ORF307 – Optimization

6. Constrained least squares

Bartolomeo Stellato — Spring 2024

Ed Forum

CONTROL $y = Ax + b$ GOAL $\min \|y - y^{ref}\|^2 + \lambda \|x\|^2$

ESTIMATION $y = Ax + v$ GOAL $\min_x \|Ax - y\|^2 + \lambda \|x\|^2$

- What is the difference between control and estimation?
- What is the definition of pareto frontier? Why can't you do better?

Recap

Multi-objective least squares

Goal choose n-vector x such that
k norm squared objectives are small

$$J_1 = \|A_1 x - b_1\|^2$$

$$\vdots$$

$$J_k = \|A_k x - b_k\|^2$$

A_i are $m_i \times n$ matrices and b_i are m_i-vectors for $i = 1, \ldots, k$

J_i are the objectives in a *multi-objective (-criterion) optimization problem*

Could choose x to minimize
any one J_i, but we want
tie make them all small

Weighted sum objective

Choose positive weights $\lambda_1, \ldots, \lambda_k$ and form *weighted sum objective*

$$\begin{aligned} J &= \lambda_1 J_1 + \cdots + \lambda_k J_k \\ &= \lambda_1 \|A_1 x - b_1\|^2 + \cdots + \lambda_k \|A_k x - b_k\|^2 \end{aligned}$$

Choose x to minimize J

Primary objective

- Often $\lambda_1 = 1$ and J_1 is the **primary objective**
- **Interpretation** λ_i is how much we care about J_i being small, relative to J_1

Bi-criterion optimization

$$J_1 + \lambda J_2 = \|A_1 x - b_1\|^2 + \lambda \|A_2 x - b_2\|^2$$

Optimal trade-off curve

Bi-criterion problem

$$\text{minimize} \quad J_1(x) + \lambda J_2(x) \quad \longrightarrow \quad x^\star(\lambda)$$

Pareto optimal $x^\star(\lambda)$

There is no point z that satisfies

$$J_1(z) \leq J_1(x^\star(\lambda)) \quad \text{and} \quad J_2(z) \leq J_2(x^\star(\lambda))$$

with one of the inequalities holding strictly
(no other point beats $x^\star$ on both objectives)

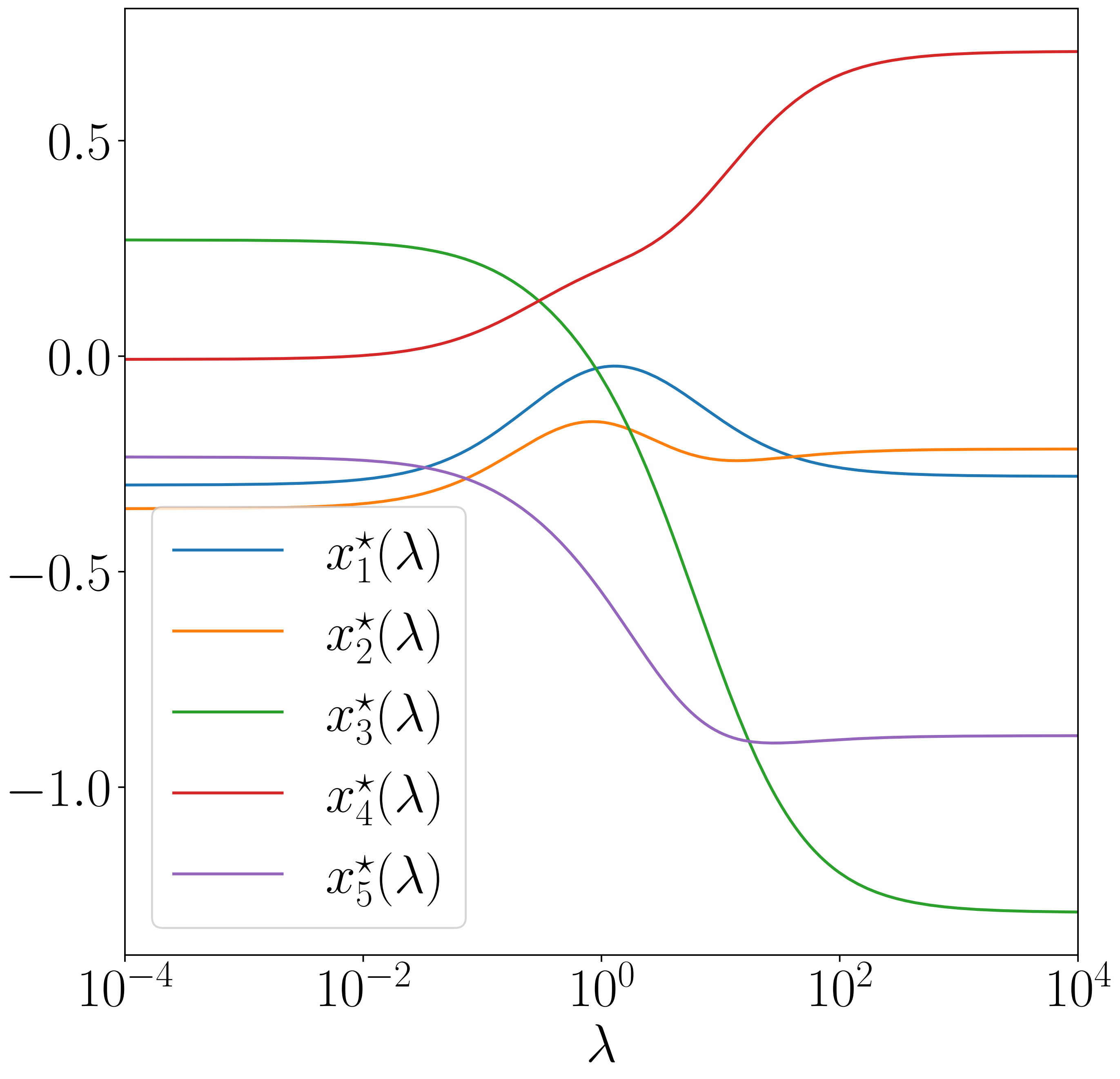
Optimal trade-off curve

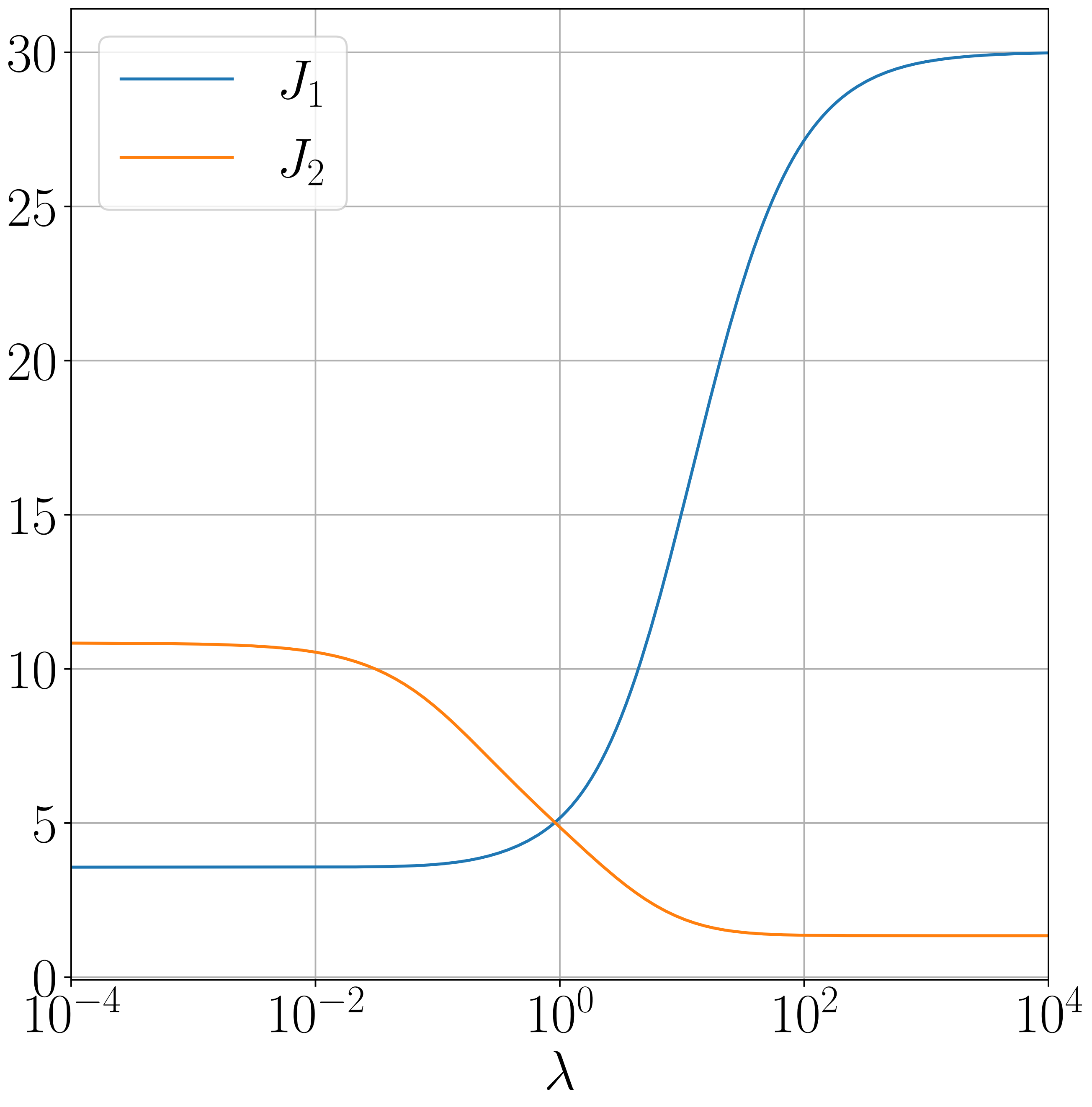
$$(J_1(x^\star(\lambda)), J_2(x^\star(\lambda)), \qquad \lambda > 0$$

Optimal trade-off curve

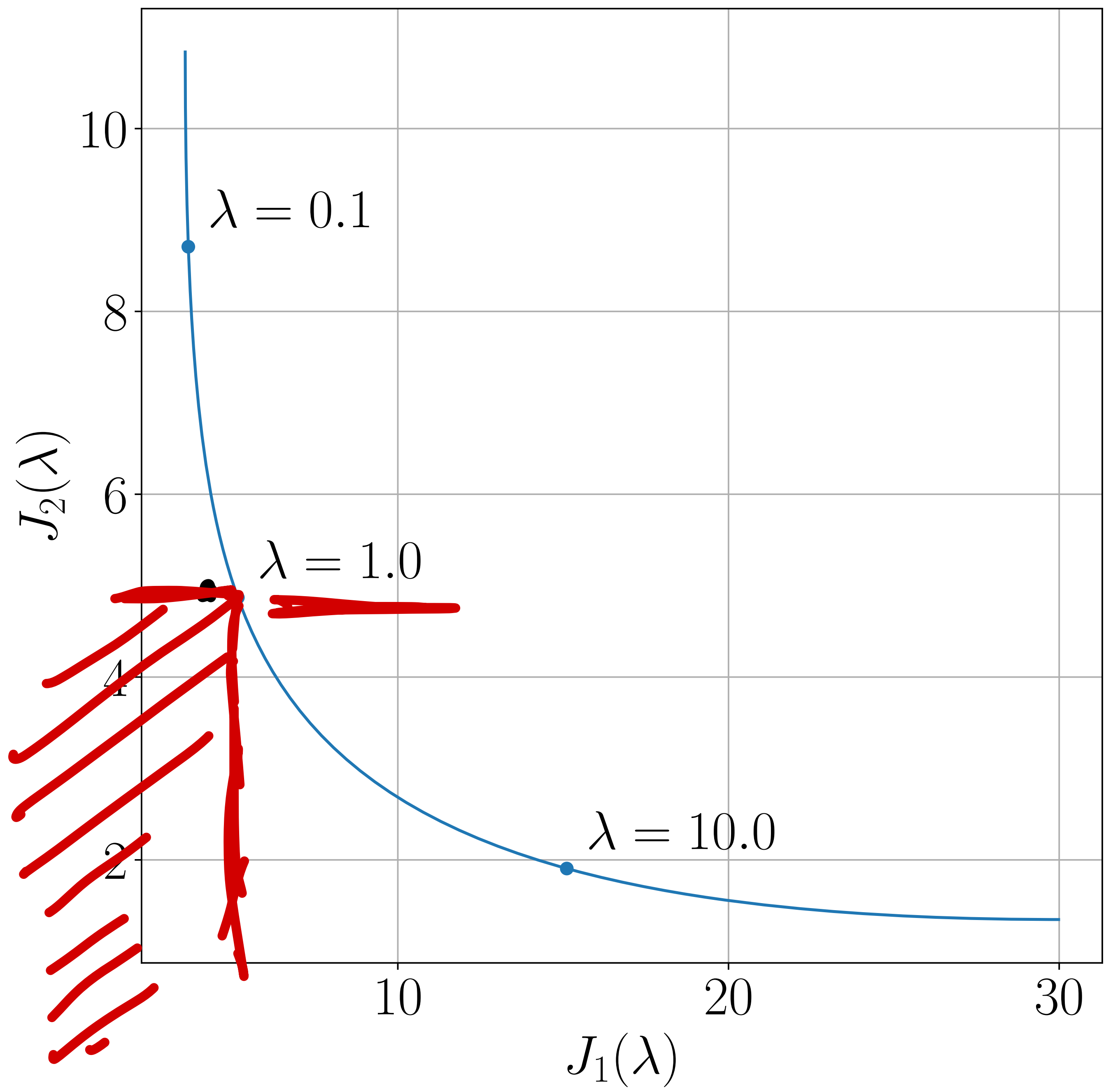
Example

minimize $\quad J_1(x) + \lambda J_2(x)$ $\qquad$ $(A_1, A_2$ are both $10 \times 5)$





Trade-off curve



Today's lecture

Constrained least squares

- Linearly constrained least squares
- Solving the constrained least squares problem
- Portfolio optimization

Linearly constrained least squares

Least squares with equality constraints

The (linearly) constrained least squares problem is

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|^2 \\ \text{subject to} & Cx = d \end{array}$$

Problem data

- $m \times n$ matrix A, m-vector b
- $p \times n$ matrix C, p-vector d

Least squares with equality constraints

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x is *feasible* if $Cx = d$

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$x^\star$ is a *solution* if

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- $\|Ax^\star - b\|^2 \leq \|Ax - b\|^2$ for any x satisfying $Cx = d$

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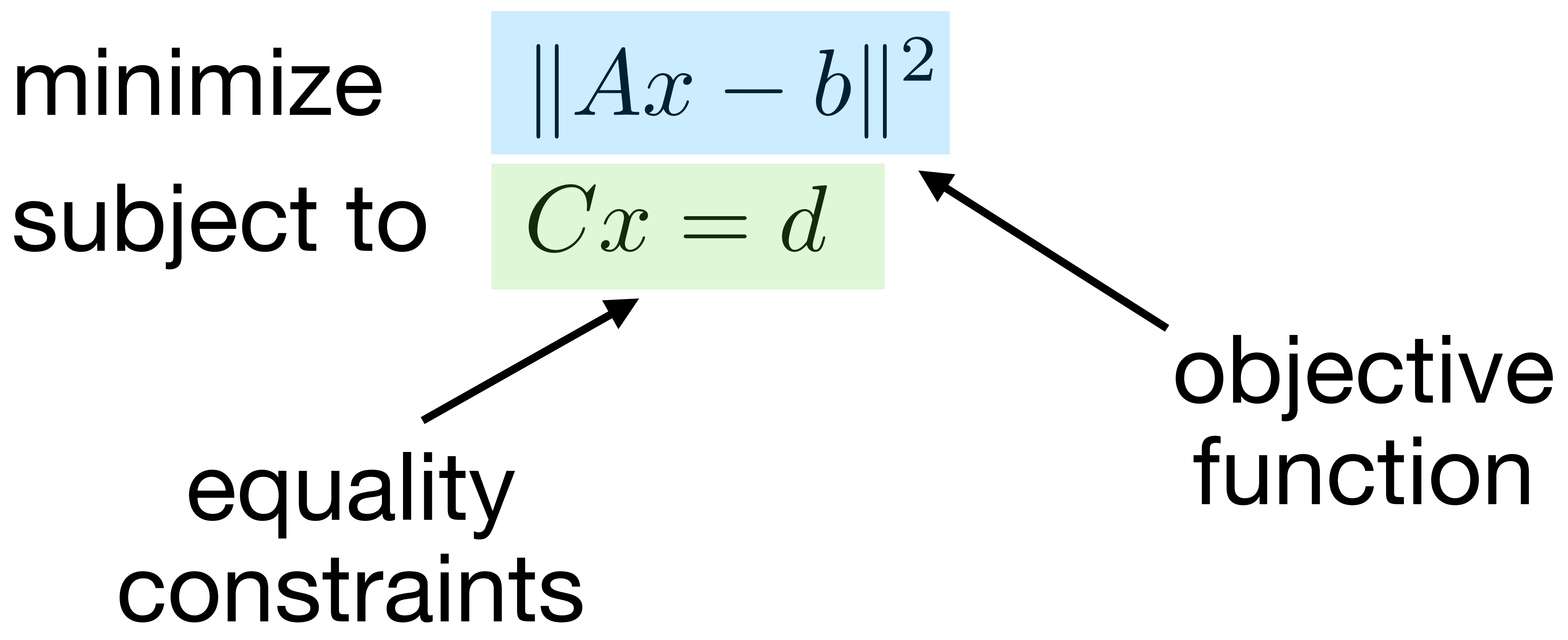

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Interpretations

- Combine solving linear equations with least squares.
- Like a bi-objective least squares with ∞ weight on second objective, $\|Cx - d\|^2$.

Optimal advertising with budget

m demographic groups
we want to advertise to

→

v^{des} is the m-vector
of desired views/impressions

n advertising channels
(web publishers, radio, print, etc.)

→

s is the n-vector
of purchases

$m \times n$ matrix A gives
demographic reach of channels

→

A_{ij} is the number of views
for group i and dollar spent
on channel j $(1000/\$)$

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Views across
demographic groups

$$v = As$$

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Views across demographic groups

$$v = As$$

Goal

$$\begin{array}{ll} \text{minimize} & \|As - v^{\text{des}}\|^2 \\ \text{subject to} & \mathbf{1}^T s = B \end{array}$$

Optimal advertising with budget

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$$\begin{aligned} \text{minimize} \quad & \|As - v^{\text{des}}\|^2 \\ \text{subject to} \quad & \mathbf{1}^T s = B \end{aligned}$$

← allocated budget

Optimal advertising with budget

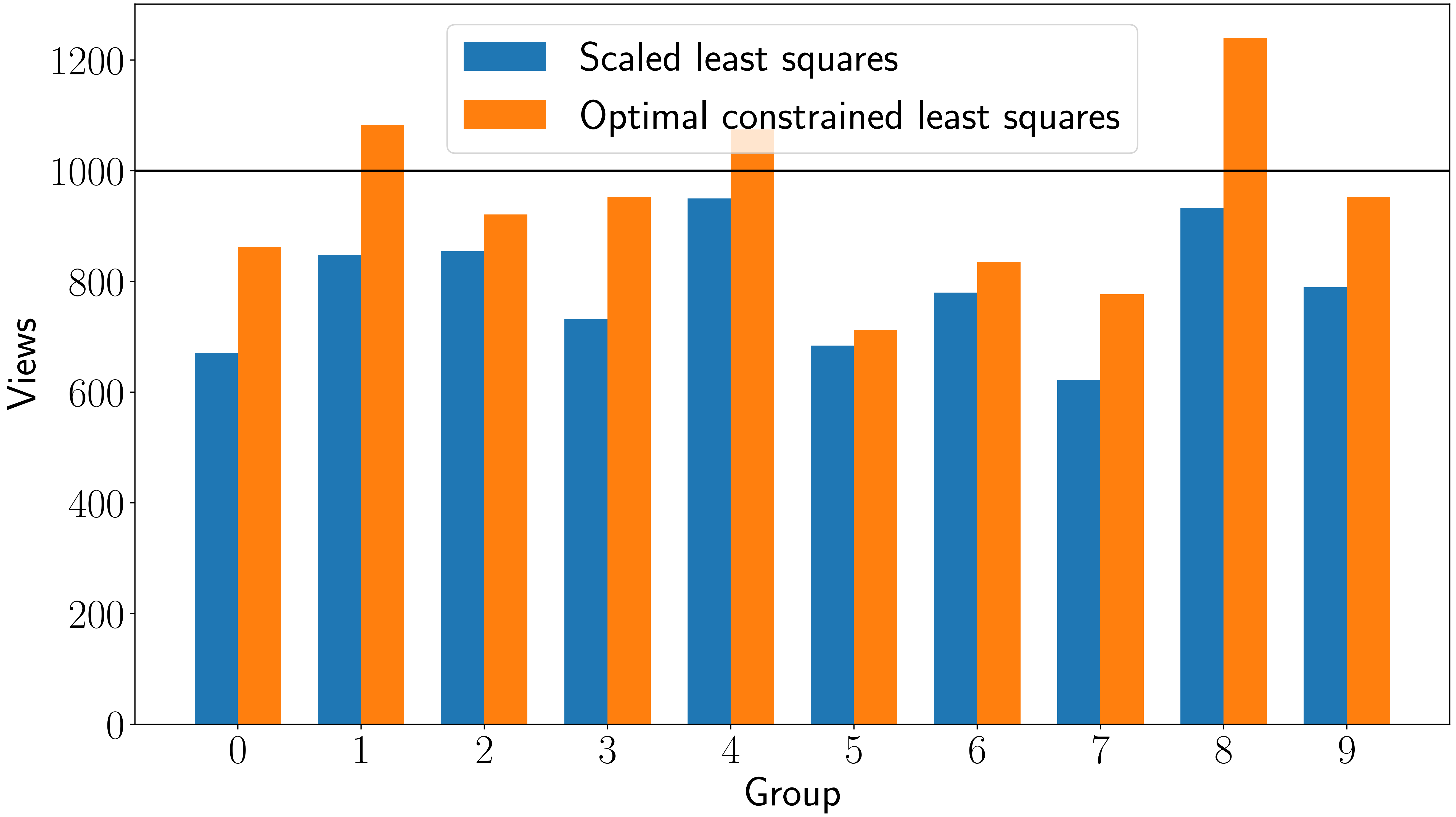
Results

$m = 10$ groups, $n = 3$ channels

budget $B = 1284$

desired views vector $v^{\text{des}} = (10^3)\mathbf{1}$

$$\begin{array}{ll} \text{minimize} & \|As - v^{\text{des}}\|^2 \\ \text{subject to} & \mathbf{1}^T s = B \end{array}$$



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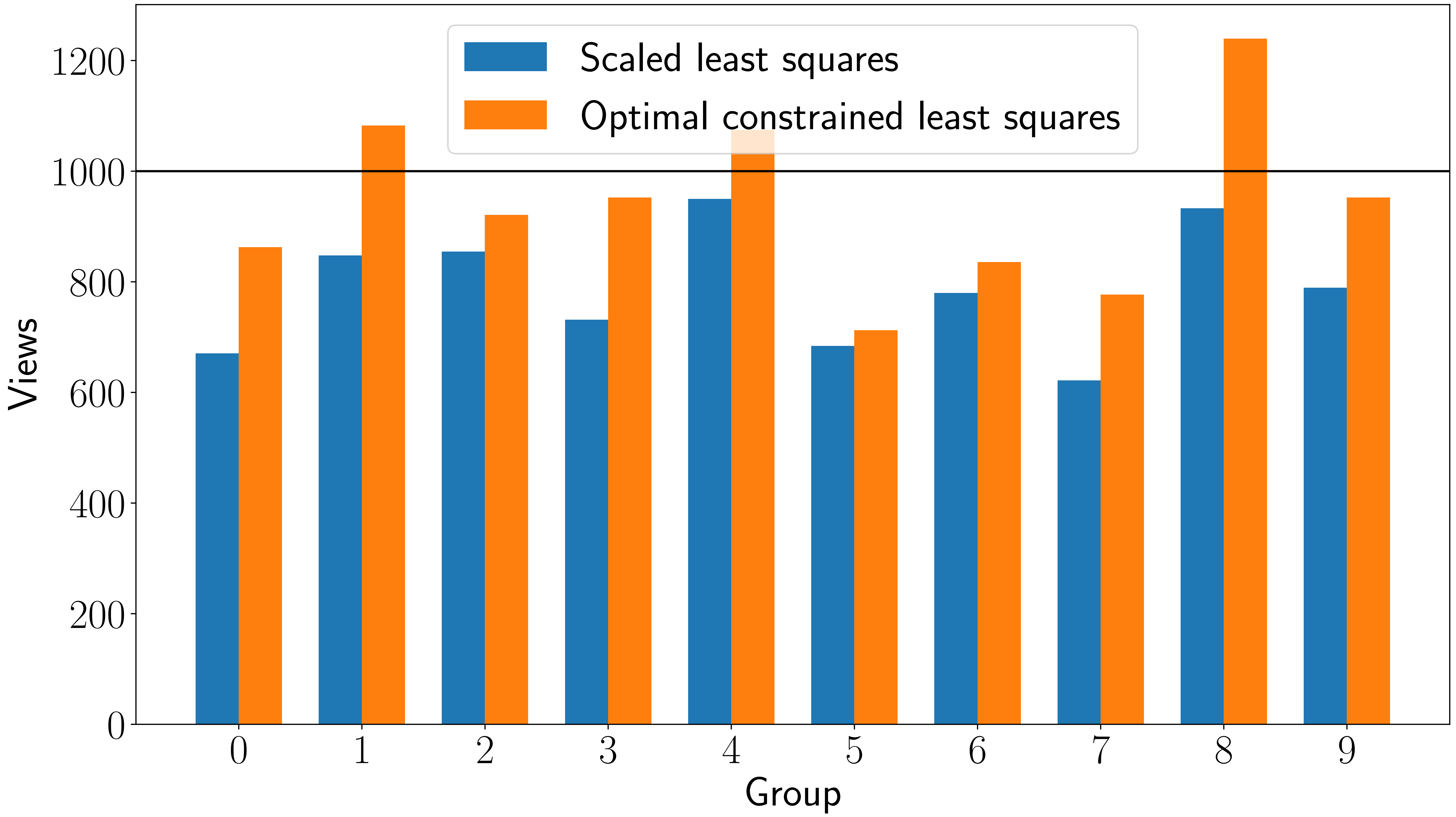

optimal spending $s^\star = (315, 110, 859)$ ⟶ RMS 16.10%

Optimal advertising with budget

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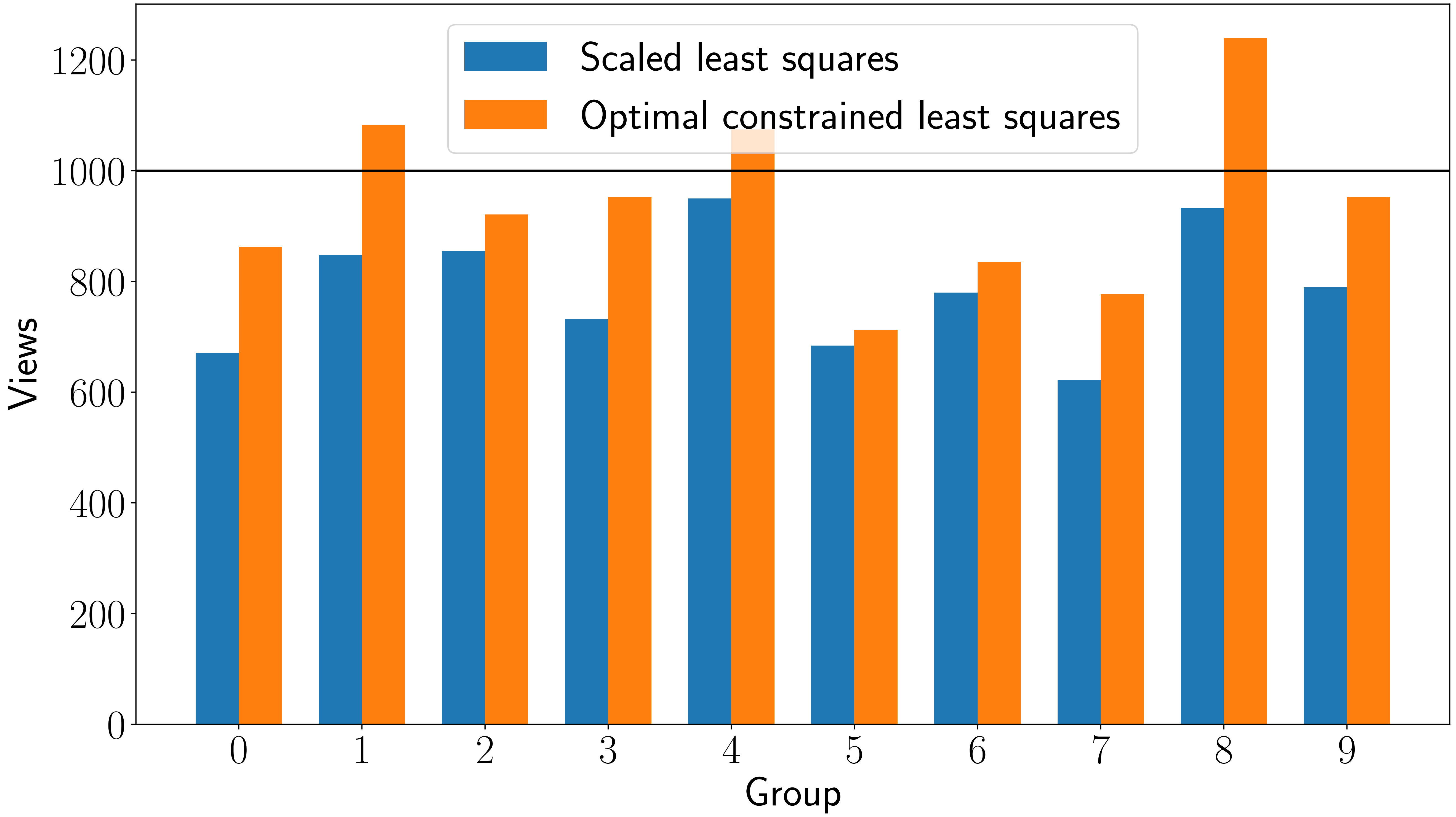


optimal spending $s^\star = (315, 110, 859)$ $\longrightarrow$ RMS 16.10%

rescaled least squares spending $s^\star = (50, 80, 1154)$ $\longrightarrow$ RMS 23.85%

Least norm problem

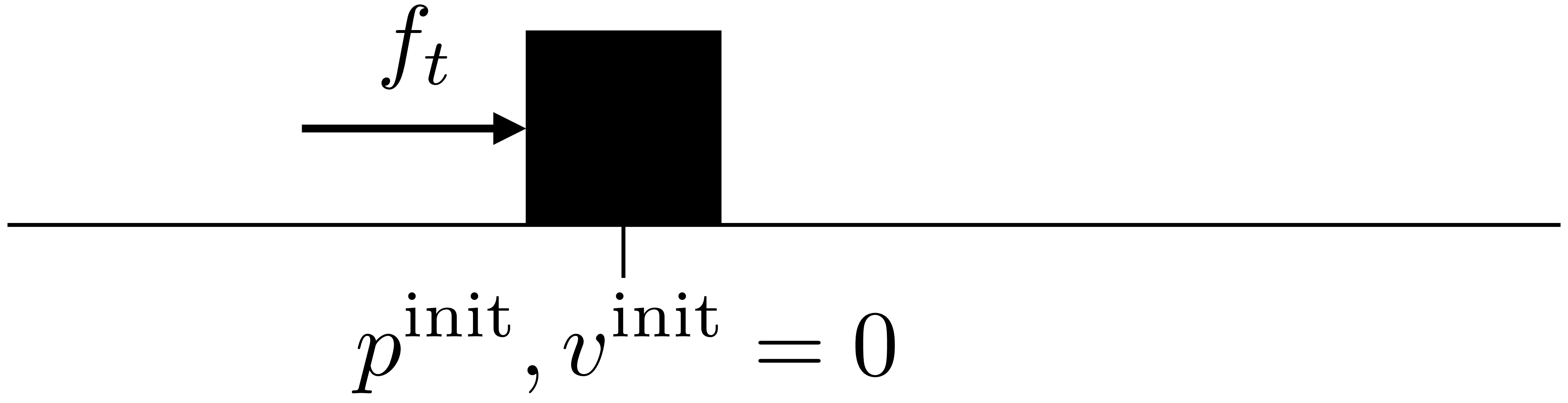
Special case of constrained least squares problem with $A = I$ and $b = 0$

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|^2 \\ \text{subject to} & Cx = d \end{array} \quad \longrightarrow \quad \begin{array}{ll} \text{minimize} & \|x\|^2 \\ \text{subject to} & Cx = d \end{array}$$

Find the smallest vector that satisfies a set of linear equations

Force sequence

Unit mass on frictionless surface, initially at rest



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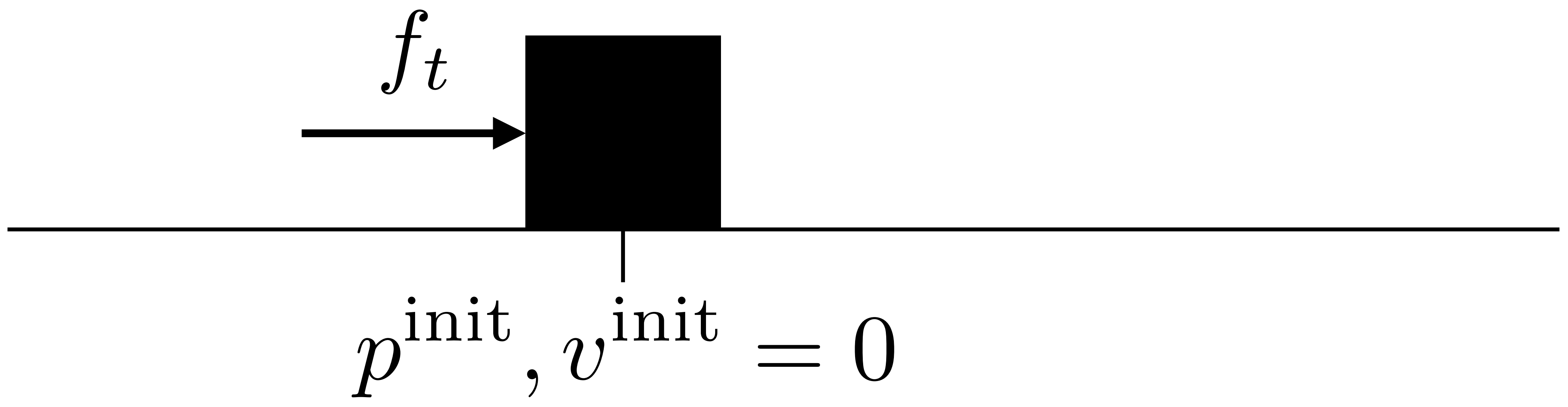

$v^{\text{init}} = 0$

$p^{\text{init}} = 0$

$m = 1$

$a = f$

$\Delta t = 1$

EXERCISE

DERIVE EQUALITIES

10-vector f gives the forces applied for one second each

Final velocity and position (Newton's laws)

$$v^{\text{fin}} = f_1 + f_2 + \cdots + f_{10}$$

$$p^{\text{fin}} = (19/2)f_1 + (17/2)f_2 + \cdots + (1/2)f_{10}$$

Force sequence

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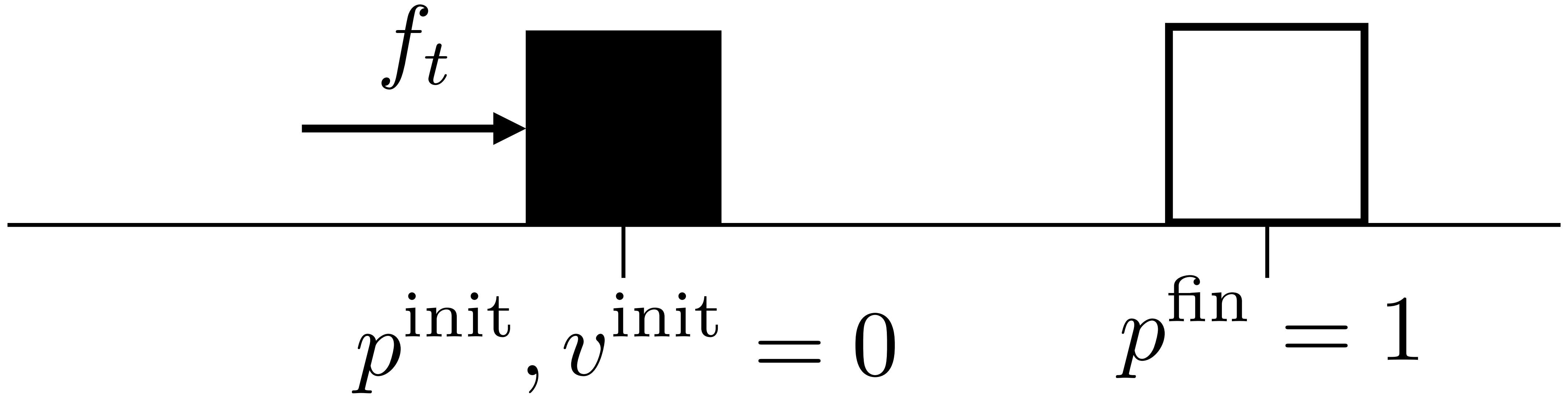

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$$v^{\text{fin}} = f_1 + f_2 + \cdots + f_{10}$$

$$p^{\text{fin}} = (19/2)f_1 + (17/2)f_2 + \cdots + (1/2)f_{10}$$

Goal

Let's find f such that $v^{\text{fin}} = 0$ and $p^{\text{fin}} = 1$

Least norm force sequence

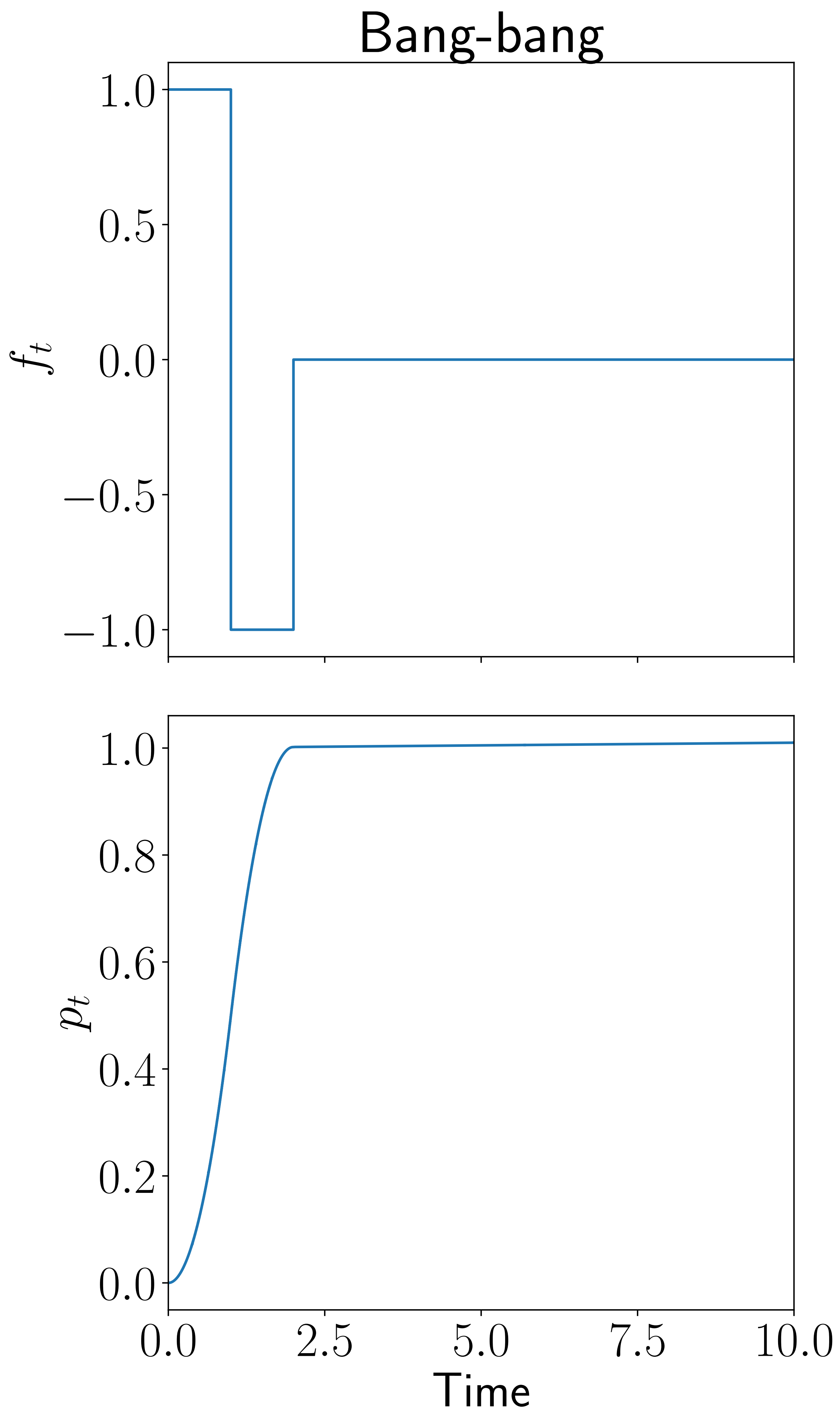
Find f that brings to $p^{\text{fin}} = 1$, $v^{\text{fin}} = 0$

Least norm force sequence

Find f that brings to $p^{\text{fin}} = 1$, $v^{\text{fin}} = 0$

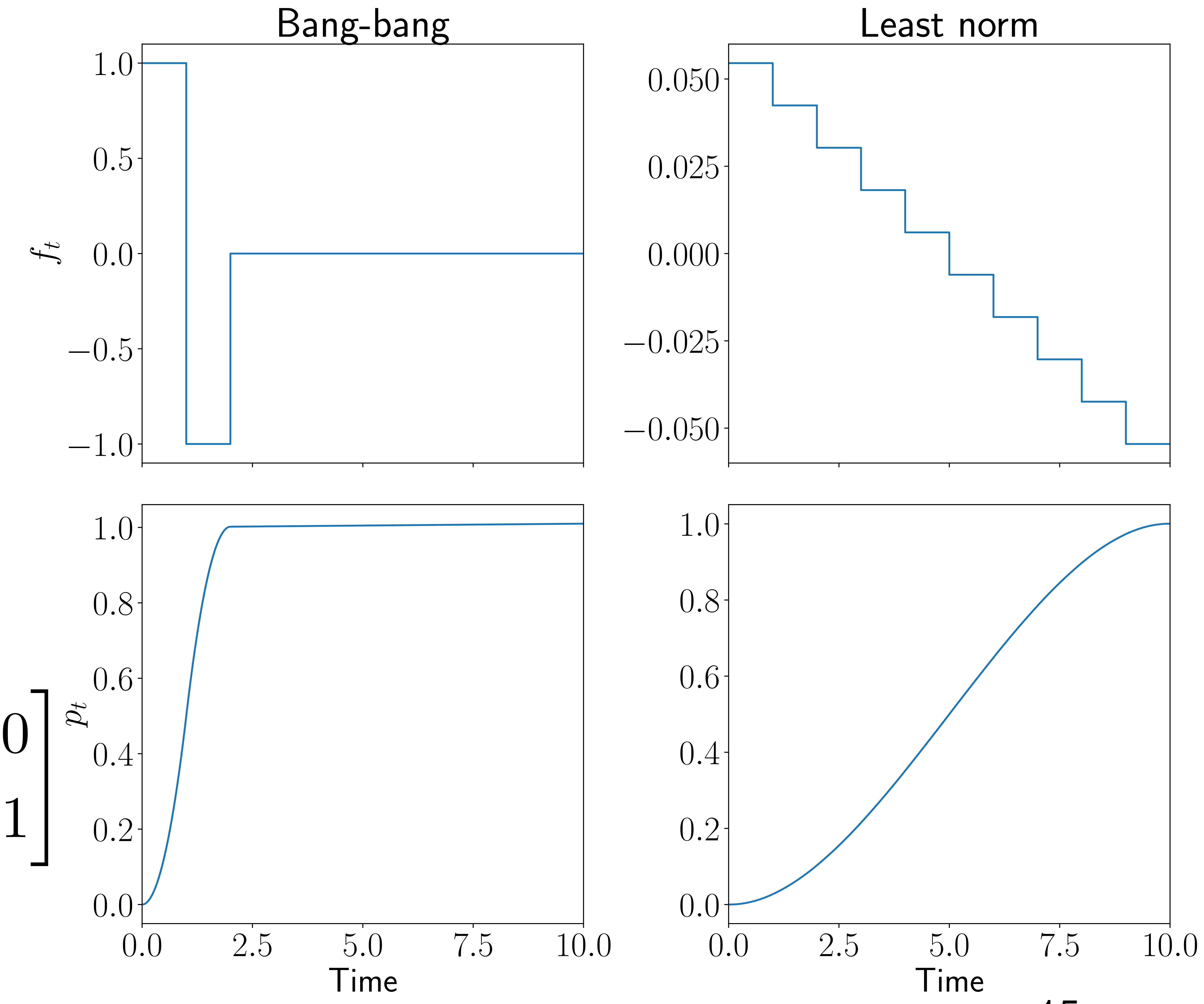
Bang-bang solution

$$f^{\text{bb}} = (1, -1, 0, \ldots, 0) \qquad \|f^{\text{bb}}\|^2 = 2$$



Least norm force sequence

Find f that brings to $p^{\text{fin}} = 1$, $v^{\text{fin}} = 0$



Bang-bang solution

$$f^{\text{bb}} = (1, -1, 0, \ldots, 0) \qquad \|f^{\text{bb}}\|^2 = 2$$

Least norm solution

minimize $\quad \|f\|^2$

subject to $\quad \begin{bmatrix} 1 & 1 & \ldots & 1 \\ 19/2 & 17/2 & \ldots & 1/2 \end{bmatrix} f = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\|f^{\text{ln}}\|^2 = 0.012$ **Much cheaper effort!**

Solving the constrained least squares problem

Optimality conditions via calculus

minimize $\quad f(x) = \|Ax - b\|^2$

subject to $\quad Cx = d$

Optimality conditions via calculus

$$\begin{array}{ll} \text{minimize} & f(x) = \|Ax - b\|^2 \\ \text{subject to} & Cx = d \end{array} \quad \longrightarrow \quad \begin{array}{ll} \text{minimize} & f(x) = \|Ax - b\|^2 \\ \text{subject to} & c_i^T x = d_i, \quad i = 1, \ldots, p \end{array}$$

Optimality conditions via calculus

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Lagrangian function

$$L(x, z) = f(x) + z_1(c_1^T x - d_1) + \cdots + z_p(c_p^T x - d_p)$$

Optimality conditions via calculus

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Optimality conditions

$$\left[\begin{aligned} &\frac{\partial L}{\partial x_i}(x^\star, z) = 0, \quad i = 1, \ldots, n, \\ &\frac{\partial L}{\partial z_i}(x^\star, z) = 0, \quad i = 1, \ldots, p \end{aligned}\right.$$

Optimality conditions via calculus

$$L(x, z) = x^T A^T A x - 2(A^T b)^T x + b^T b + z_1(c_1^T x - d_1) + \cdots + z_p(c_p^T x - d_p)$$

Optimality conditions via calculus

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Optimality conditions

$$\frac{\partial L}{\partial z_i}(x^\star, z) = c_i^T x^{*} - d_i = 0$$ (we already knew)

Optimality conditions via calculus

$$L(x, z) = x^T A^T A x - 2(A^T b)^T x + b^T b + z_1(c_1^T x - d_1) + \cdots + z_p(c_p^T x - d_p)$$

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$$\frac{\partial L}{\partial z_i}(x^\star, z) = c_i^T x - d_i = 0$$

(we already knew)

$$\frac{\partial L}{\partial x_i}(x^\star, z) = \underbrace{2\sum_{j=1}^{n}(A^T A)_{ij} x_j^\star - 2(A^T b)_i}_{2A^T(Ax-b)_i} + \sum_{j=1}^{p} z_j (c_j)_i = 0$$

Optimality conditions via calculus

$$C = \begin{bmatrix} \text{—}\; c_1^T \;\text{—} \\ \text{—}\; c_2^T \;\text{—} \\ \vdots \\ \text{—}\; c_p^T \;\text{—} \end{bmatrix}$$

$$L(x,z) = x^TA^TAx - 2(A^Tb)^Tx + b^Tb + z_1(c_1^Tx - d_1) + \cdots + z_p(c_p^Tx - d_p)$$

Optimality conditions

$$\frac{\partial L}{\partial z_i}(x^\star, z) = c_i^Tx - d_i = 0$$ (we already knew)

$$\frac{\partial L}{\partial x_i}(x^\star, z) = 2\sum_{j=1}^{n}(A^TA)_{ij}x_j^\star - 2(A^Tb)_i + \sum_{j=1}^{p} z_j(c_j)_i = 0$$

$$\longrightarrow$$

Vector form

$$Cx = d$$

$$2A^TAx^\star - 2A^Tb + C^Tz = 0$$

$$\begin{bmatrix} | & | & | & & | \\ c_1 & c_2 & c_3 & \cdots & c_p \\ | & | & | & & | \end{bmatrix}$$

Optimality conditions via calculus

$$L(x, z) = x^T A^T A x - 2(A^T b)^T x + b^T b + z_1(c_1^T x - d_1) + \cdots + z_p(c_p^T x - d_p)$$

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Vector form

$$Cx = d$$

$$2A^T A x^\star - 2A^T b + C^T z = 0$$

Karush-Kuhn-Tucker (KKT) conditions

$$\begin{bmatrix} 2A^T A & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} x^\star \\ z \end{bmatrix} = \begin{bmatrix} 2A^T b \\ d \end{bmatrix}$$

(square set of $n + p$ linear equations)

Optimality conditions via calculus

$$L(x, z) = x^T A^T A x - 2(A^T b)^T x + b^T b + z_1(c_1^T x - d_1) + \cdots + z_p(c_p^T x - d_p)$$

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(square set of $n + p$ linear equations)

Note KKT equations are extension of normal equations to constrained least squares

Invertibility of KKT matrix

no longer positive definite in general ⟶

$$\begin{bmatrix} 2A^TA & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} x^\star \\ z \end{bmatrix} = \begin{bmatrix} 2A^Tb \\ d \end{bmatrix}$$

Invertibility of KKT matrix

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The KKT matrix is invertible if and only if

- C has linearly independent rows
- $\begin{bmatrix} A \\ C \end{bmatrix}$ has linearly independent columns (true when A has linearly indep. cols)

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The KKT matrix is invertible if and only if

- C has linearly independent rows ⟶ $p \leq n$ (C is wide)
- $\begin{bmatrix} A \\ C \end{bmatrix}$ has linearly independent columns ⟶ $m + p \geq n$ $\left(\begin{bmatrix} A \\ C \end{bmatrix} \text{ is tall} \right)$
  (true when A has linearly indep. cols)

Invertibility of KKT matrix

no longer positive definite in general ⟶

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Complexity (with $p \leq n \leq m$)

- Factor + solve: $2mn^2 + (2/3)(n+p)^3 + 2(n+p)^2 \approx 2mn^2$
- Solve given a new b (prefactored): $2mn + 2(n+p)^2 \approx 2mn$

same as unconstrained

Optimality from KKT solution

For $x^\star$ and $z^\star$ such that

$$2A^TAx^\star + C^Tz^\star = 2A^Tb, \quad Cx^\star = d$$

Optimality from KKT solution

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Given a feasible x and z, we can write the objective (just as least squares)

$$\begin{aligned}\|Ax - b\|^2 &= \|(Ax - Ax^\star) + (Ax^\star - b)\|^2 \\ &= \|A(x - x^\star)\|^2 + \|Ax^\star - b\|^2 + 2(x - x^\star)^TA^T(Ax^\star - b)\end{aligned}$$

Optimality from KKT solution

For $x^\star$ and $z^\star$ such that

$$2A^TAx^\star + C^Tz^\star = 2A^Tb, \quad Cx^\star = d$$

Given a feasible x and z, we can write the objective (just as least squares)

$$\|Ax - b\|^2 = \|(Ax - Ax^\star) + (Ax^\star - b)\|^2$$

$$= \|A(x - x^\star)\|^2 + \|Ax^\star - b\|^2 + \overset{=0}{\xcancel{2(x - x^\star)^TA^T(Ax^\star - b)}}$$

We can expand last term, using $2A^T(Ax^\star - b) = -C^Tz^\star$ and $Cx = Cx^\star = d$

$$2(x - x^\star)^TA^T(Ax^\star - b) = -(x - x^\star)^TC^Tz^\star = -(C(x - x^\star))^Tz^\star \boxed{= 0}$$

Optimality from KKT solution

For $x^\star$ and $z^\star$ such that

$$2A^TAx^\star + C^Tz^\star = 2A^Tb, \quad Cx^\star = d$$

Given a feasible x and z, we can write the objective (just as least squares)

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We can expand last term, using $2A^T(Ax^\star - b) = -C^Tz^\star$ and $Cx = Cx^\star = d$

$$2(x - x^\star)^TA^T(Ax^\star - b) = -(x - x^\star)^TC^Tz^\star = -(C(x - x^\star))^Tz^\star = 0$$

$$\|Ax - b\|^2 = \overbrace{\|A(x - x^\star)\|^2}^{\geq 0} + \|Ax^\star - b\|^2 \boxed{\geq} \|Ax^\star - b\|^2$$

$x^\star$ is **optimal**

Portfolio optimization

Portfolio allocation weights

We want to invest V dollars in n different *assets* (stocks, bonds, ...) over periods $t = 1, \ldots, T$

Portfolio allocation weights

n-vector w gives the fraction of our total portfolio held in each asset

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Portfolio allocation weights

n-vector w gives the fraction of our total portfolio held in each asset

Properties

- Vw_j dollar value hold in asset j
- $\mathbf{1}^T w = 1$ (normalized)
- $w_j < 0$ means short positions (you borrow) (must be returned at time T)
- Example: $w = (-0.2, 0.0, 1.2)$

Portfolio allocation weights

We want to invest V dollars in n different *assets* (stocks, bonds, ...)
over periods $t = 1, \ldots, T$

Portfolio allocation weights

n-vector w gives the fraction of our total portfolio held in each asset

Properties

- Vw_j dollar value hold in asset j
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Short position
of $0.2V$ on asset 1

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Short position of $0.2V$ on asset 1

Don't hold any of asset 2

Hold $1.2V$ in asset 3

Leverage, long-only portfolios, and cash

Leverage

$$L = |w_1| + \cdots + |w_n| = \|w\|_1$$

$L = 1$ when all weights are nonnegative (“long only portfolio”)

Leverage, long-only portfolios, and cash

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$$w = \mathbf{1}/n$$

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Uniform portfolio

$$w = \mathbf{1}/n$$

Risk free asset

We often assume asset n is ”risk-free” (e.g., cash)

if $w = e_n$, it means the portfolio is all cash

Return over a period

Asset returns

$\tilde{r}_t$ is the (fractional) return of each asset over period t

example: $\tilde{r}_t = (0.01, -0.023, 0.02)$
(often expressed as percentage)

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Portfolio return

$$r_t = \tilde{r}_t^T w$$

It is the (fractional) return for the entire portfolio over period t

Return over a period

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Portfolio return

$$r_t = \tilde{r}_t^T w$$

It is the (fractional) return
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Total portfolio value after a period

$$V_{t+1} = V_t + V_t \tilde{r}_t^T w = V_t(1 + r_t)$$

Return matrix

Hold constant portfolio
with weights w over T periods

R is the $T \times n$ matrix of **asset returns**

R_{tj} is the return of asset j in period t

$$R = \begin{array}{c|cccc|l} & \text{AAPL} & \text{GOOG} & \text{MMM} & \text{US \$} & \\ \hline & 0.00219 & 0.0006 & -0.00113 & 0.00005 & \text{Mar 1, 2016} \\ & 0.00744 & -0.00894 & -0.00019 & 0.00005 & \text{Mar 2, 2016} \\ & 0.01488 & -0.00215 & 0.00433 & 0.00005 & \text{Mar 3, 2016} \end{array}$$

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Row t is $\tilde{r}_t$ is the asset
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Portfolio returns (time series)

$$r = Rw \quad (T\text{-vector})$$

Returns over multiple periods

r is time series T-vector of portfolio returns

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average return
(or just return)

$$\mathbf{avg}(r) = \mathbf{1}^T r/T$$

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Total portfolio value

$$\begin{aligned} V_{T+1} &= V_1(1 + r_1) \cdots (1 + r_T) \\ &\approx V_1 + V_1(r_1 + \cdots + r_T) \\ &= V_1 + T\mathbf{avg}(r)V_1 \end{aligned}$$

(for $|r_t|$ small, e.g., ≤ 0.01
ignore higher order terms)

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(for $|r_t|$ small, e.g., ≤ 0.01
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For high portfolio value we need large $\mathbf{avg}(r)$

Annualized return and risk

Mean return and and risk are often expressed in **annualized form** (per year)

Annualized return and risk

Mean return and and risk are often expressed in **annualized form** (per year)

Given P trading periods per year (i.e., 250 days)

annualized return $= P\mathbf{avg}(r)$, annualized risk $= \sqrt{P}\mathbf{std}(r)$

Portfolio optimization

How shall we choose the portfolio weight vector w?

Portfolio optimization

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Goals

High (mean) return
$\mathbf{avg}(r)$

Low risk
$\mathbf{std}(r)$

Portfolio optimization

How shall we choose the portfolio weight vector w?

Goals

High (mean) return	Low risk
$\mathbf{avg}(r)$	$\mathbf{std}(r)$

Data

- We know **realized asset returns** but not future ones
- **Optimization.** We choose w that would have worked well in the past
- **True goal.** Hope it will work well in the future (just like data fitting)

$r = Rw$

Portfolio optimization

Minimize risk given a target return

Chose n-vector w to solve

$$\begin{array}{ll} \text{minimize} & \mathbf{std}(Rw)^2 = (1/T)\|Rw - \rho\mathbf{1}\|^2 \\ \text{subject to} & \mathbf{1}^T w = 1 \\ & \mathbf{avg}(Rw) = \rho \end{array}$$

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(past) portfolio
returns time series

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(past) target
mean return

(past) portfolio
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Solutions w are **Pareto optimal**

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(past) target mean return

(past) portfolio returns time series

Solutions w are **Pareto optimal**

Our question

what would have been the best constant allocation w,
had we known future returns?

Example allocations

Annual return 1% (risk-free asset has 1% return)

$$w = (0.00, 0.00, 0.00, \ldots, 0.00, 0.00, 1.00)$$

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Annual return 13%

$$w = (0.02, -0.07, -0.05, \ldots, -0.03, 0.06, 0.56)$$

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Annual return 25%

$$w = (0.05, -0.143, -0.09, \ldots, -0.07, 0.12, 0.12)$$

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Annual return 25%

$$w = (0.05, -0.143, -0.09, \dots, -0.07, 0.12, 0.12)$$

Asking for higher annual returns yields

- More invested in risky, but high return assets
- Larger short positions (”leveraging”)

Portfolio optimization

As constrained least squares

$$\begin{array}{ll} \text{minimize} & \|Rw - \rho\mathbf{1}\|^2 \\ \text{subject to} & \begin{bmatrix} \mathbf{1}^T \\ \mu^T \end{bmatrix} w = \begin{bmatrix} 1 \\ \rho \end{bmatrix} \end{array}$$

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Portfolio optimization

As constrained least squares

$$\begin{array}{ll} \text{minimize} & \|\overbrace{Rw}^{A} - \overbrace{\rho\mathbf{1}}^{b}\|^2 \\ \text{subject to} & \underbrace{\begin{bmatrix}\mathbf{1}^T \\ \mu^T\end{bmatrix}}_{C} w = \left.\begin{bmatrix}1 \\ \rho\end{bmatrix}\right\} d \end{array}$$

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Solution via KKT linear system

$$C\begin{bmatrix} \overbrace{2R^TR}^{A^TA} & \overset{C^T}{\boxed{\mathbf{1} \quad \mu}} \\ \boxed{\begin{matrix}\mathbf{1}^T \\ \mu^T\end{matrix}} & \begin{matrix}0 & 0 \\ 0 & 0\end{matrix} \end{bmatrix} \begin{bmatrix} w \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 2\rho T\mu \\ 1 \\ \rho \end{bmatrix} \begin{matrix} \}\, 2A^Tb \\ \left.\begin{matrix}\\ \\\end{matrix}\right\} d \end{matrix}$$

Optimal portfolios

Rewrite right-hand side

$$\begin{bmatrix} 2\rho T\mu \\ 1 \\ \rho \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \rho \begin{bmatrix} 2T\mu \\ 0 \\ 0 \end{bmatrix}$$

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Optimal portfolios

Two fund theorem

Optimal portfolio w is an affine function of ρ

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We can rewrite the first n-components as
the combination of two portfolios (funds)

$$w = w_0 + \rho v$$

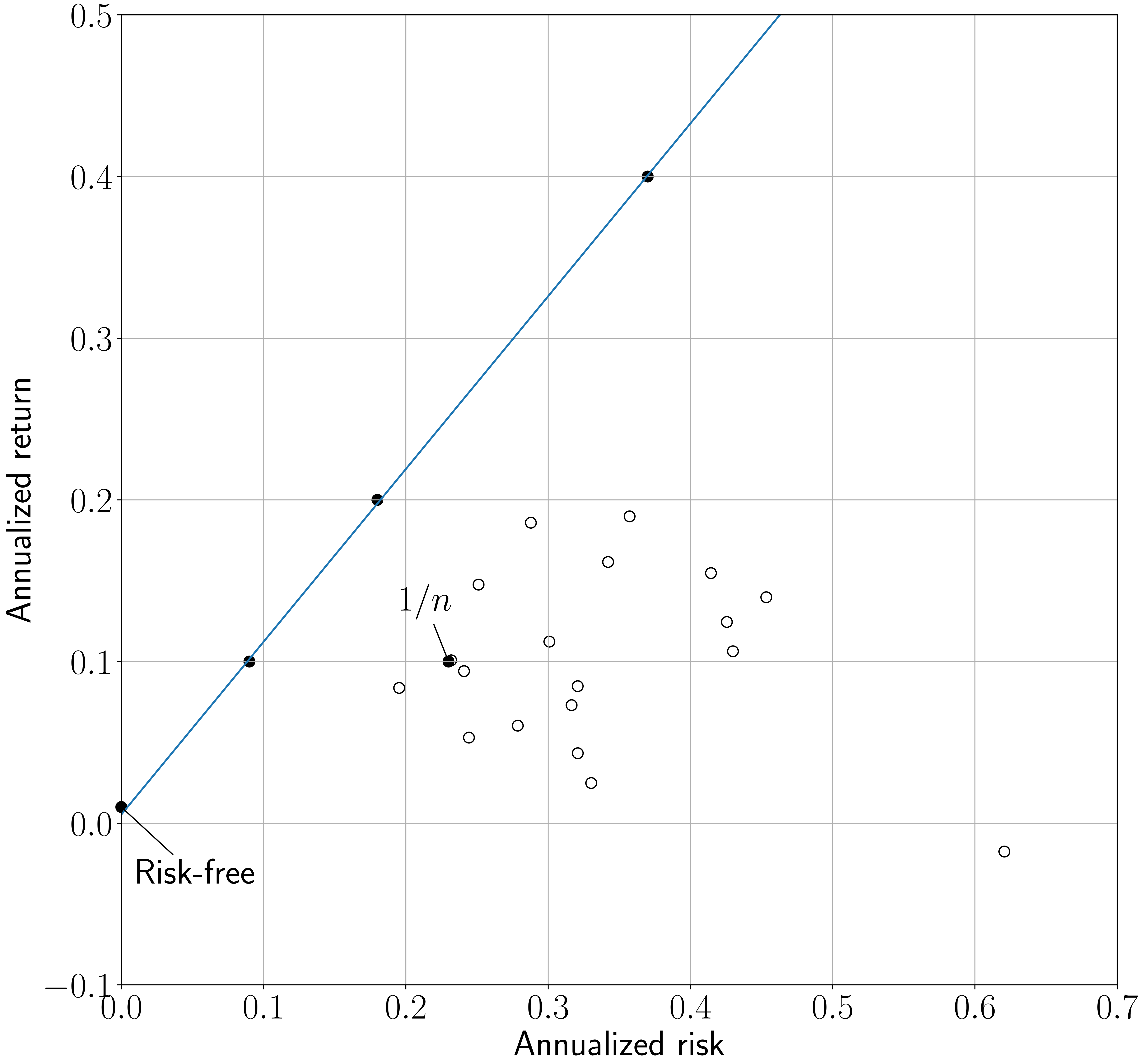
Risk-free
($\rho = 0$)

Other optimal portfolio

Example

20 assets over 2000 days (past)

- Optimal portfolios on a **straight line**
- Line starts at risk-free portfolio ($\rho = 0$)
- $1/n$ much better than single portfolios



The big assumption

Future returns will look like past ones

- You are warned this is false, every time you invest
- It is often reasonable
- During crisis, market shifts, other big events not true

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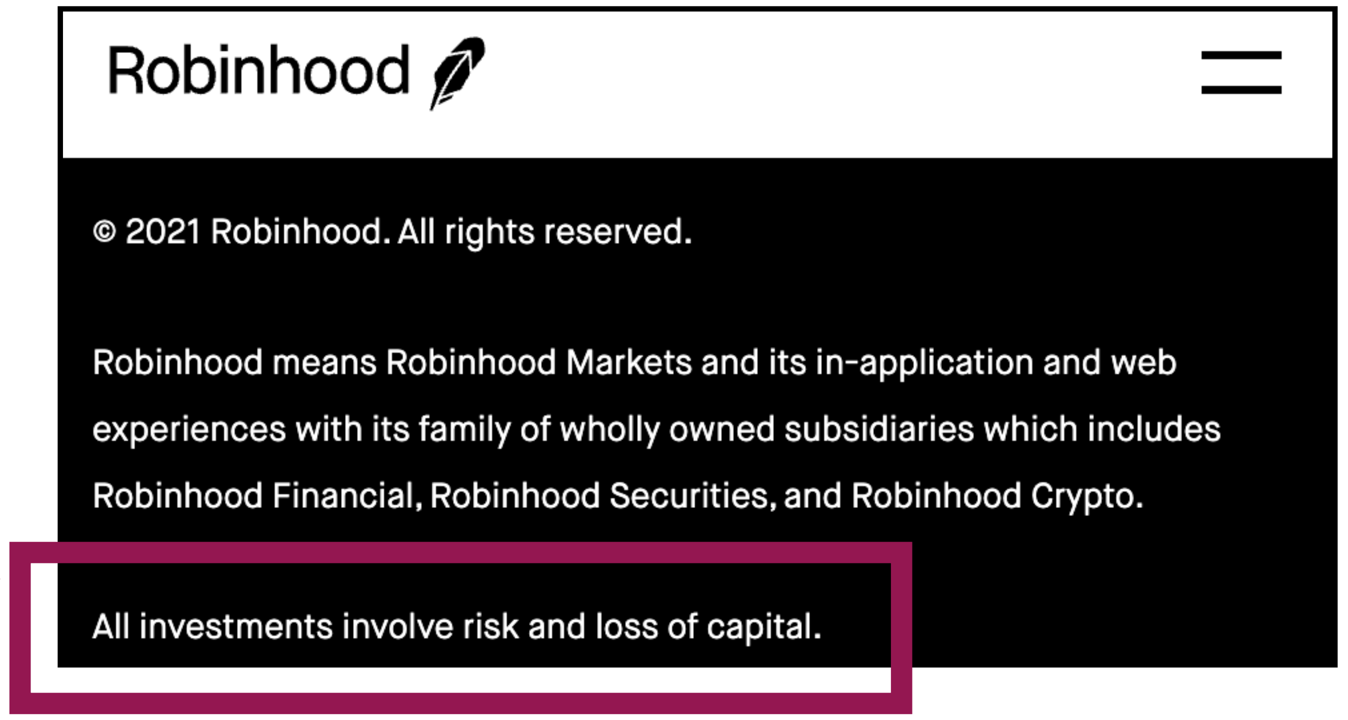

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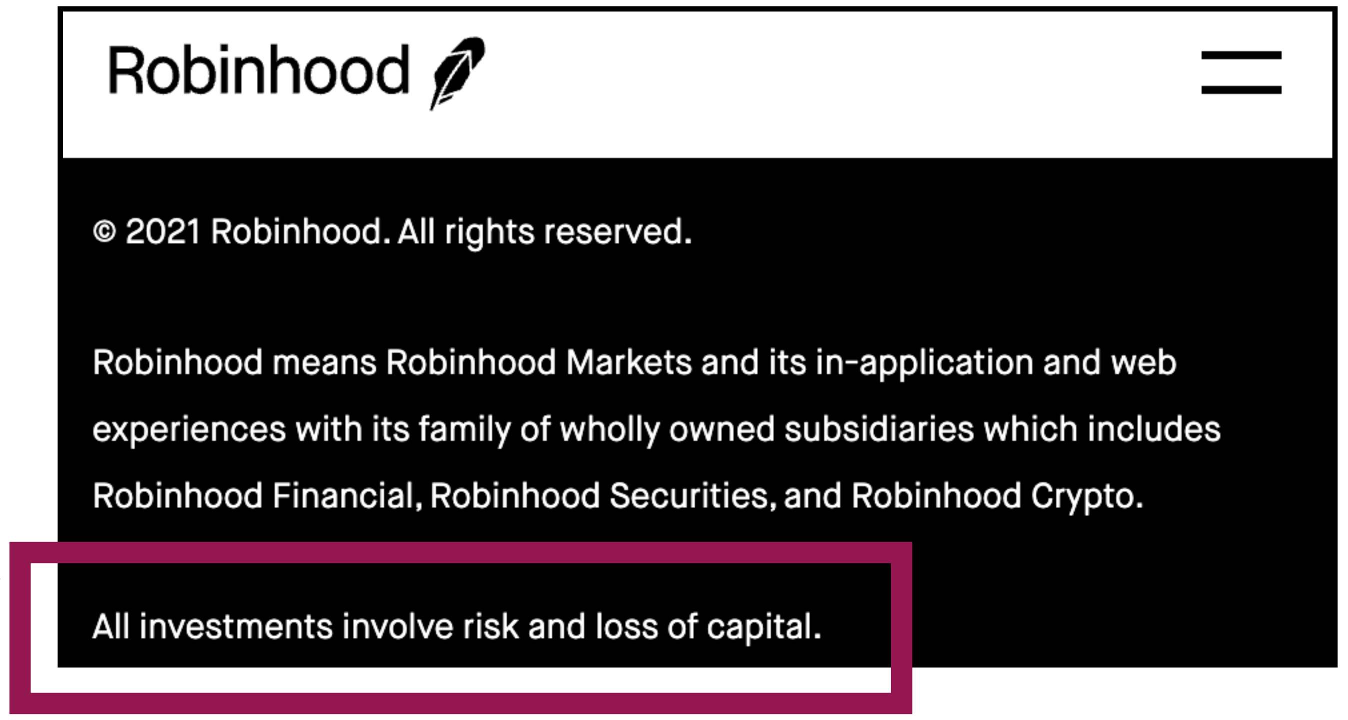

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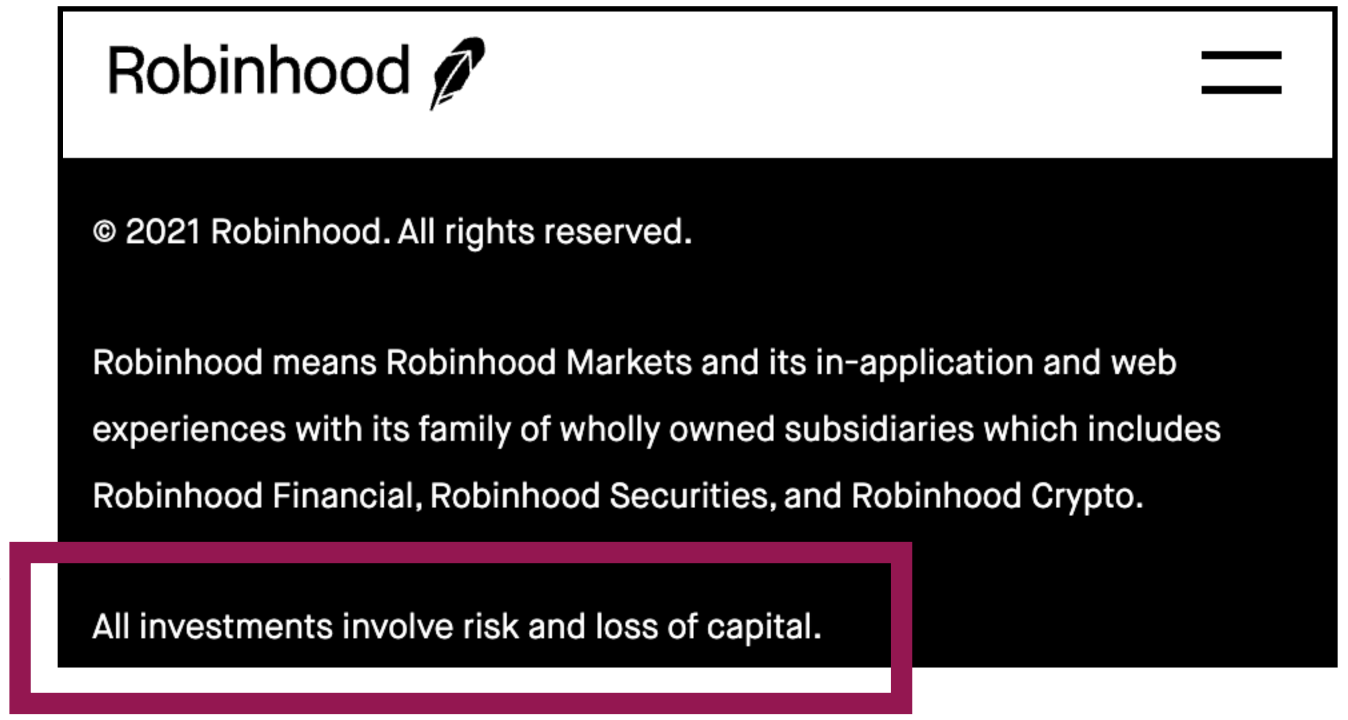

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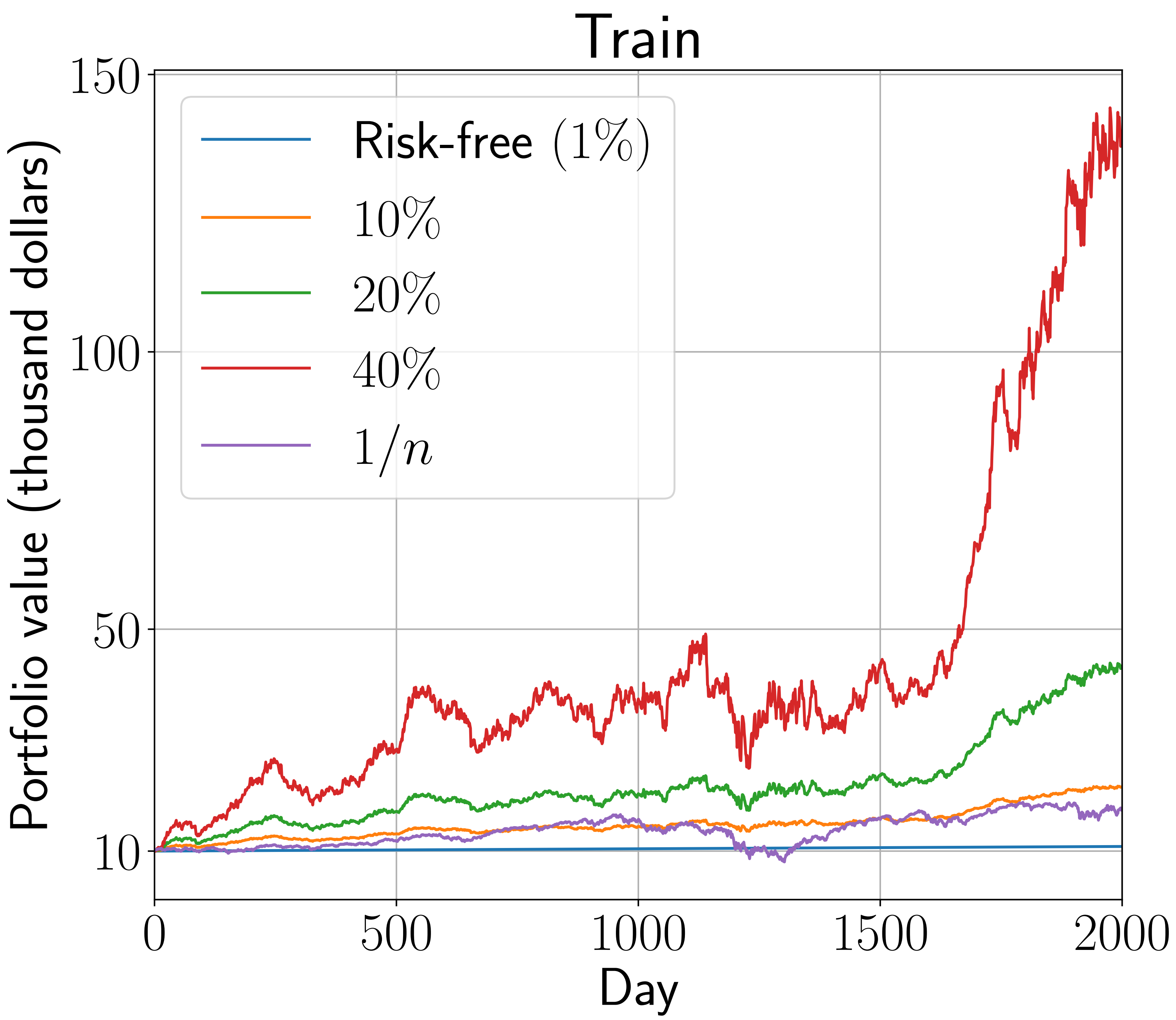
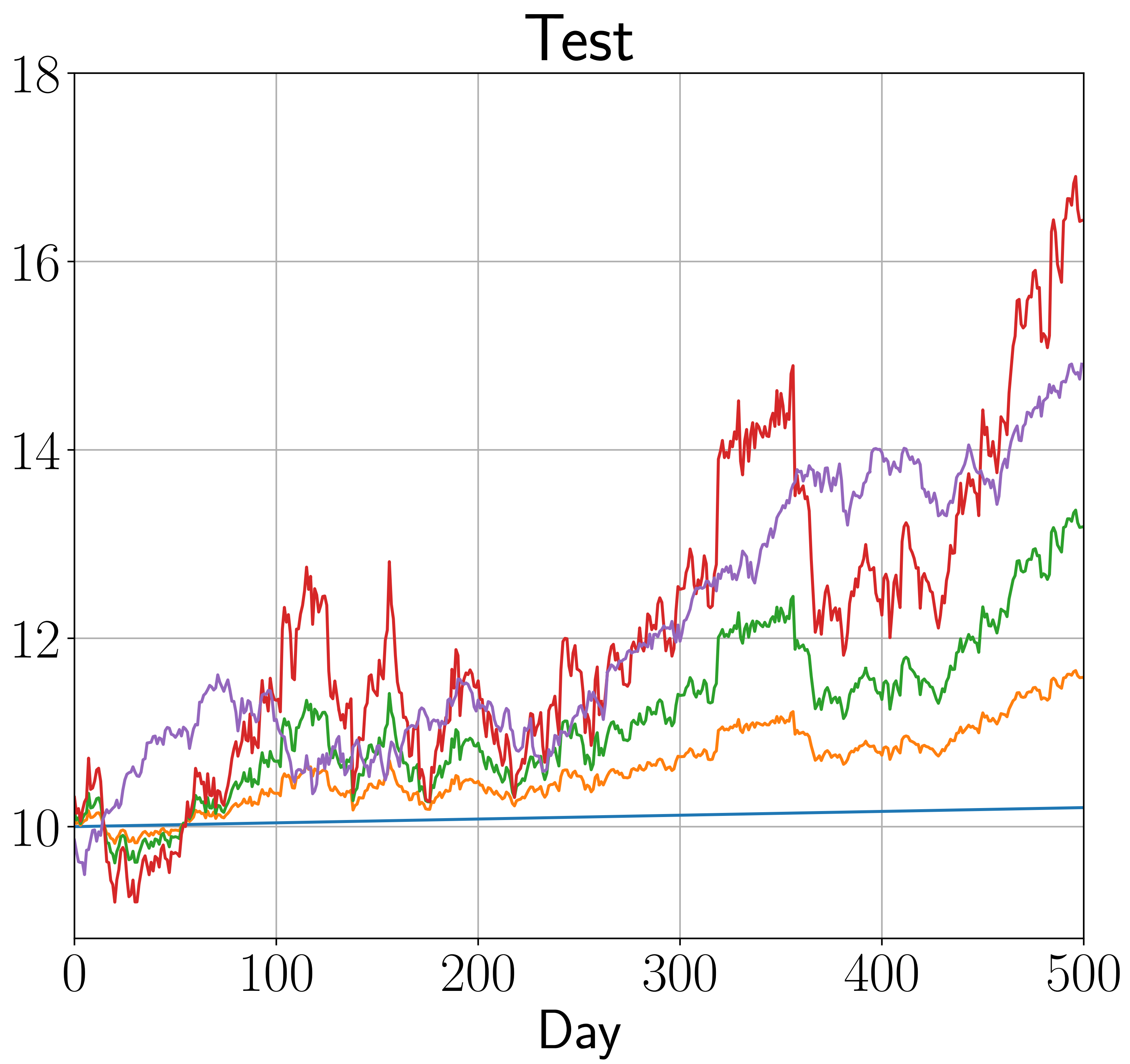
Example

- Pick w based on last 2 years of returns
- Use w during next 6 months

Total portfolio value

	Return		Risk		
	Train	Test	Train	Test	Leverage
Risk-free (1%)	0.01	0.01	0.00	0.00	1.00
10%	0.10	0.08	0.09	0.07	1.96
20%	0.20	0.15	0.18	0.15	3.03
40%	0.40	0.30	0.37	0.31	5.48
$1/n$	0.10	0.21	0.23	0.13	1.00





Build your quantitative hedge fund

Rolling portfolio optimization

For each period t, find weight w_t using L past returns

$$r_{t-1}, \dots, r_{t-L}$$

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Variations

- Update w every K periods (monthly, quarterly, ...)
- Add secondary objective $\lambda \|w_t - w_{t-1}\|^2$ to discourage turnover, reduce transaction cost
- Add logic to detect when the future is likely to not look like the past
- Add "signals" that predict future return of assets (Twitter sentiment analysis)

Constrained least squares

Today, we learned to:

- **Formulate** (linearly) and **solve** constrained least squares problems
- **Solve** portfolio allocations problems
- **Understand** the difference between past and future returns (be careful!)

References

- S. Boyd, L. Vandenberghe: Introduction to Applied Linear Algebra – Vectors, Matrices, and Least Squares
  - Chapter 16 and 17: constrained least squares

Next lecture

- Linear optimization