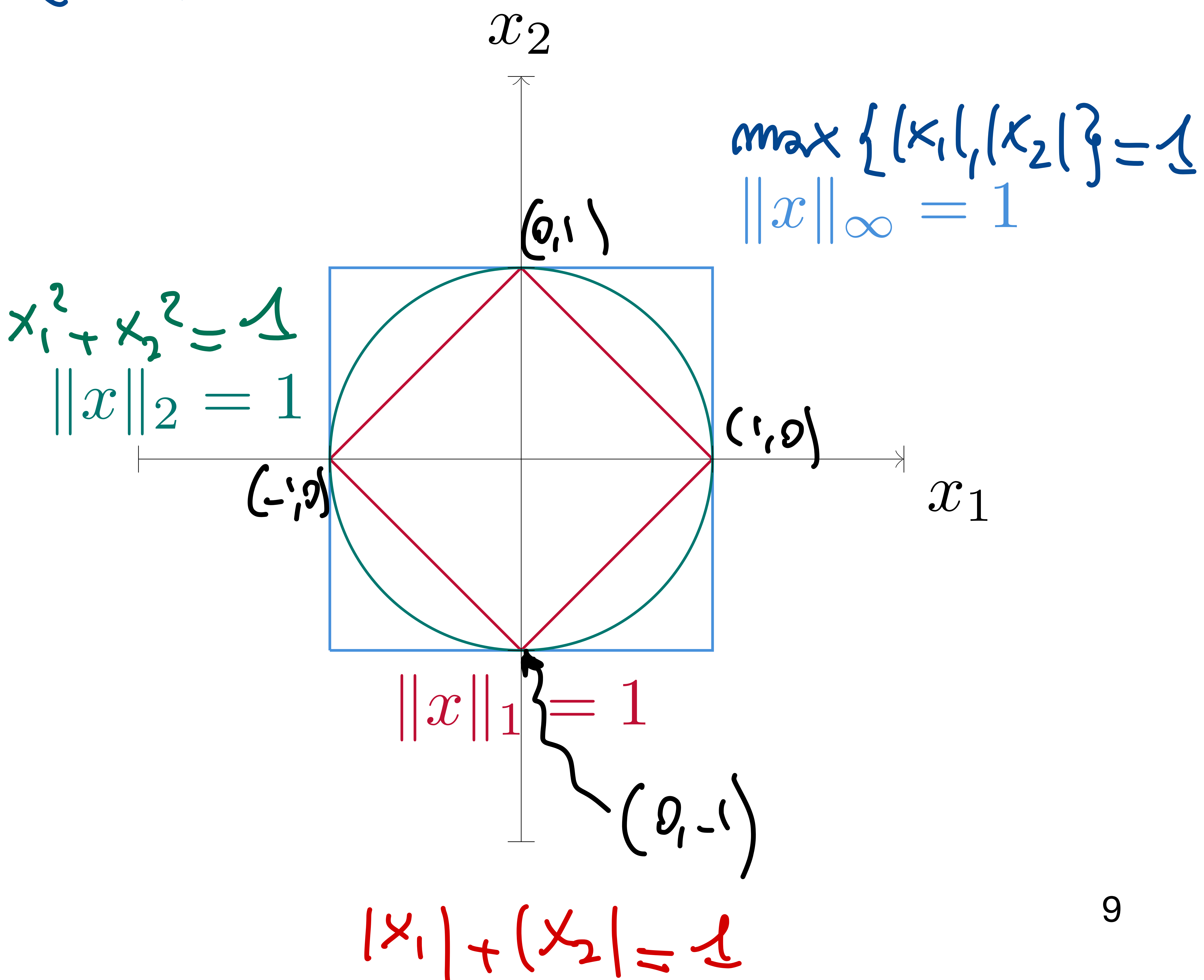
$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$$

1-norm (Manhattan norm)

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

∞-norm (max-norm)

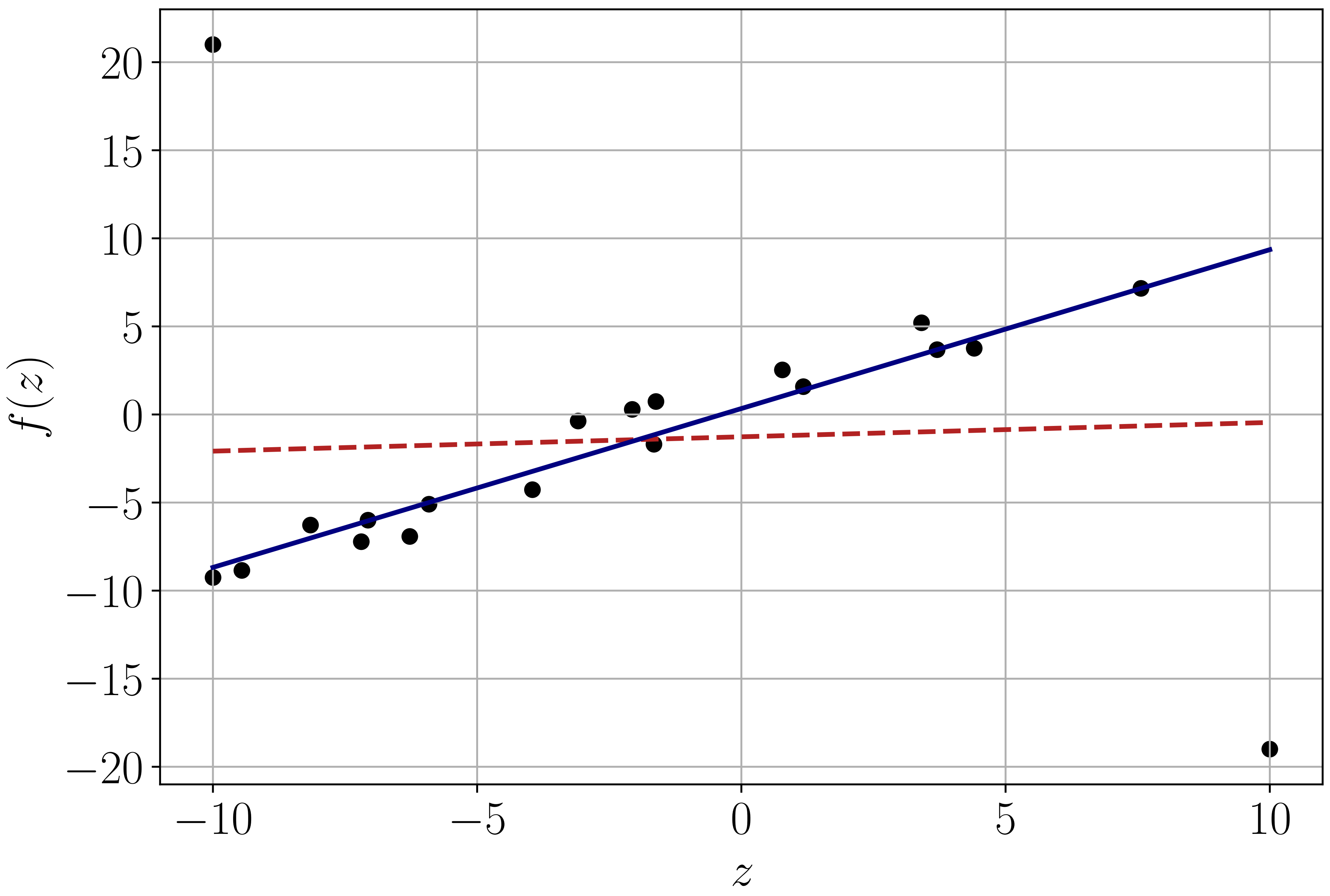
$$\|x\|_\infty = \max_i |x_i|$$



Data-fitting example

Fit a linear function $f(z) = x_1 + x_2 z$ to m data points (z_i, f_i):

Approximation problem $Ax \approx b$ where $\underbrace{\begin{bmatrix} 1 & z_1 \\ \vdots & \vdots \\ 1 & z_m \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} \approx \underbrace{\begin{bmatrix} f_1 \\ \vdots \\ f_m \end{bmatrix}}_{b}$



Recall our regression problem:

minimize $\quad \displaystyle\sum_{i=1}^{m} |Ax - b|_i = \|Ax - b\|_1$

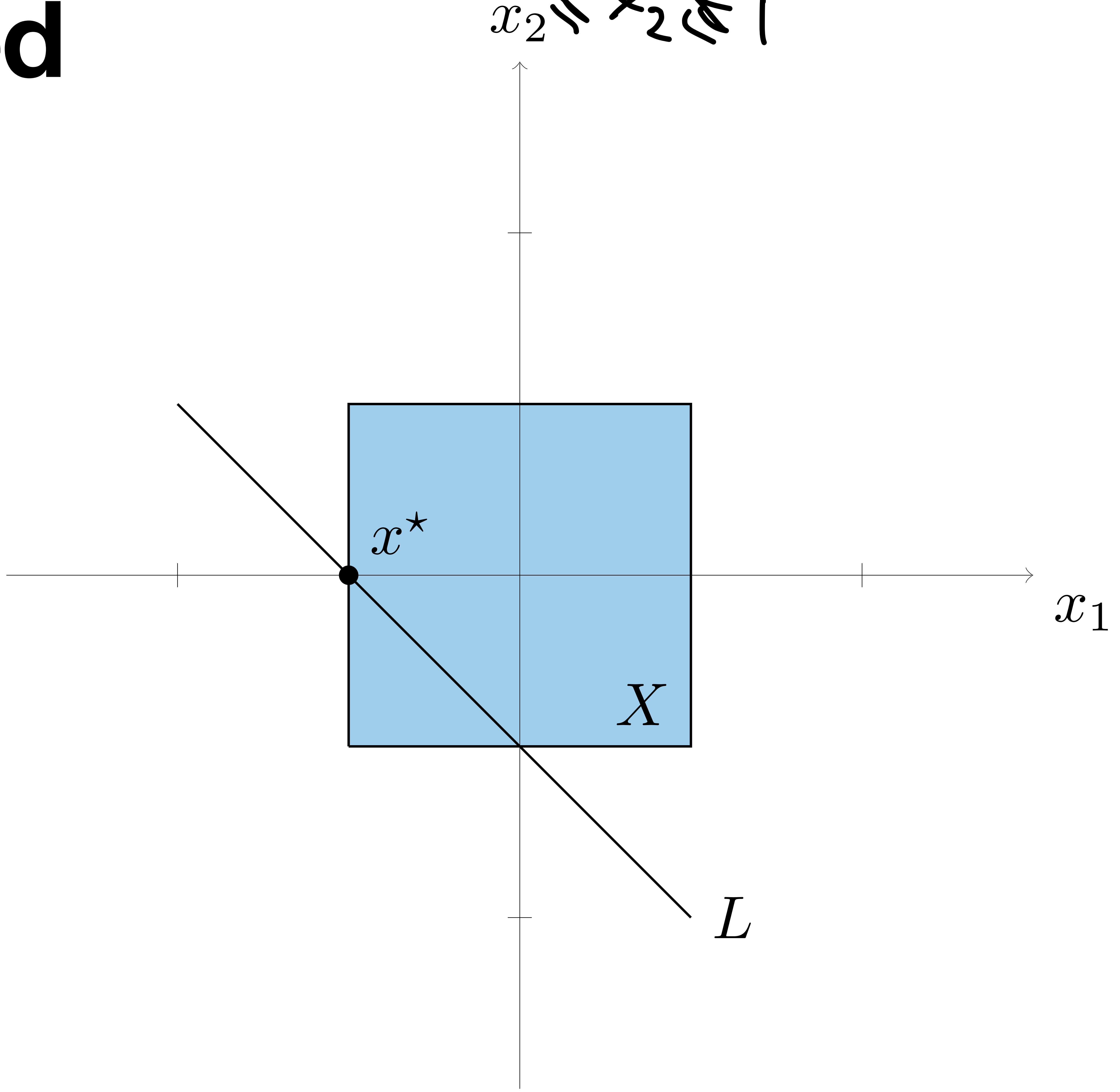
Why is it a linear program?

Simple example revisited

$-1 \leq x_1 \leq 1$
$-1 \leq x_2 \leq 1$

Goal find point as far left as possible,
in the unit box X,
and restricted to the line L

$$
\begin{array}{ll}
\text{minimize} & x_1 \\
\text{subject to} & \|x\|_\infty \leq 1 \\
 & x_1 + x_2 = -1
\end{array}
$$



Simple example revisited

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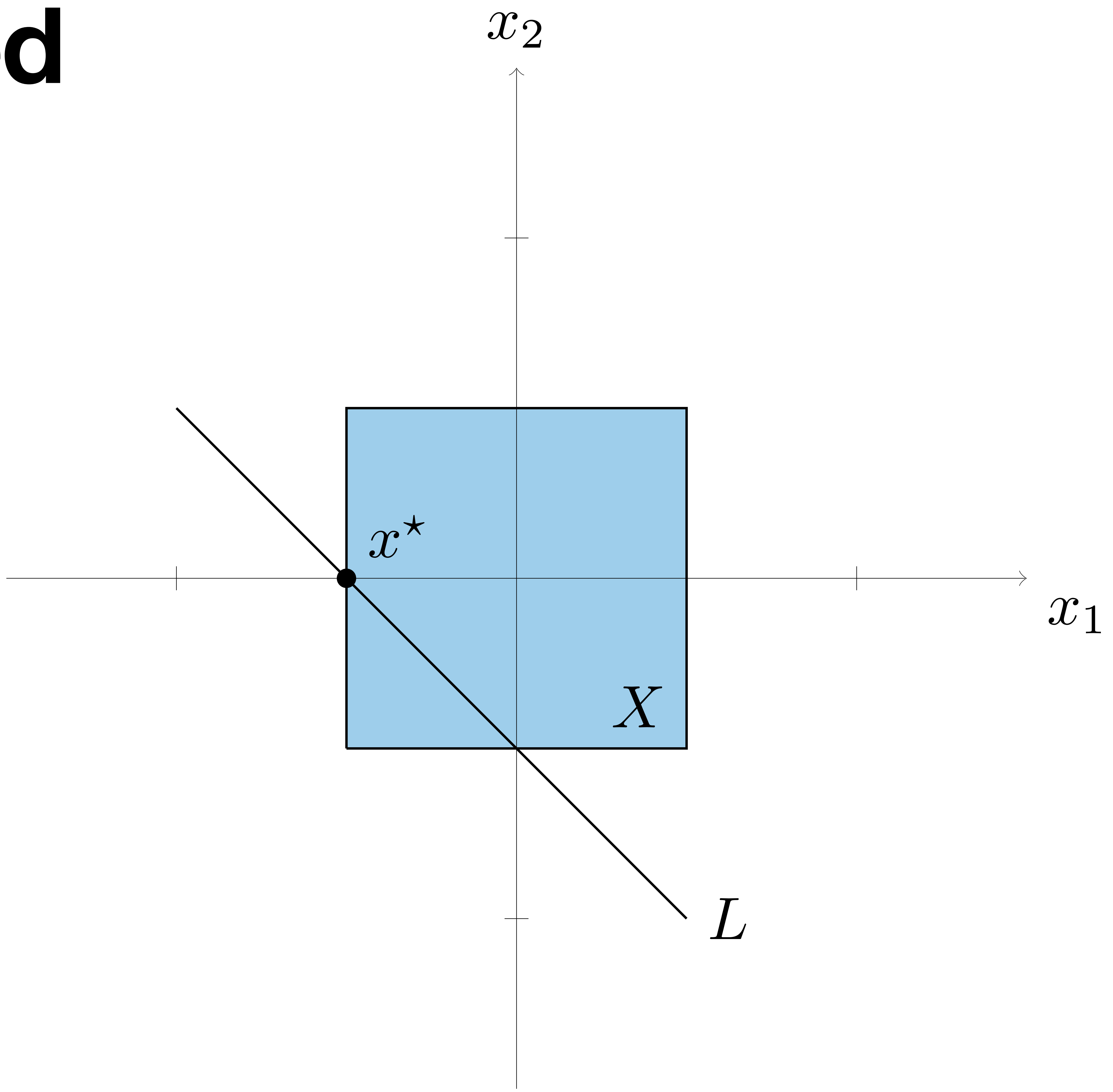
$$

The (nonlinear) norm function
appears in the constraints

Why is it a linear progam?



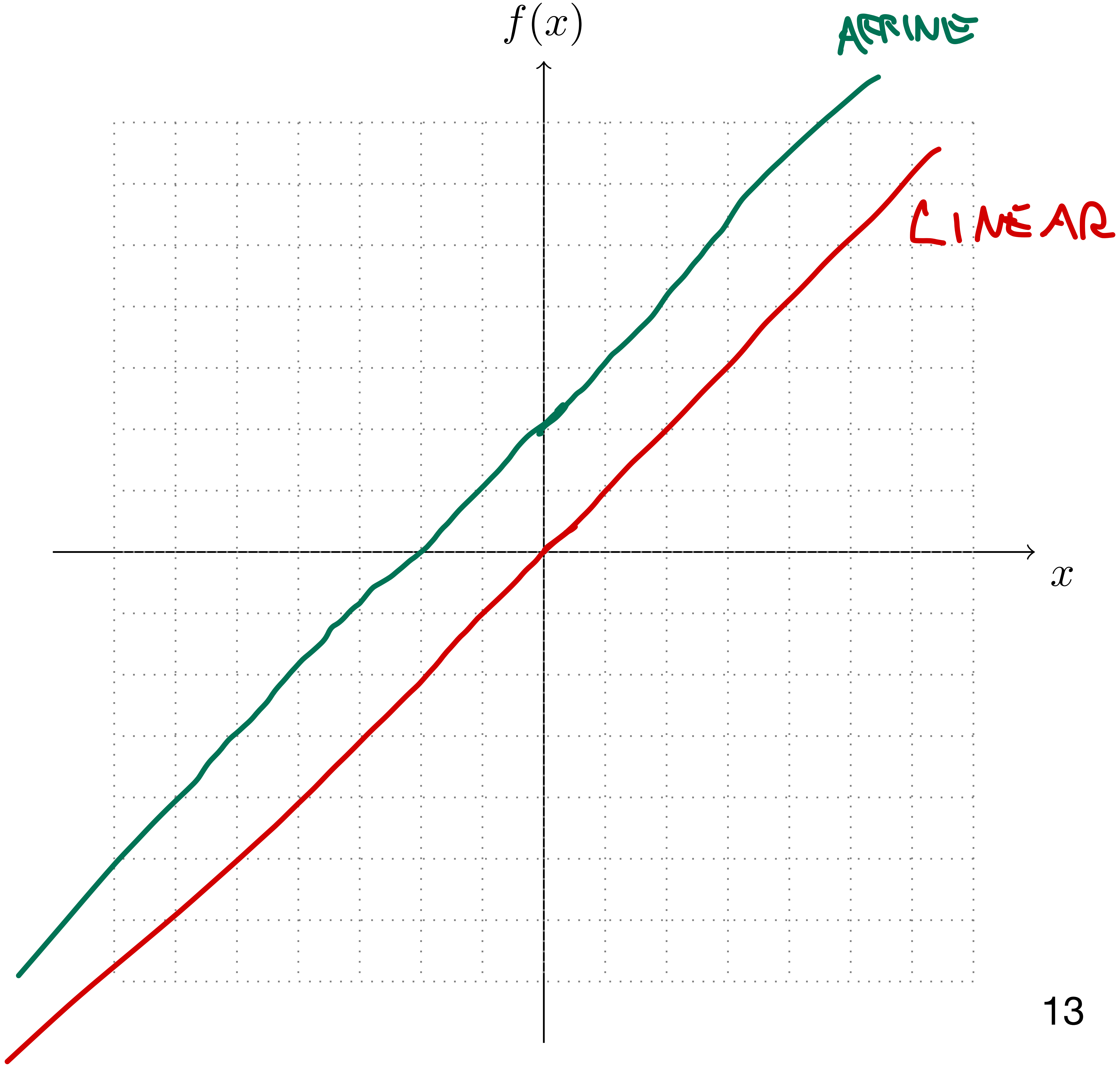
Piecewise linear optimization

Linear, affine and convex functions

min $c^T x \boxed{+ b}$

Linear function: $f(x) = a^T x$

Affine function: $f(x) = a^T x + b$

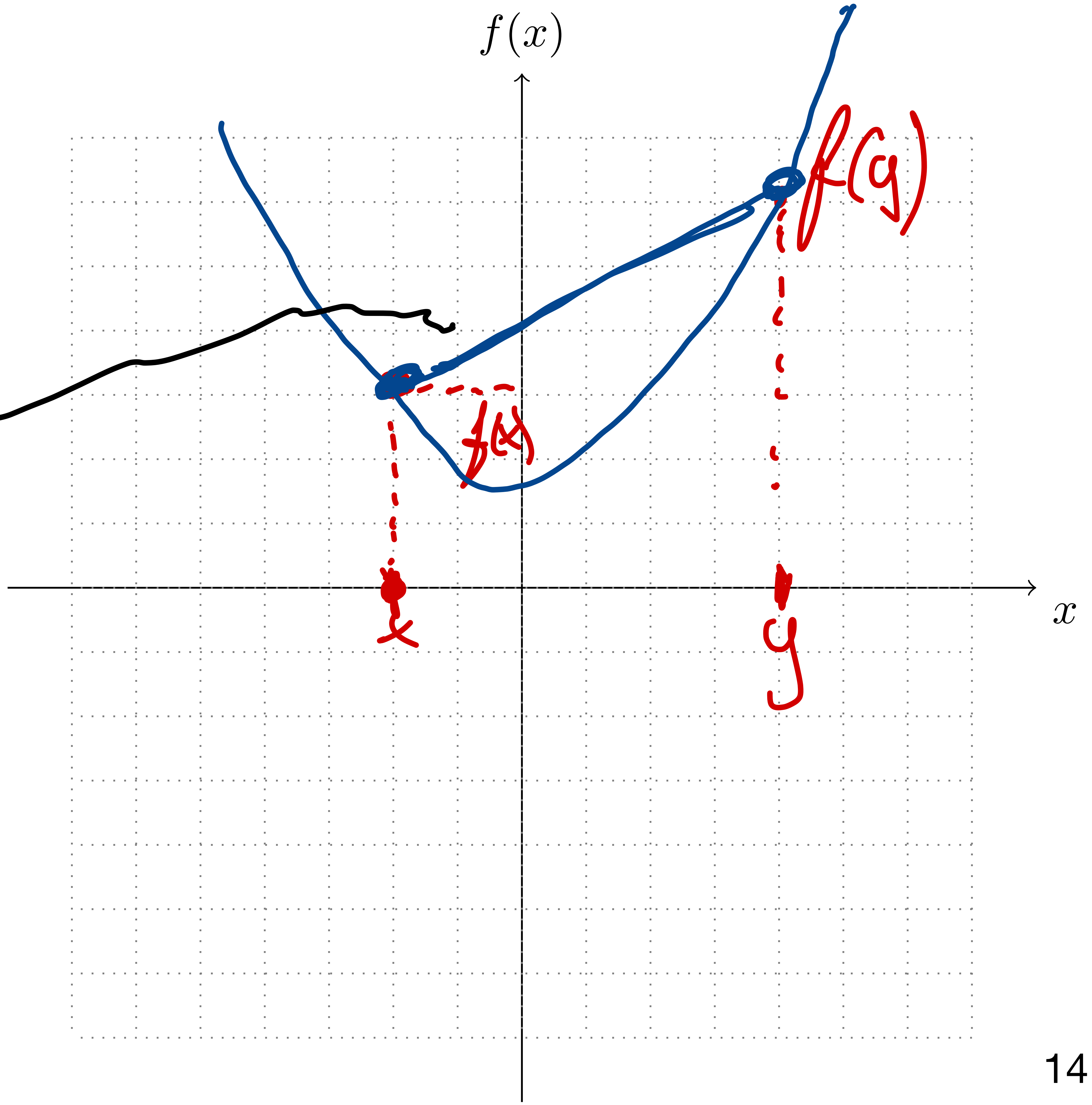


Linear, affine and convex functions

Convex function:

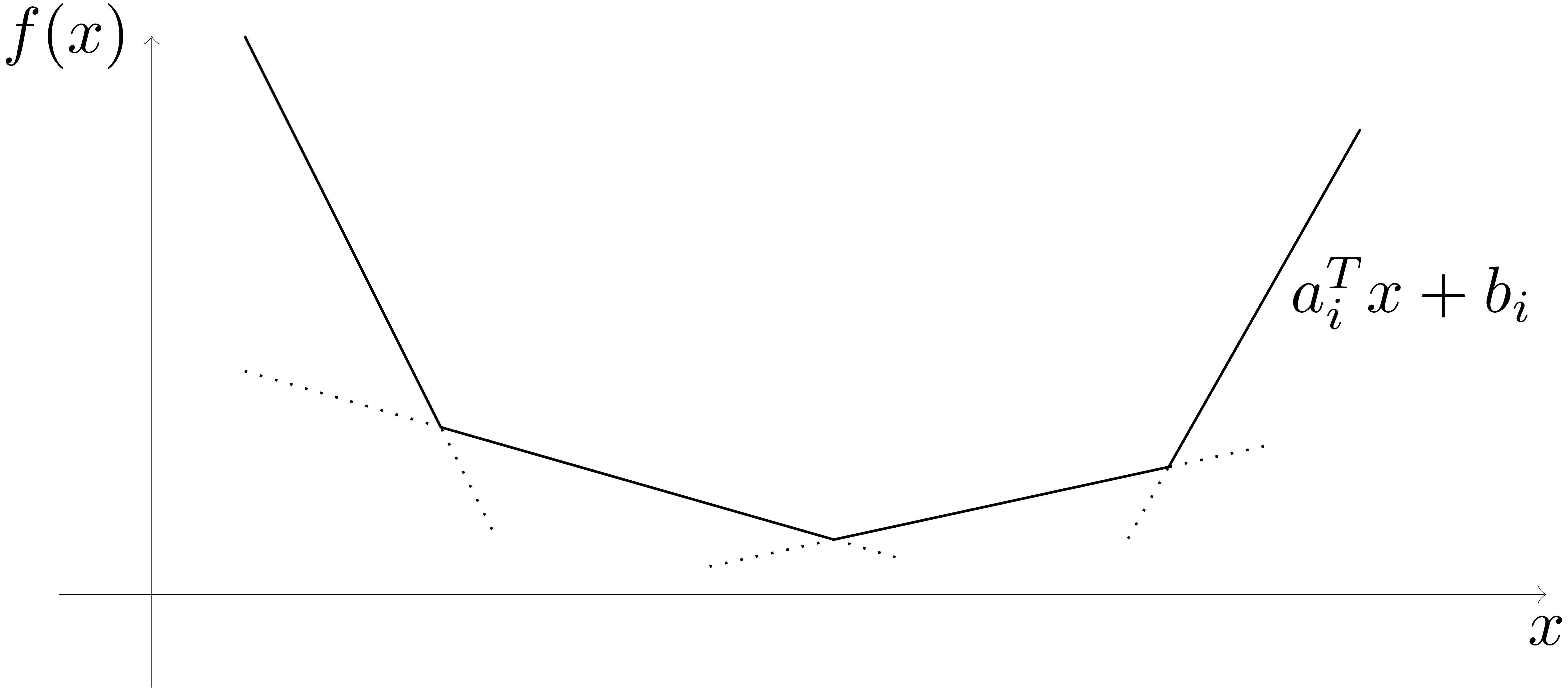
$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y),$$

$$\forall x, y \in \mathbf{R}^n, \ \alpha \in [0,1]$$



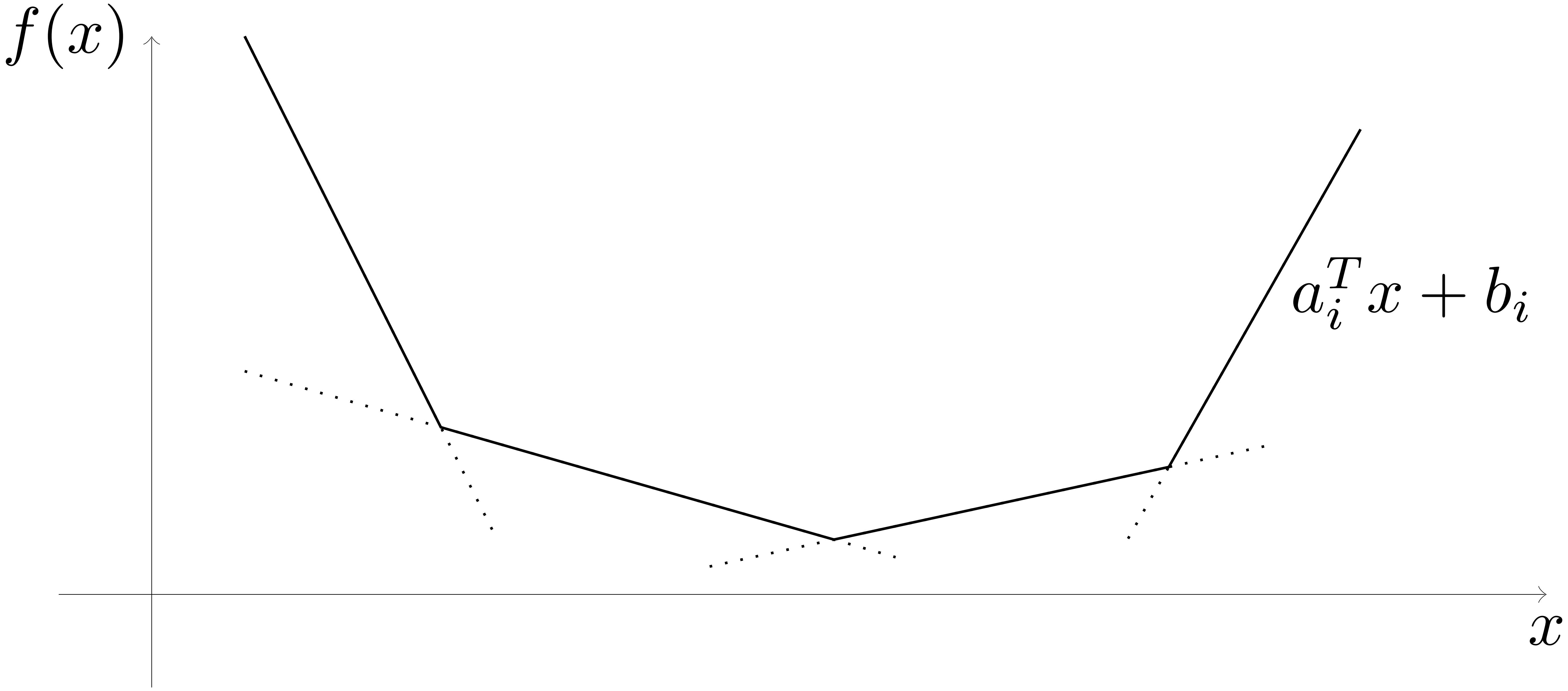
Convex piecewise-linear functions (PWL)

$$f(x) = \max_{i=1,\ldots,m} (a_i^T x + b_i)$$



Convex piecewise-linear minimization

minimize $\max_{i=1,\ldots,m} (a_i^T x + b_i)$



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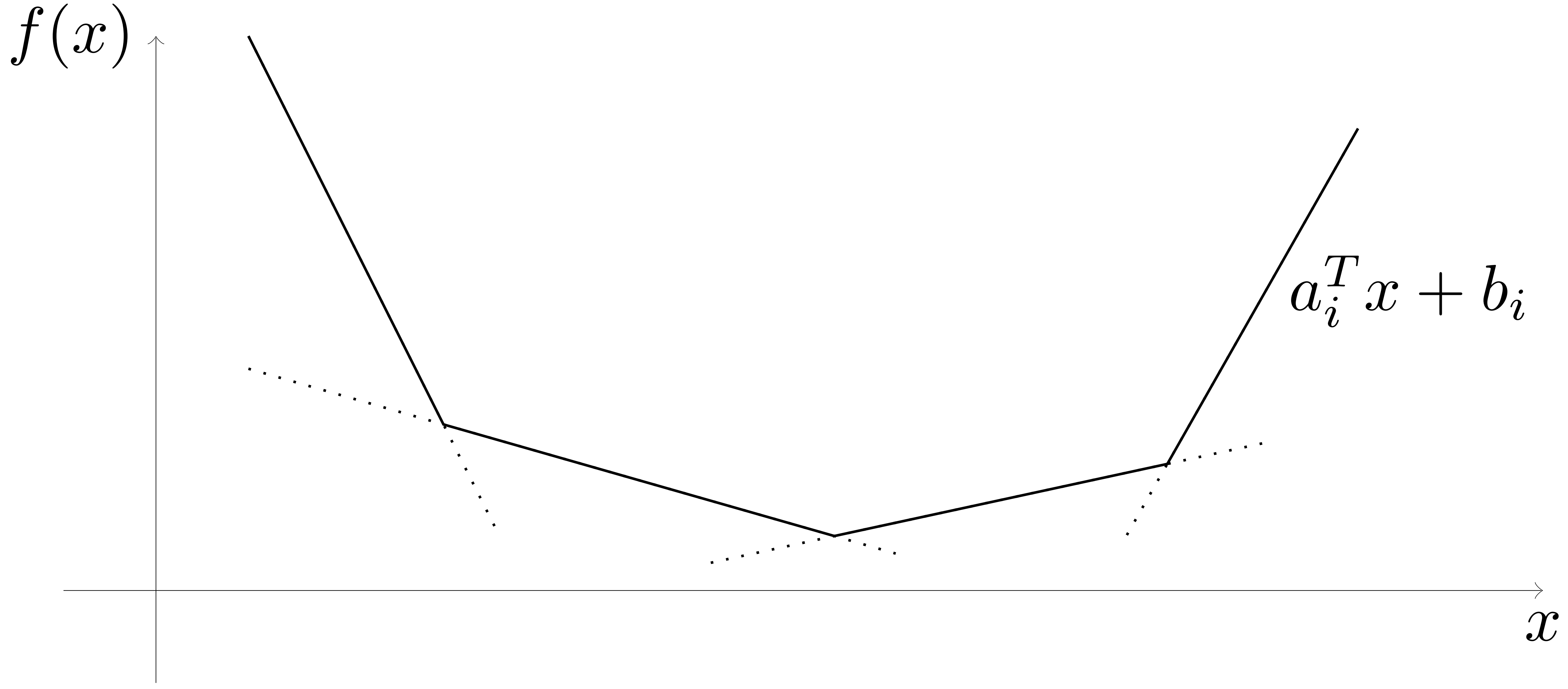
minimize $\quad \max_{i=1,\ldots,m} (a_i^T x + b_i)$



Equivalent linear optimization

$$
\begin{array}{ll}
\text{minimize} & t \\
\text{subject to} & a_i^T x + b_i \leq t, \quad i = 1, \ldots, m
\end{array}
$$

Convex piecewise-linear minimization

Equivalent linear optimization

$$
\begin{array}{ll}
\text{minimize} & t \\
\text{subject to} & a_i^T x + b_i \leq t, \quad i = 1, \ldots, m
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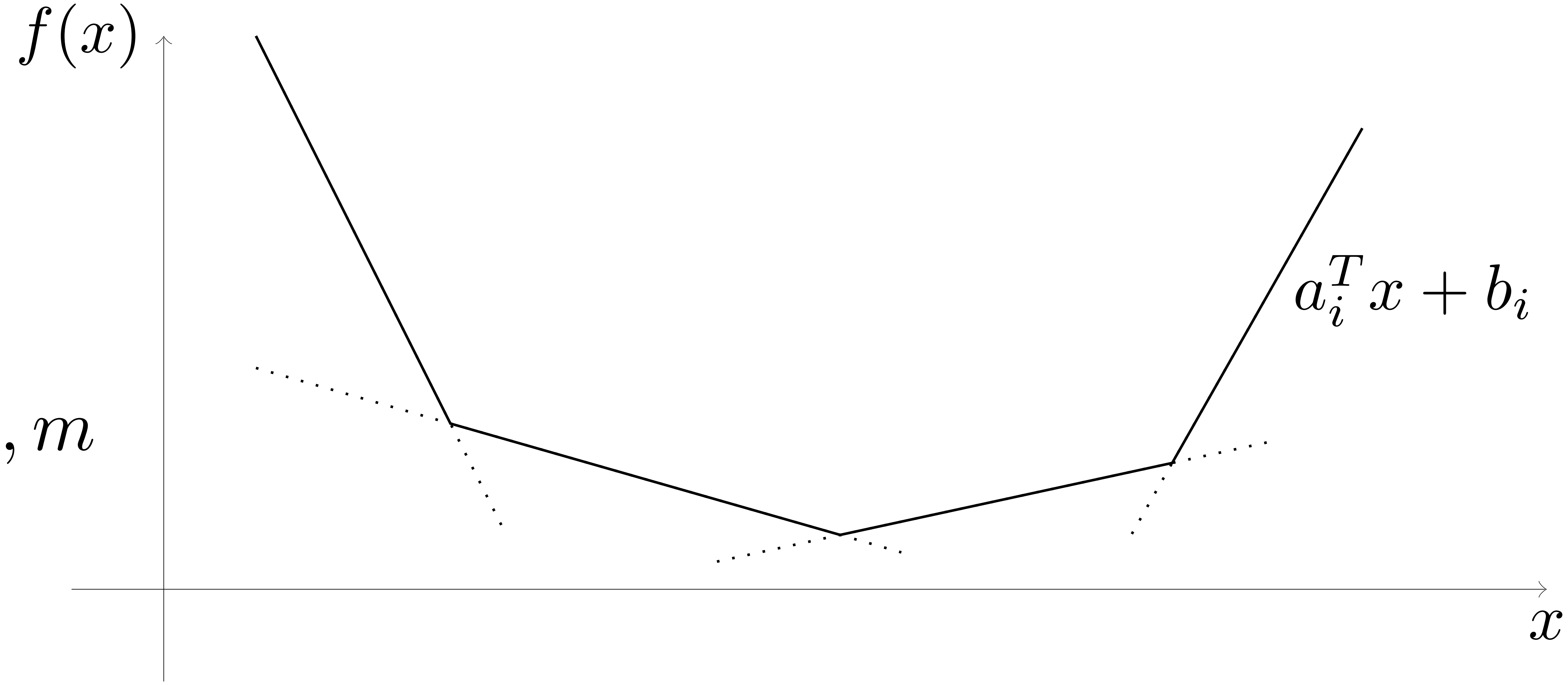
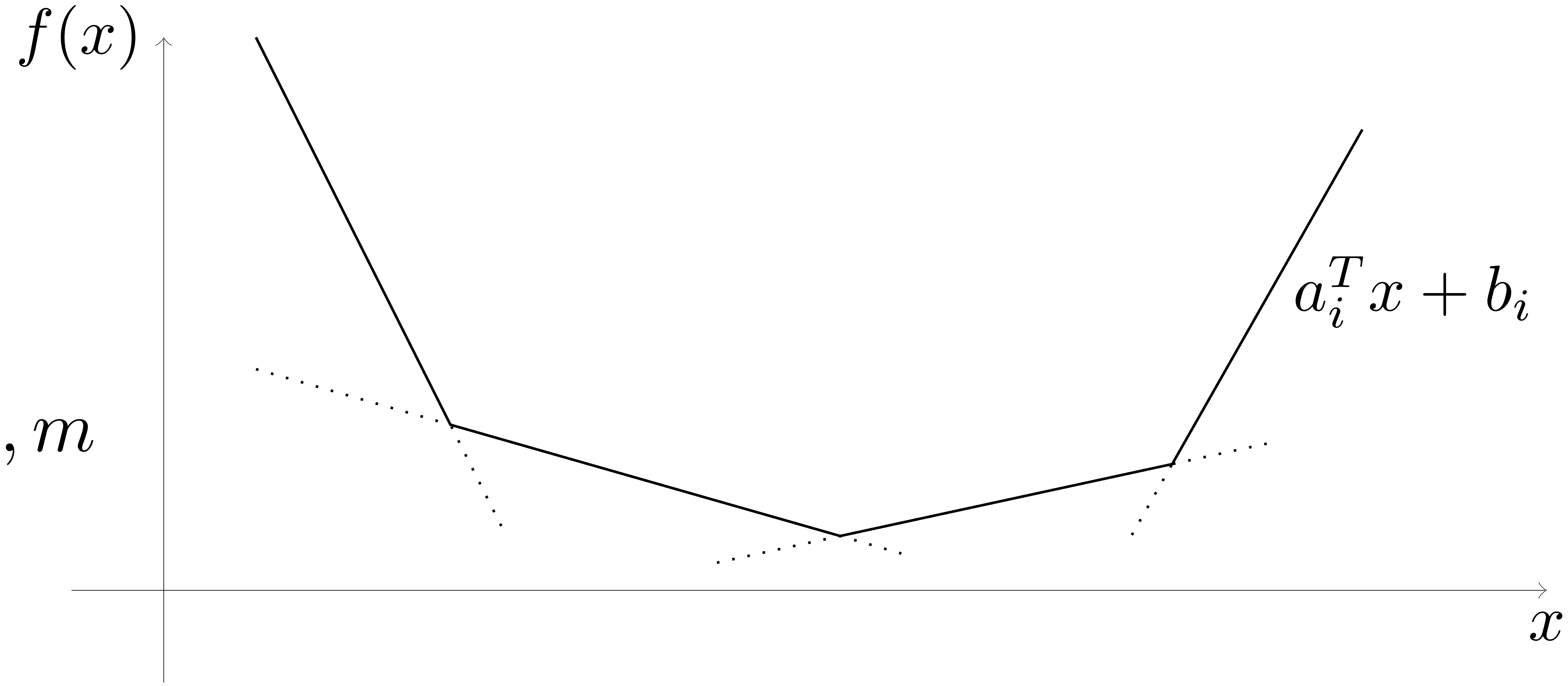


Convex piecewise-linear minimization



Equivalent linear optimization

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & a_i^T x + b_i \le t, \quad i = 1, \dots, m \end{array}$$

$$\rightarrow a_i^T x - t \le -b_i$$

$$\begin{bmatrix} a_i^T & -1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \le -b_i$$

Matrix notation

$$\begin{array}{ll} \text{minimize} & \tilde{c}^T \tilde{x} \\ \text{subject to} & \tilde{A}\tilde{x} \le \tilde{b} \end{array}$$

$$\tilde{x} = \begin{bmatrix} x \\ t \end{bmatrix}, \quad \tilde{c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \}\mathbb{R}^n, \quad \tilde{A} = \begin{bmatrix} a_1^T & -1 \\ \vdots & \vdots \\ a_m^T & -1 \end{bmatrix}, \quad \tilde{b} = \begin{bmatrix} -b_1 \\ \vdots \\ -b_m \end{bmatrix}$$

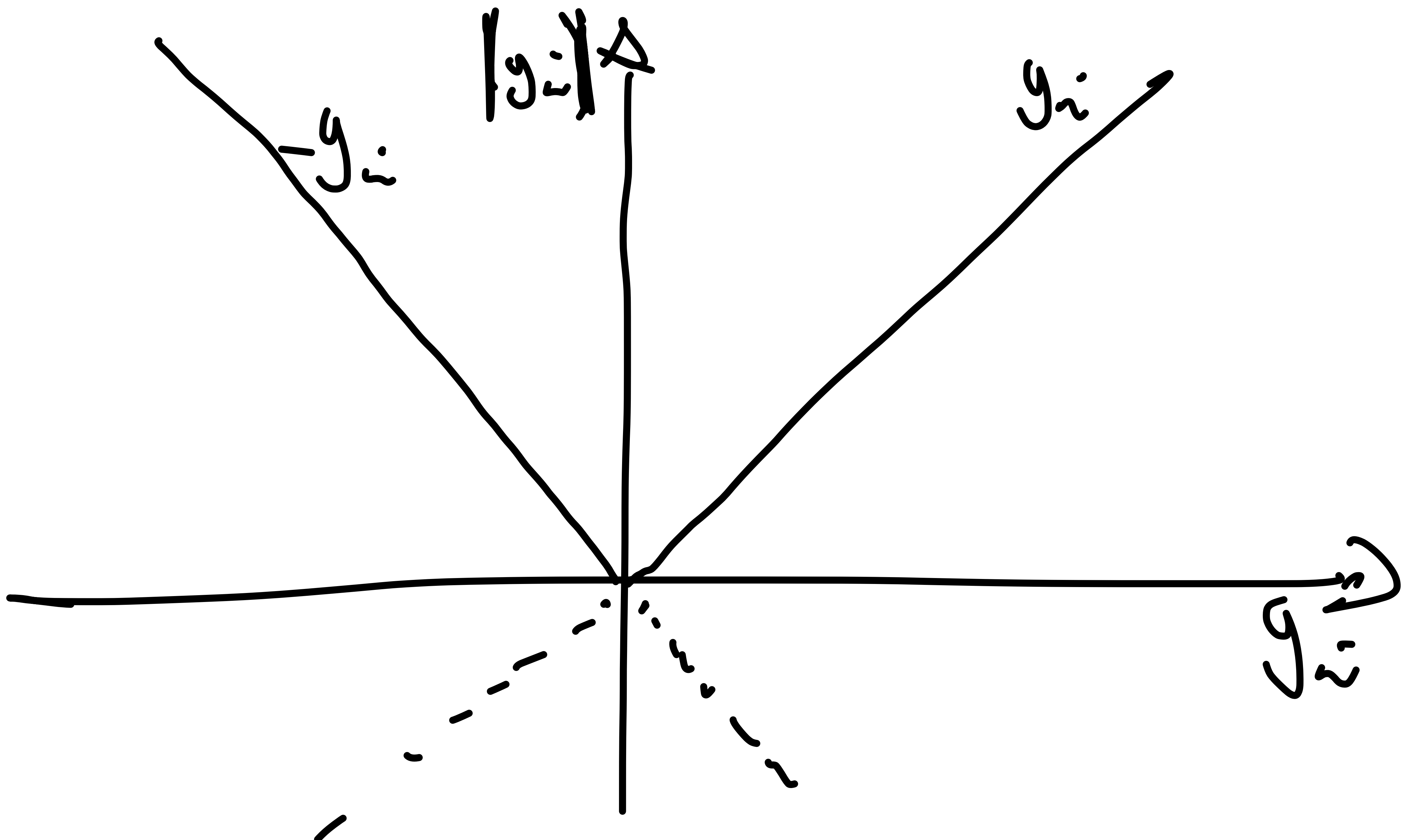
Vector norm problems as linear optimization

∞-norm regression

$$\text{minimize} \quad \|Ax - b\|_\infty$$

The ∞**-norm** of m-vector y is

$$\|y\|_\infty = \max_{i=1,\dots,m} |y_i| = \max_{i=1,\dots,m} \max\{y_i, -y_i\}$$



min t

s.t. $|(Ax-b)_i| \leq t \quad \forall i$

∞-norm regression

$$\text{minimize} \quad \|Ax - b\|_\infty$$

The ∞**-norm** of m-vector y is

$$\|y\|_\infty = \max_{i=1,\dots,m} |y_i| = \max_{i=1,\dots,m} \max\{y_i, -y_i\}$$

Equivalent problem

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & (Ax-b)_i \leq t, \quad i = 1,\dots,m \\ & -(Ax-b)_i \leq t, \quad i = 1,\dots,m \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & t \\ \text{subject to} & Ax - b \leq t\mathbf{1} \\ & -(Ax-b) \leq t\mathbf{1} \end{array}$$

∞-norm regression

$$\text{minimize} \quad \|Ax - b\|_\infty$$

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Equivalent problem

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & Ax - b \leq t\mathbf{1} \\ & -(Ax - b) \leq t\mathbf{1} \end{array}$$

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$$\|y\|_\infty = \max_{i=1,\dots,m} |y_i| = \max_{i=1,\dots,m} \max\{y_i, -y_i\}$$

Equivalent problem

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & Ax - b \le t\mathbf{1} \rightarrow Ax - t\mathbf{1} \le b \\ & -(Ax - b) \le t\mathbf{1} \end{array}$$

Matrix notation

$$\begin{array}{ll} \text{minimize} & \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} x \\ t \end{bmatrix} \\ \text{subject to} & \begin{bmatrix} A & -\mathbf{1} \\ -A & -\mathbf{1} \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \le \begin{bmatrix} b \\ -b \end{bmatrix} \end{array}$$

Sum of piecewise-linear functions

minimize $\quad f(x) + g(x) = \max_{i=1,\ldots,m} (a_i^T x + b_i) + \max_{i=1,\ldots,p} (c_i^T x + d_i)$

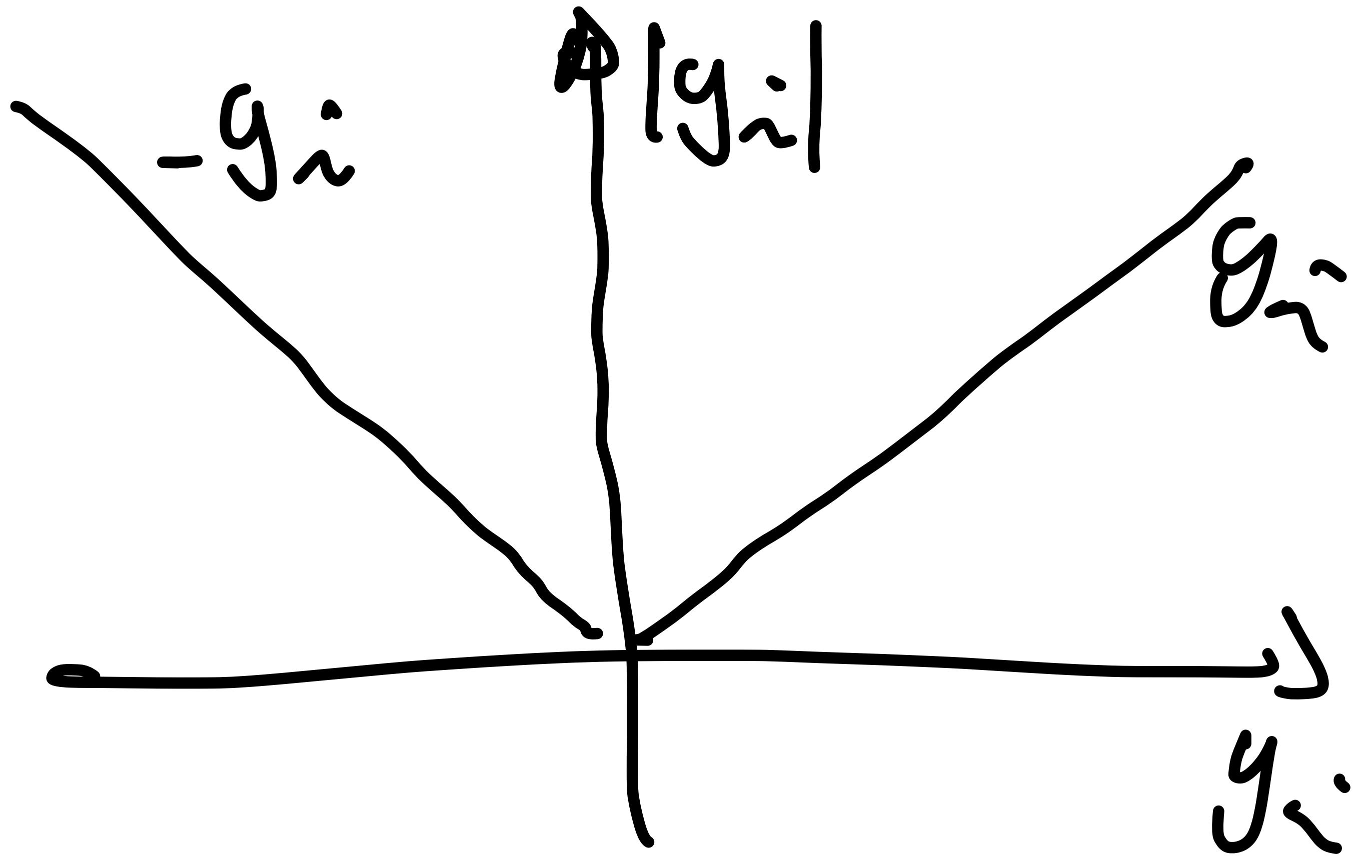
Sum of piecewise-linear functions

minimize $\quad f(x) + g(x) = \max_{i=1,\dots,m} (a_i^T x + b_i) + \max_{i=1,\dots,p} (c_i^T x + d_i)$

Equivalent linear optimization

$$
\begin{array}{ll}
\text{minimize} & t_1 + t_2 \\
\text{subject to} & a_i^T x + b_i \leq t_1, \quad i = 1, \dots, m \\
& c_i^T x + d_i \leq t_2, \quad i = 1, \dots, p
\end{array}
$$

1-norm regression



$$\text{minimize} \quad \|Ax - b\|_1$$

The 1-**norm** of m-vector y is

$$\|y\|_1 = \sum_{i=1}^{m} |y_i| = \sum_{i=1}^{m} \max\{y_i, -y_i\}$$

1-norm regression

$$\text{minimize} \quad \|Ax - b\|_1$$

The 1-**norm** of m-vector y is

$$\|y\|_1 = \sum_{i=1}^{m} |y_i| = \sum_{i=1}^{m} \max\{y_i, -y_i\}$$

$$\tilde{x} = (\overset{\mathbb{R}^n}{x}, \overset{\mathbb{R}^m}{u})$$

Equivalent problem

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^{m} u_i \\ \text{subject to} & (Ax - b)_i \leq u_i, \quad i = 1, \dots, m \\ & -(Ax - b)_i \leq u_i, \quad i = 1, \dots, m \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & \mathbf{1}^T u \\ \text{subject to} & Ax - b \leq u \\ & -(Ax - b) \leq u \end{array}$$

1-norm regression

$$\text{minimize} \quad \|Ax - b\|_1$$

The 1**-norm** of m-vector y is

$$\|y\|_1 = \sum_{i=1}^{m} |y_i| = \sum_{i=1}^{m} \max\{y_i, -y_i\}$$

Equivalent problem

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T u \\ \text{subject to} & Ax - b \leq u \\ & -(Ax - b) \leq u \end{array}$$

1-norm regression

$$\text{minimize} \quad \|Ax - b\|_1$$

The 1-**norm** of m-vector y is

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$Ax - u \le b$

$\begin{bmatrix} A & -I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \le b$

Equivalent problem

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T u \\ \text{subject to} & \boxed{Ax - b \le u} \\ & -(Ax - b) \le u \end{array}$$

Matrix notation

$$\begin{array}{ll} \text{minimize} & \begin{bmatrix} 0 \\ \mathbf{1} \end{bmatrix}^T \begin{bmatrix} x \\ u \end{bmatrix} \\ \text{subject to} & \begin{bmatrix} A & -I \\ -A & -I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \le \begin{bmatrix} b \\ -b \end{bmatrix} \end{array}$$

Summary: 1 and ∞-norm regression

∞-norm

minimize $\quad \|Ax - b\|_\infty$

Equivalent to

$$
\begin{array}{ll}
\text{minimize} & t \\
\text{subject to} & Ax - b \leq t\mathbf{1} \\
& -(Ax - b) \leq t\mathbf{1}
\end{array}
$$

Absolute value of every element $(Ax - b)_i$ is

bounded by the same **scalar** t

Summary: 1 and ∞-norm regression

∞-norm

minimize $\quad \|Ax - b\|_\infty$

Equivalent to

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & Ax - b \leq t\mathbf{1} \\ & -(Ax - b) \leq t\mathbf{1} \end{array}$$

Absolute value of every element $(Ax - b)_i$ is bounded by the same **scalar** t

1-norm

minimize $\quad \|Ax - b\|_1$

Equivalent to

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T u \\ \text{subject to} & Ax - b \leq u \\ & -(Ax - b) \leq u \end{array}$$

Absolute value of every element $(Ax - b)_i$ is bounded by a component of the **vector** u

Example : converting to an LP

minimize $\quad \|Ax - b\|_\infty$

subject to $\quad \|x\|_1 \leq k$

$\tilde{x} = (x, t, u)$

min $\quad t$

s.t. $\quad Ax - b \leq t\mathbb{1}$

$\quad -(Ax - b) \leq t\mathbb{1}$

$\quad \mathbb{1}^T u \leq K$

$\quad x \leq u$

$\quad -x \leq u$

$$\min \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ t \\ u \end{bmatrix}$$

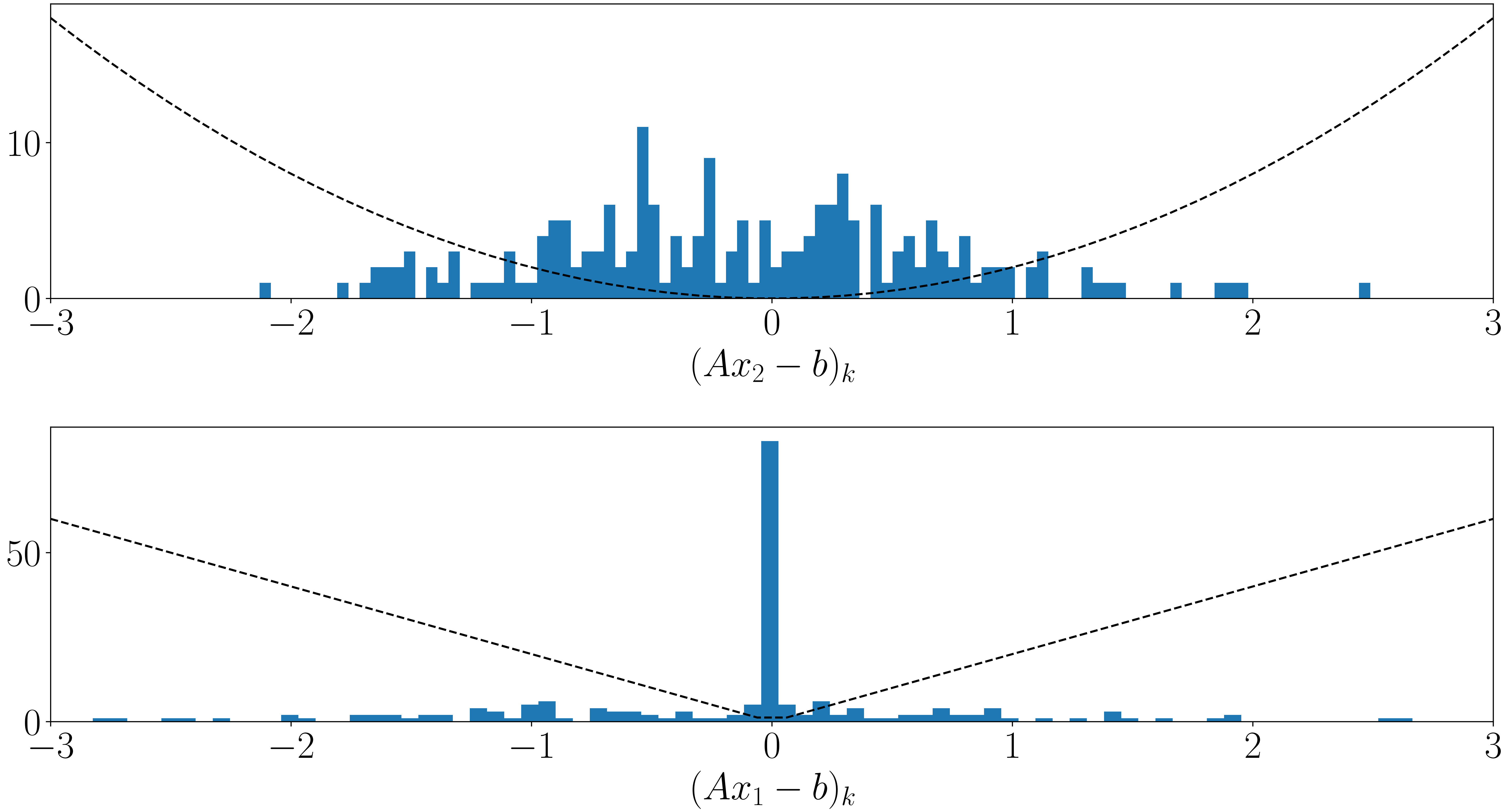
$$\text{st.} \begin{bmatrix} A & -\mathbb{1} & 0 \\ -A & -\mathbb{1} & 0 \\ 0 & 0 & \mathbb{1}^T \\ I & 0 & -I \\ -I & 0 & -I \end{bmatrix} \begin{bmatrix} x \\ t \\ u \end{bmatrix} \leq \begin{bmatrix} b \\ -b \\ K \\ 0 \\ 0 \end{bmatrix}$$

Comparison with least-squares

Histogram of residuals $Ax - b$ with randomly generated $A \in \mathbf{R}^{200\times 80}$

$$x_2 = \operatorname{argmin} \|Ax - b\|_2^2, \qquad x_1 = \operatorname{argmin} \|Ax - b\|_1$$

$r = Ax - b$



1-norm distribution is **wider** with a **high peak at zero**

Modeling software does most of this for you

∞-norm

minimize $\quad ||Ax - b||_\infty$

```python
import numpy as np
import cvxpy as cp

m = 200; n = 80

A = np.random.randn(200, 80)
b = np.random.randn(200)
x = cp.Variable(80)

objective = cp.norm(A @ x - b, np.inf)
problem = cp.Problem(cp.Minimize(objective))
problem.solve()
```

Modeling software does most of this for you

∞-norm

minimize $\quad \|Ax - b\|_\infty$

```
import numpy as np
import cvxpy as cp

m = 200; n = 80

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b = np.random.randn(200)
x = cp.Variable(80)

objective = cp.norm(A @ x - b, np.inf)
problem = cp.Problem(cp.Minimize(objective))
problem.solve()
```

1-norm

minimize $\quad \|Ax - b\|_1$

```
import numpy as np
import cvxpy as cp

m = 200; n = 80

A = np.random.randn(200, 80)
b = np.random.randn(200)
x = cp.Variable(80)

objective = cp.norm(A @ x - b, 1)
problem = cp.Problem(cp.Minimize(objective))
problem.solve()
```

Sparse signal recovery

Sparse signal recovery via 1–norm minimization

$\hat{x} \in \mathbf{R}^n$ is unknown signal, known to be sparse

We make linear measurements $y = A\hat{x}$ with $A \in \mathbf{R}^{m \times n}, m < n$

Estimate signal with smallest ℓ_1-norm, consistent with measurements

$$
\begin{array}{ll}
\text{minimize} & \|x\|_1 \\
\text{subject to} & Ax = y
\end{array}
$$

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We make linear measurements $y = A\hat{x}$ with $A \in \mathbf{R}^{m \times n}, m < n$

Estimate signal with smallest ℓ_1-norm, consistent with measurements

$$\begin{array}{ll} \text{minimize} & \|x\|_1 \\ \text{subject to} & Ax = y \end{array}$$

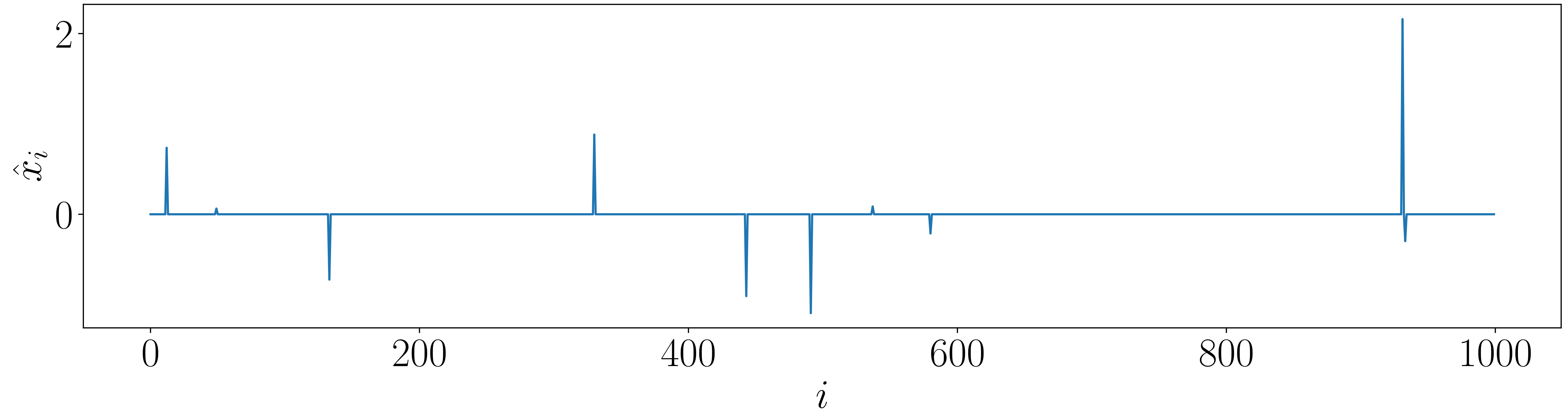
Equivalent linear optimization

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T u \\ \text{subject to} & -u \leq x \leq u \\ & Ax = y \end{array}$$

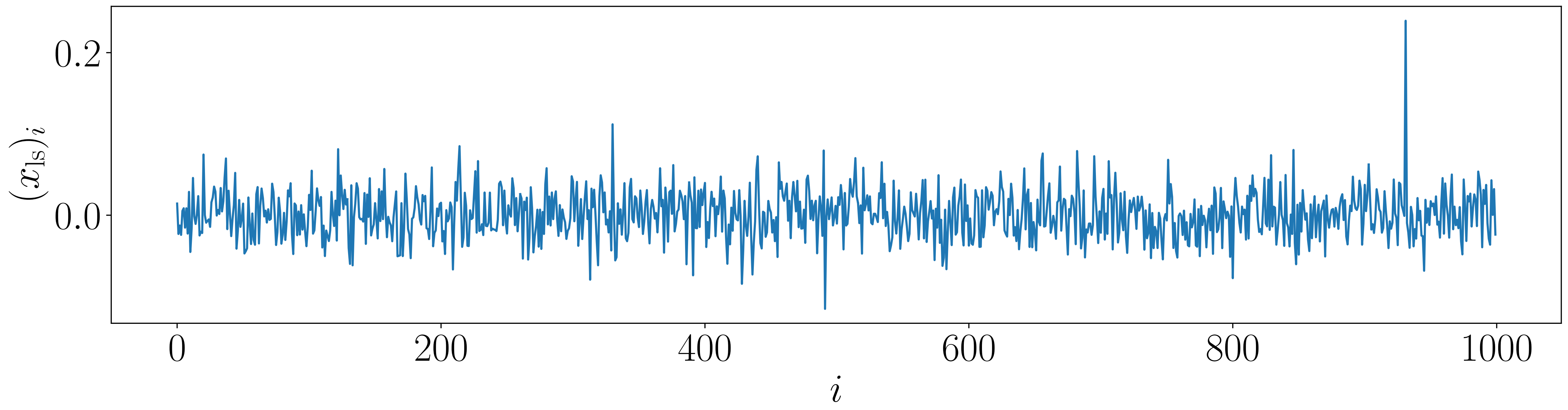
Sparse signal recovery via 1–norm minimization

Example

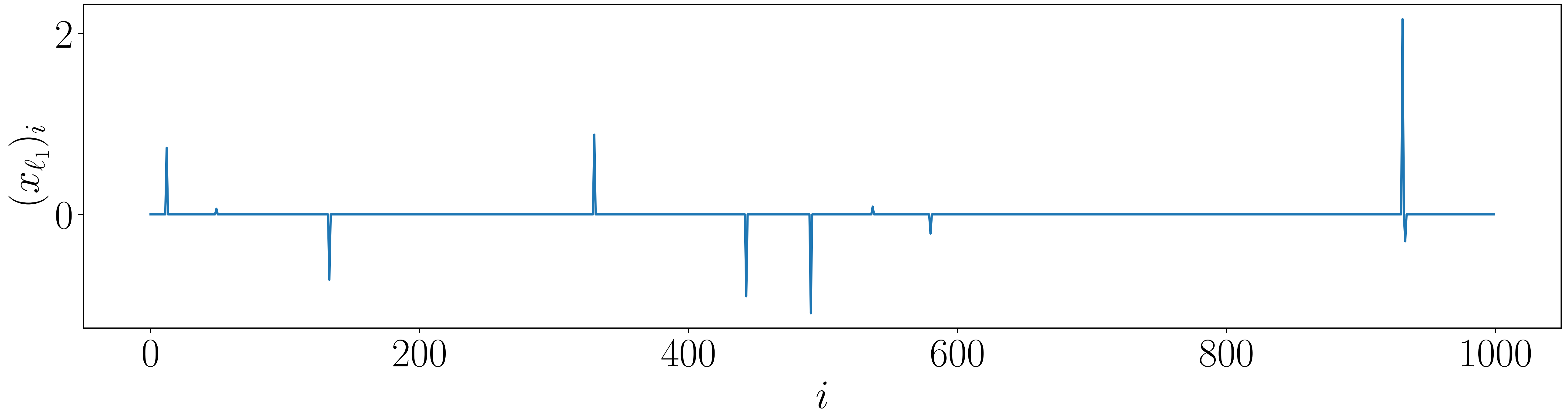
Exact signal $\hat{x} \in \mathbf{R}^{1000}$
10 nonzero components
Random $A \in \mathbf{R}^{100 \times 1000}$



The least squares estimate cannot recover the sparse signal



The 1-norm estimate is **exact**



Support vector machines

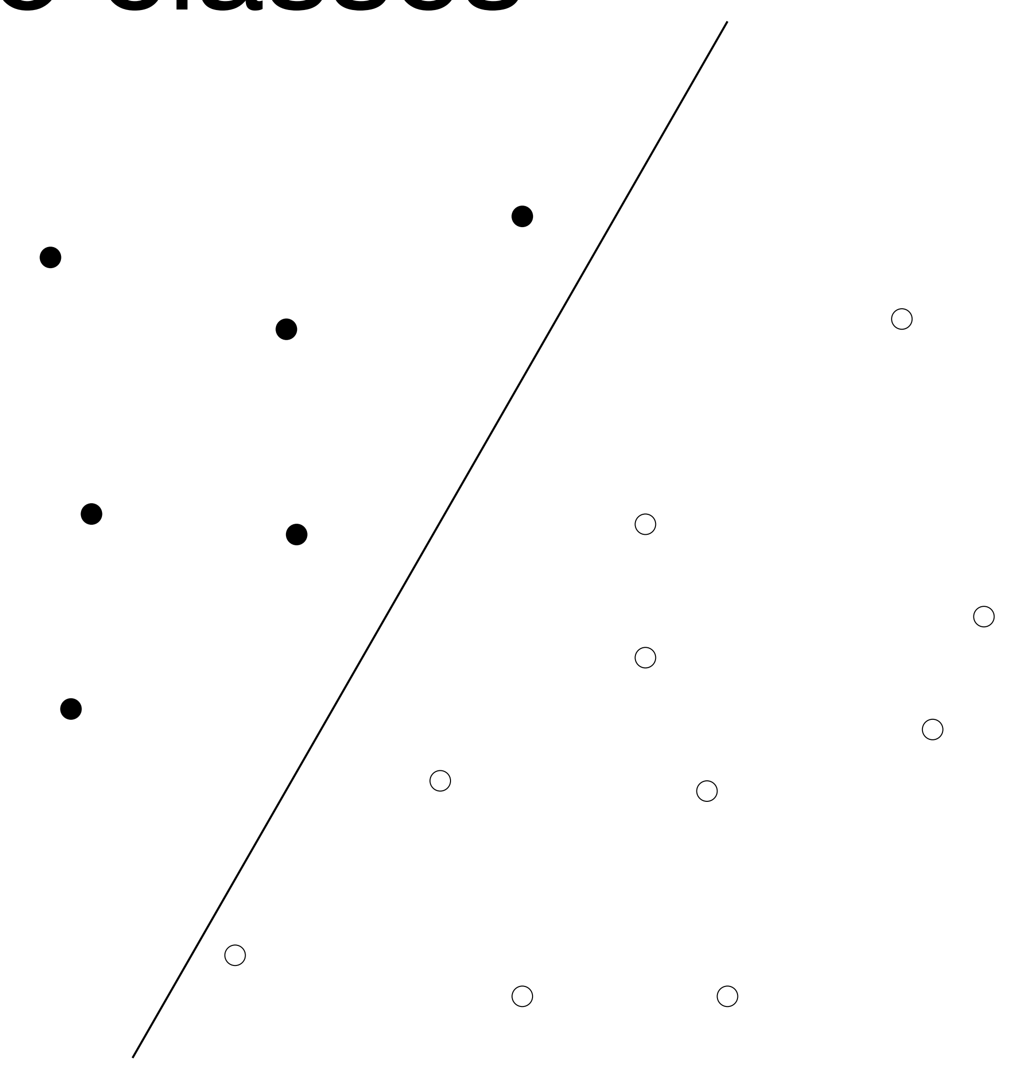
Linear classification

Support vector machine (linear separation)

Given a set of points $\{v_1, \ldots, v_N\}$ with binary labels $s_i \in \{-1, 1\}$

Find hyperplane that strictly separates the tho classes

$$
\begin{aligned}
a^T v_i + b &> 0 \quad \text{if} \quad s_i = 1 \\
a^T v_i + b &< 0 \quad \text{if} \quad s_i = -1
\end{aligned}
$$



$a^T v_i + b_i > 0.5$

$2a^T v_i + 2b \geq 1$

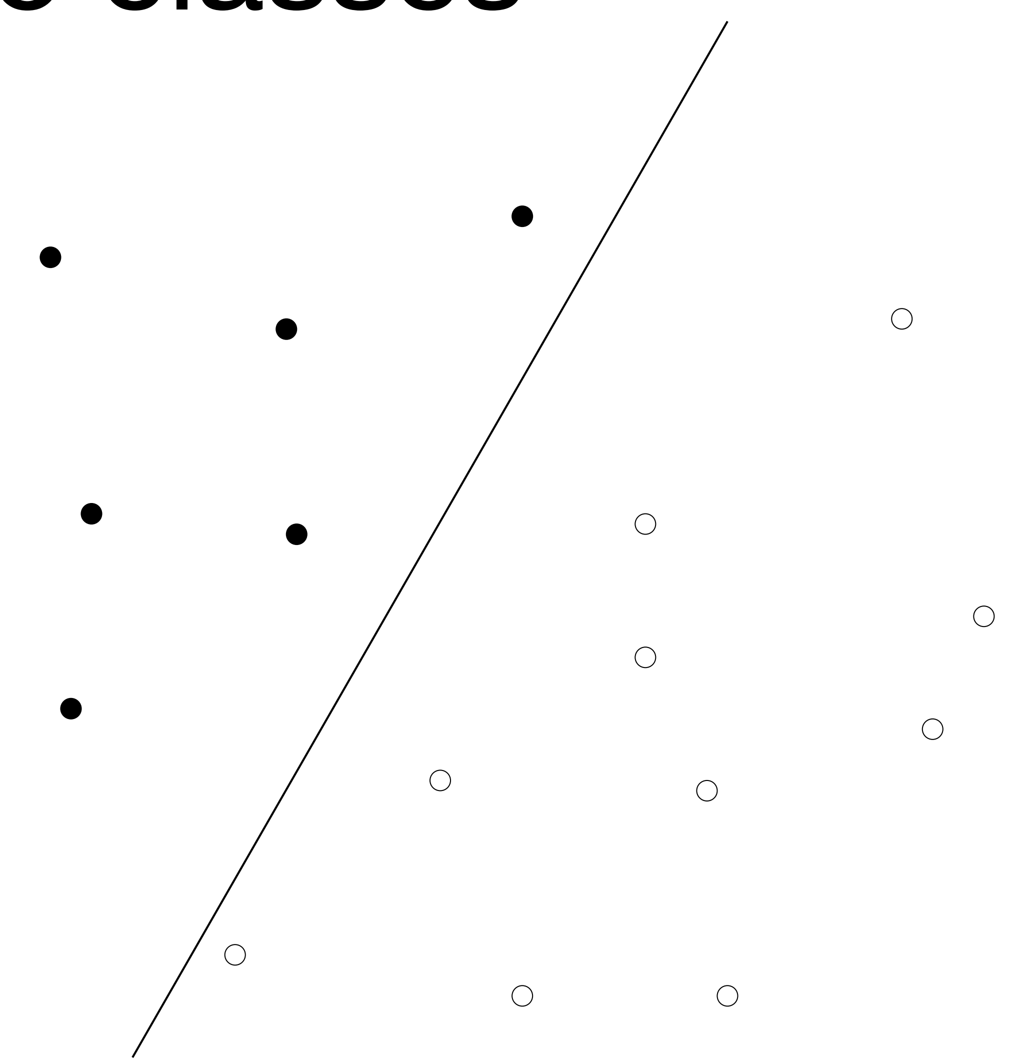
Linear classification

Support vector machine (linear separation)

Given a set of points $\{v_1, \ldots, v_N\}$ with binary labels $s_i \in \{-1, 1\}$

Find hyperplane that strictly separates the tho classes

$$a^T v_i + b > 0 \quad \text{if} \quad s_i = 1$$
$$a^T v_i + b < 0 \quad \text{if} \quad s_i = -1$$



Homogeneous in (a, b), hence equivalent to the linear inequalities (in a, b)

$$s_i(a^T v_i + b) \geq 1$$

$s_i = 1$

$a^T v_i + b \geq 1$

$-a^T v_i - b \leq -1$

Linear classification

Separable case

Feasibility problem

$$
\begin{array}{ll}
\text{find} & a, b \\
\text{subject to} & s_i(a^T v_i + b) \geq 1, \quad i = 1, \ldots, N
\end{array}
$$

Linear classification

Separable case

Feasibility problem

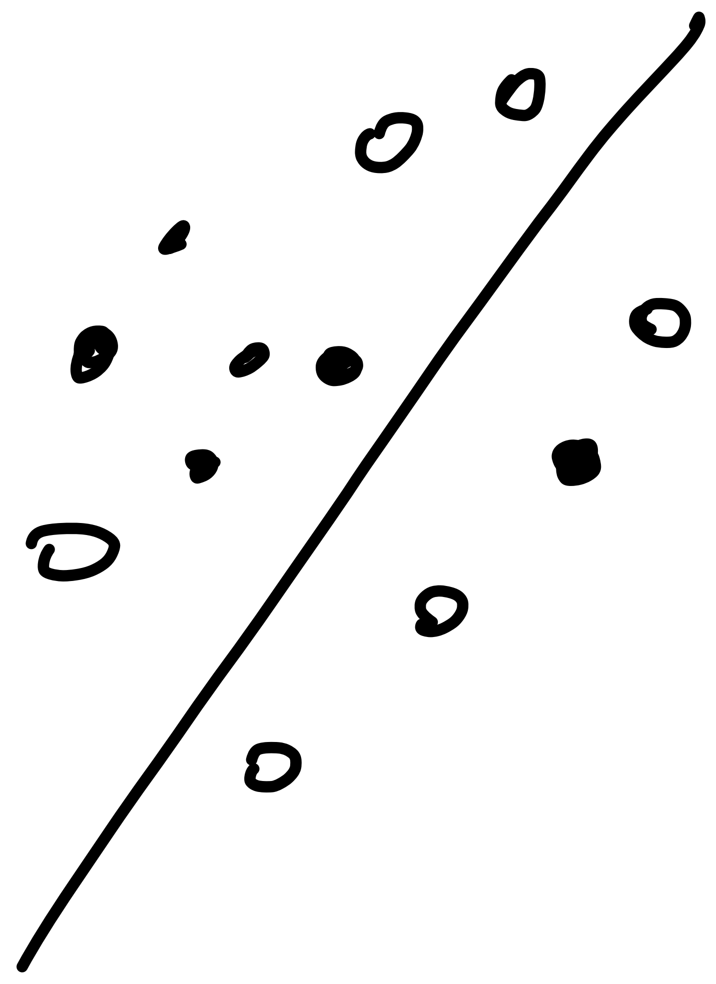
$$
\begin{array}{ll}
\text{find} & a, b \\
\text{subject to} & s_i(a^T v_i + b) \geq 1, \quad i = 1, \dots, N
\end{array}
$$

Which can be seen as a special case of LP with

$$
\begin{array}{ll}
\text{minimize} & 0 \\
\text{subject to} & s_i(a^T v_i + b) \geq 1, \quad i = 1, \dots, N
\end{array}
$$

Linear classification

Separable case



Feasibility problem

$$\begin{array}{ll} \text{find} & a, b \\ \text{subject to} & s_i(a^T v_i + b) \geq 1, \quad i = 1, \dots, N \end{array}$$

Which can be seen as a special case of LP with

$$\begin{array}{ll} \text{minimize} & 0 \\ \text{subject to} & s_i(a^T v_i + b) \geq 1, \quad i = 1, \dots, N \end{array}$$

$p^\star = 0$ if problem feasible (points separable)

$p^\star = \infty$ if problem infeasible (points not separable)

Linear classification

Separable case

Feasibility problem

$$\begin{array}{ll} \text{find} & a, b \\ \text{subject to} & s_i(a^T v_i + b) \geq 1, \quad i = 1, \dots, N \end{array}$$

Which can be seen as a special case of LP with

$$\begin{array}{ll} \text{minimize} & 0 \\ \text{subject to} & s_i(a^T v_i + b) \geq 1, \quad i = 1, \dots, N \end{array}$$

$p^\star = 0$ if problem feasible (points separable)

$p^\star = \infty$ if problem infeasible (points not separable) $\longrightarrow$ **What then?**

Linear classification

Approximate linear separation of non-separable points

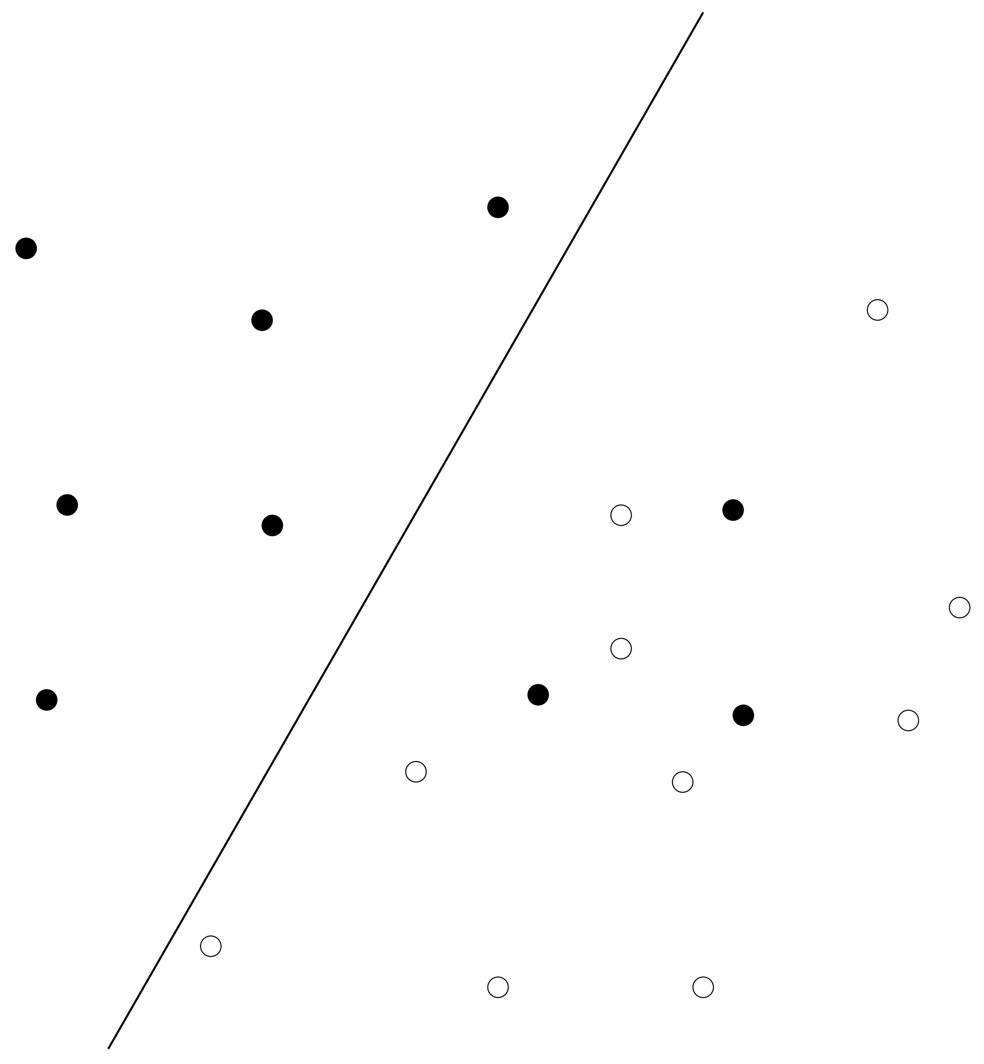
Each of our constraints is

$$s_i(a^T v_i + b) \geq 1$$

$$1 - s_i(a^T v_i + b) \leq 0$$

Violation

$$\max\{0, 1 - s_i(a^T v_i + b)\}$$



Linear classification

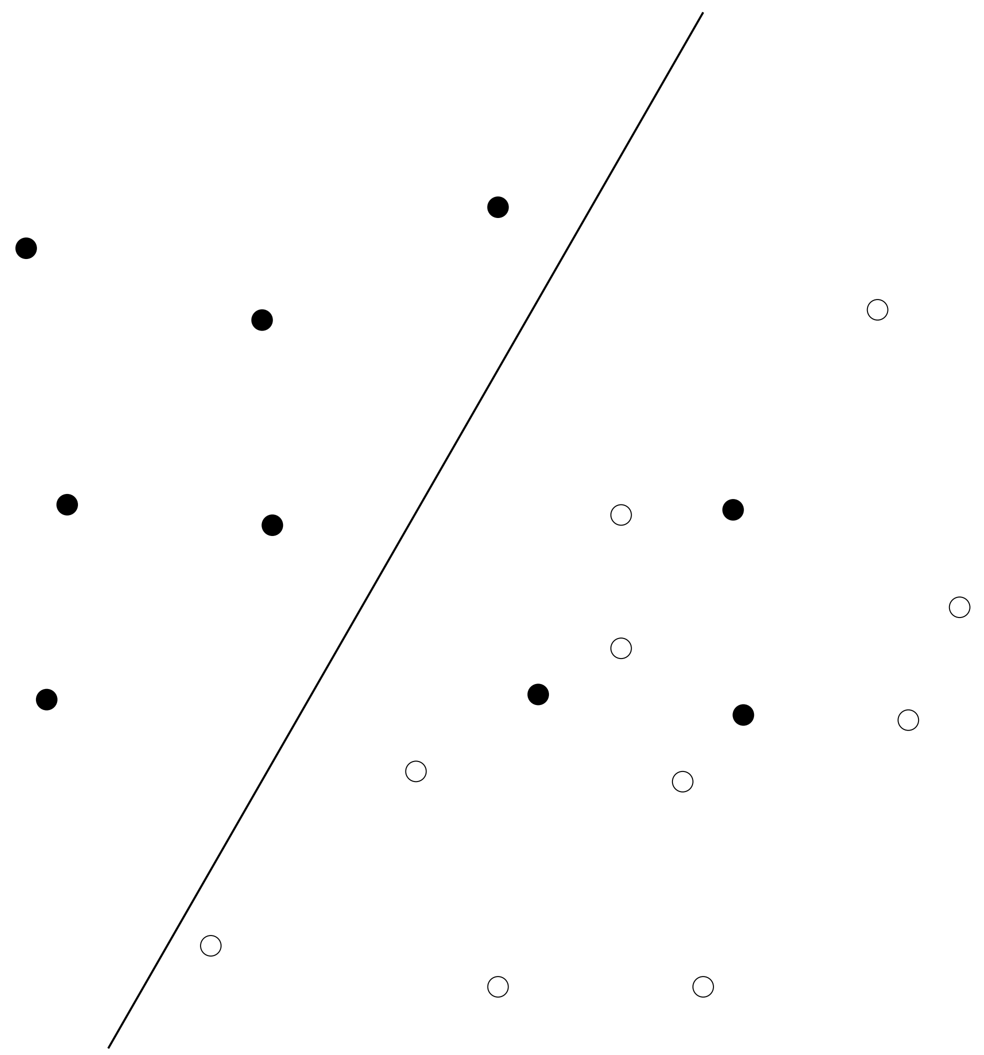
Approximate linear separation of non-separable points

Each of our constraints is

$$s_i(a^T v_i + b) \geq 1 \longrightarrow$$

Violation

$$\max\{0, 1 - s_i(a^T v_i + b)\}$$



Goal

Minimize sum of the violations

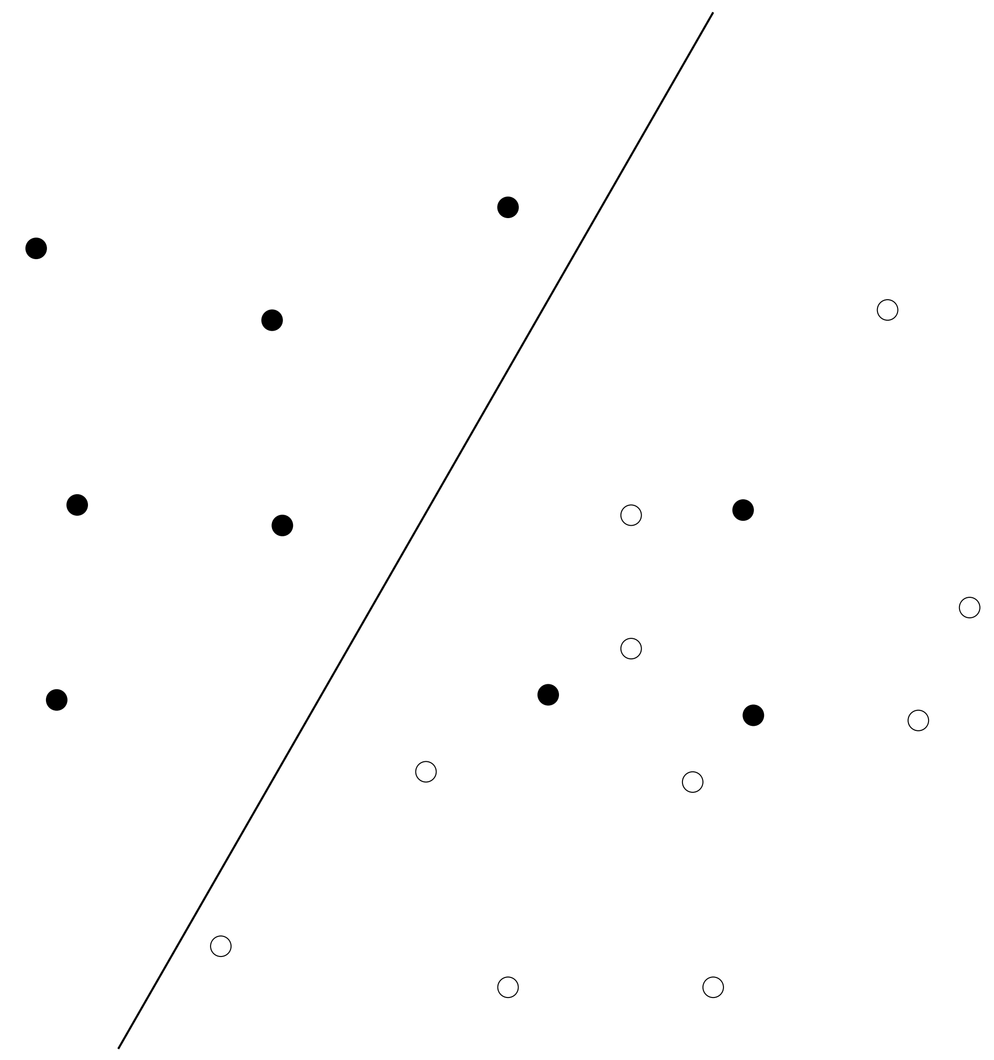
minimize $\sum_{i=1}^{N} \max\{0, 1 - s_i(a^T v_i + b)\}$

Piecewise-linear minimization problem with variables a, b

Linear classification

Approximate linear separation of non-separable points

minimize $\quad \sum_{i=1}^{N} \max\{0, 1 - s_i(a^T v_i + b)\}$



Linear classification

Approximate linear separation of non-separable points

minimize $\quad \sum_{i=1}^{N} \max\{0, 1 - s_i(a^T v_i + b)\}$

EXERCISE

$\tilde{x} = (a, b, t)$

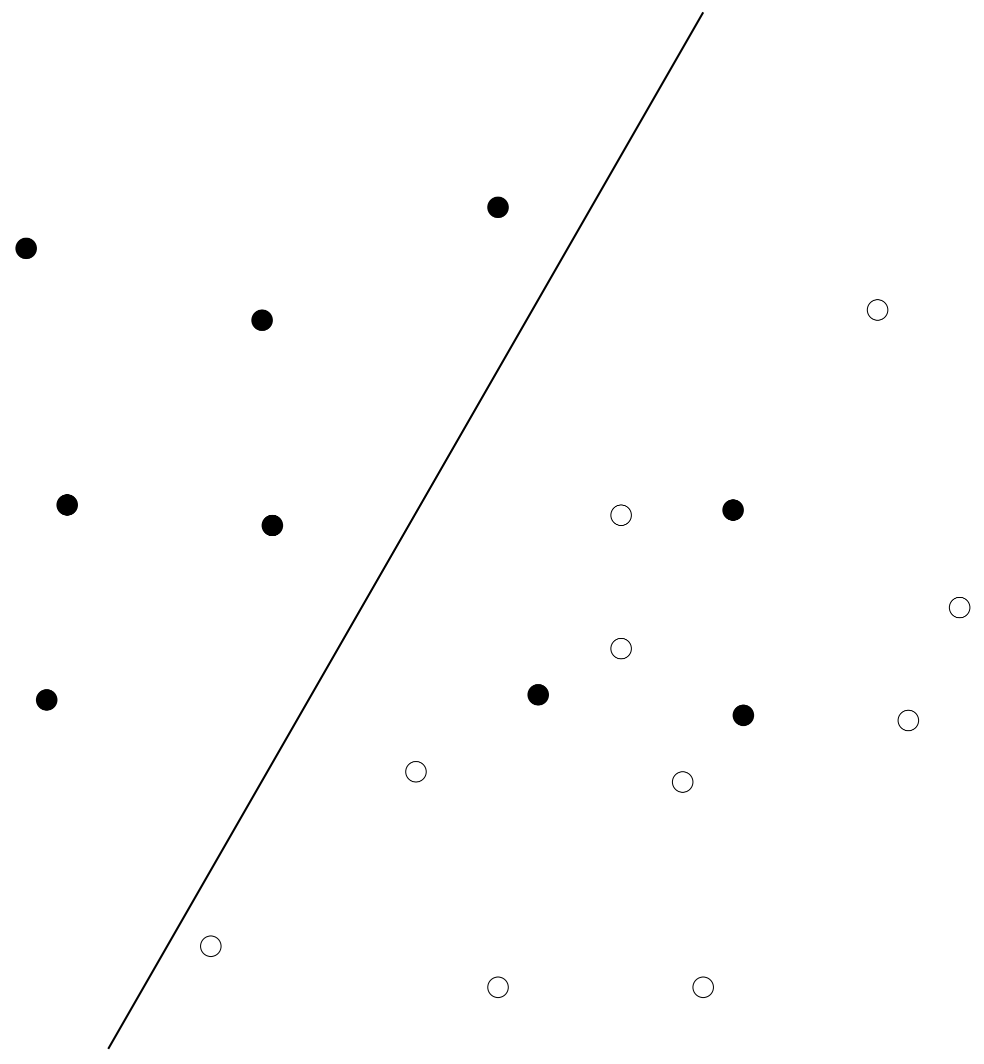
WRITE in MATRIX FORM

As a linear optimization problem

$$\min \sum_{i=1}^{N} t_i$$

$$\text{s.t.} \quad 1 - s_i(a^T v_i + b) \le t_i \qquad i = 1, \dots, N$$

$$0 \le t_i \qquad i = 1, \dots, N$$



Piecewise-linear optimization

Today, we learned to:

- **Understand** the differences between vector norms
- **Reformulate** convex piecewise linear minimization as linear optimization
- **Apply** these techniques to sparse signal recovery and classification problems

References

- Bertsimas, Tsitsiklis: Introduction to Linear Optimization
  - Chapter 1.3: piecewise linear optimization
- R. Vanderbei: Linear Programming — Foundations and Extensions
  - Chapter 12.4,12.7: 1-norm regression and SVMs

Next time

- Linear optimization geometry
- Optimality conditions