

ORF307 – Optimization

6. Constrained least squares

Ed Forum

- What is the difference between control and estimation?
- What is the definition of pareto frontier? Why can't you do better?

Recap

Multi-objective least squares

Goal choose n -vector x such that
 k norm squared objectives are small

$$J_1 = \|A_1 x - b_1\|^2$$

$$\vdots$$

$$J_k = \|A_k x - b_k\|^2$$

A_i are $m_i \times n$ matrices and b_i are m_i -vectors for $i = 1, \dots, k$

J_i are the objectives in a *multi-objective (-criterion) optimization problem*

Could choose x to minimize
any one J_i , but we want
tie make them all small

Weighted sum objective

Choose positive weights $\lambda_1, \dots, \lambda_k$ and form *weighted sum objective*

$$\begin{aligned} J &= \lambda_1 J_1 + \dots + \lambda_k J_k \\ &= \lambda_1 \|A_1 x - b_1\|^2 + \dots + \lambda_k \|A_k x - b_k\|^2 \end{aligned}$$

Choose x to minimize J

Primary objective

- Often $\lambda_1 = 1$ and J_1 is the **primary objective**
- **Interpretation** λ_i is how much we care about J_i being small, relative to J_1

Bi-criterion optimization

$$J_1 + \lambda J_2 = \|A_1 x - b_1\|^2 + \lambda \|A_2 x - b_2\|^2$$

Optimal trade-off curve

Bi-criterion problem

$$\text{minimize } J_1(x) + \lambda J_2(x) \longrightarrow x^*(\lambda)$$

Pareto optimal $x^*(\lambda)$

There is no point z that satisfies

$$J_1(z) \leq J_1(x^*(\lambda)) \quad \text{and} \quad J_2(z) \leq J_2(x^*(\lambda))$$

with one of the inequalities holding strictly
(no other point beats x^* on both objectives)

Optimal trade-off curve

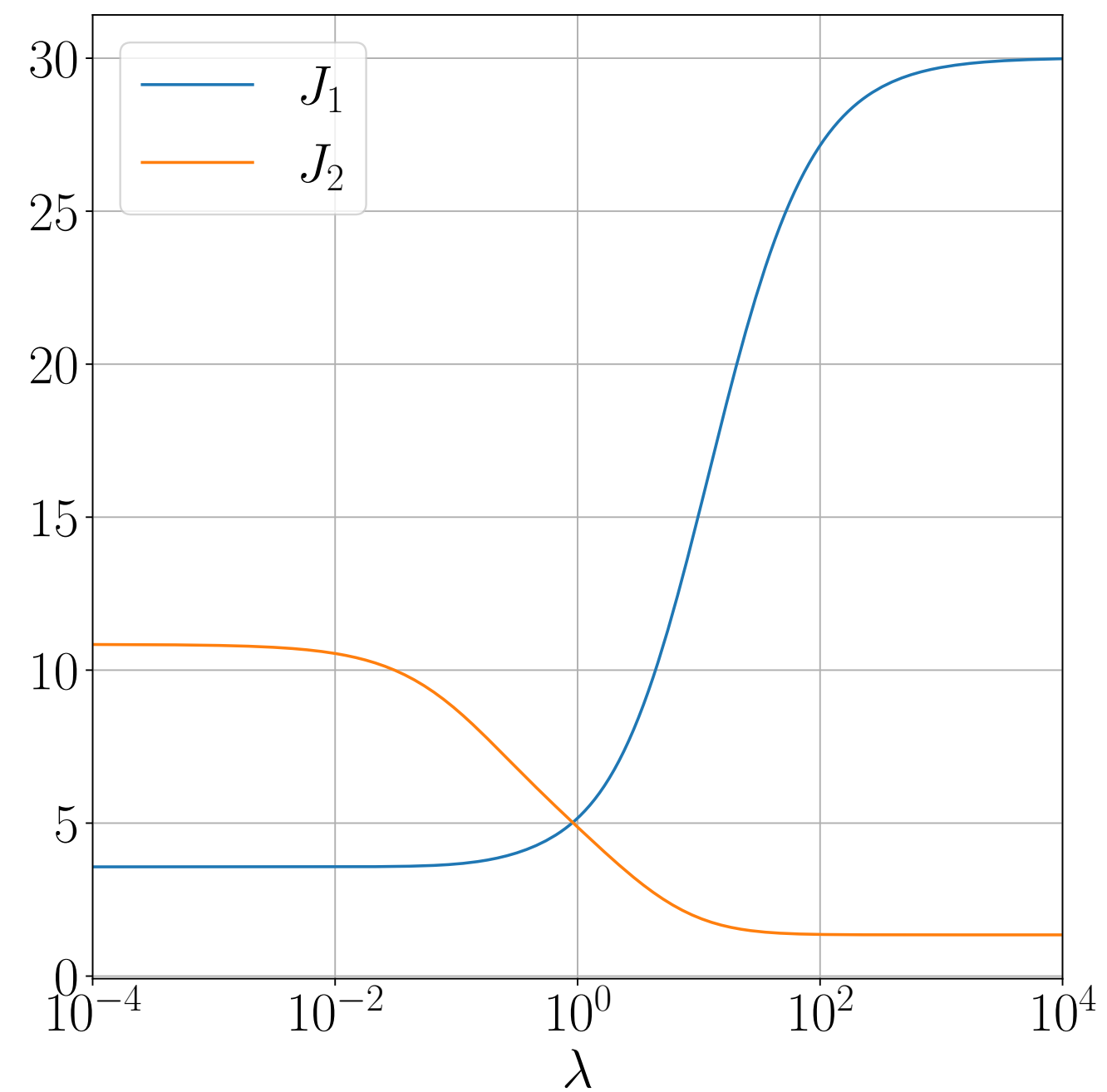
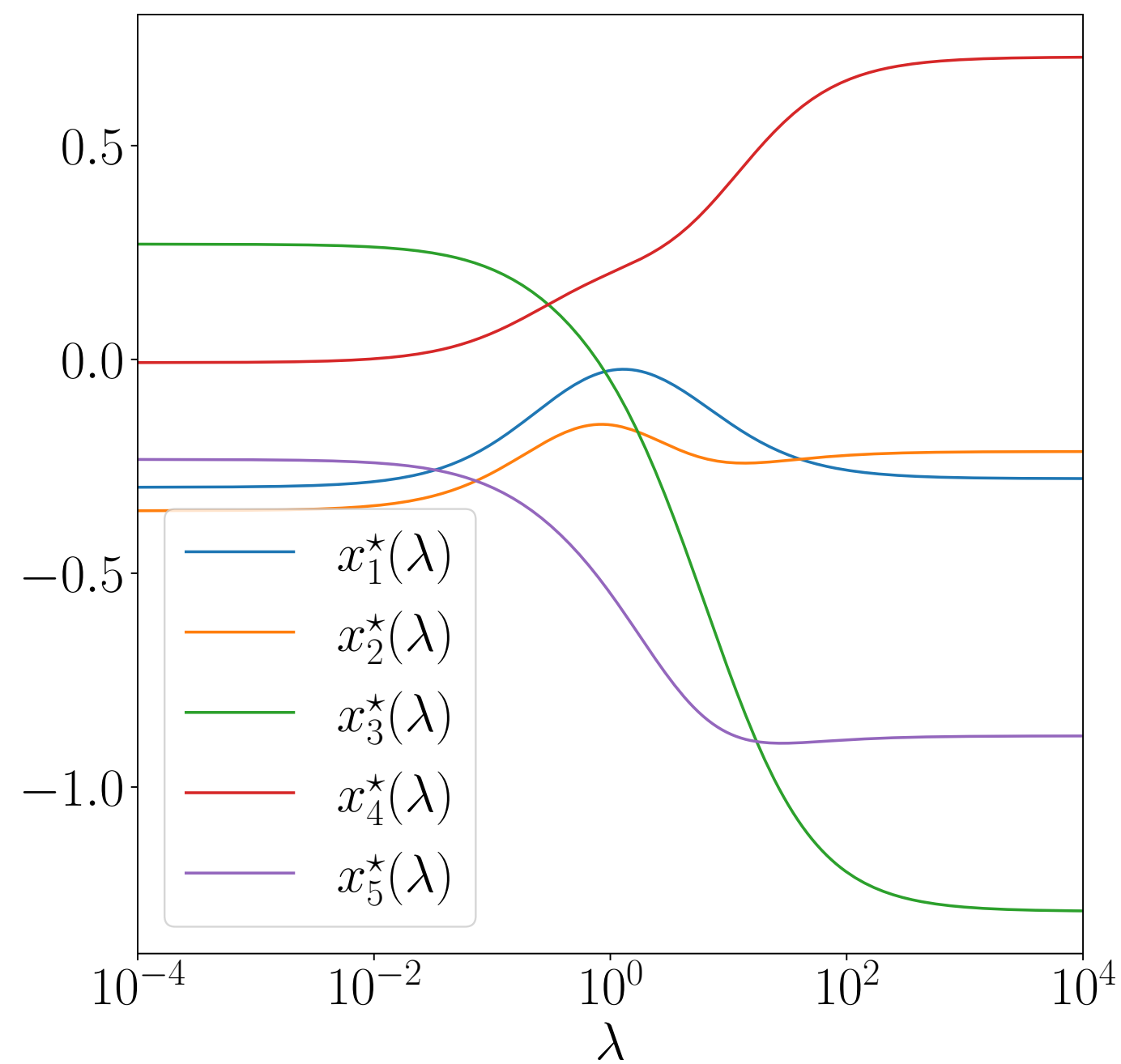
$$(J_1(x^*(\lambda)), J_2(x^*(\lambda))), \quad \lambda > 0$$

Optimal trade-off curve

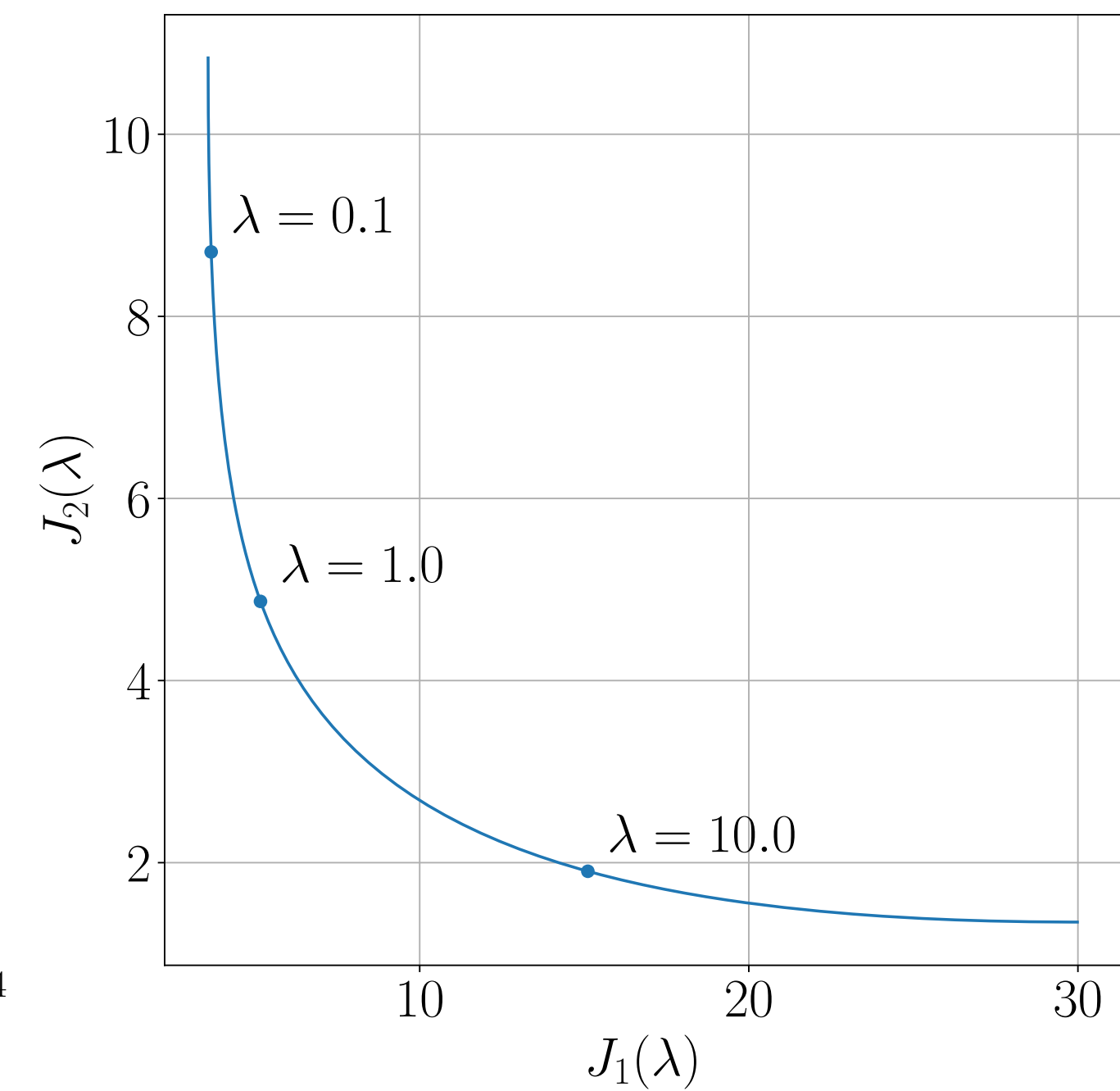
Example

$$\text{minimize } J_1(x) + \lambda J_2(x)$$

(A_1, A_2 are both 10×5)



Trade-off curve



Today's lecture

Constrained least squares

- Linearly constrained least squares
- Solving the constrained least squares problem
- Portfolio optimization

Linearly constrained least squares

Least squares with equality constraints

The (linearly) constrained least squares problem is

minimize $\|Ax - b\|^2$
subject to $Cx = d$

equality
constraints

objective
function

Problem data

- $m \times n$ matrix A , m -vector b
- $p \times n$ matrix C , p -vector d

Definitions

x is *feasible* if $Cx = d$

x^* is a *solution* if

- $Cx^* = d$
- $\|Ax^* - b\|^2 \leq \|Ax - b\|^2$
for any x satisfying $Cx = d$

Interpretations

- Combine solving linear equations with least squares.
- Like a bi-objective least squares with ∞ weight on second objective, $\|Cx - d\|^2$.

Optimal advertising with budget

m demographic groups
we want to advertise to



v^{des} is the m -vector
of desired views/impressions

n advertising channels
(web publishers, radio, print, etc.)



s is the n -vector
of purchases

$m \times n$ matrix A gives
demographic reach of channels



A_{ij} is the number of views
for group i and dollar spent
on channel j (1000/\$)

**Views across
demographic groups**

$$v = As$$

Goal

minimize $\|As - v^{\text{des}}\|^2$

subject to $\mathbf{1}^T s = B$



allocated
budget

Optimal advertising with budget

Results

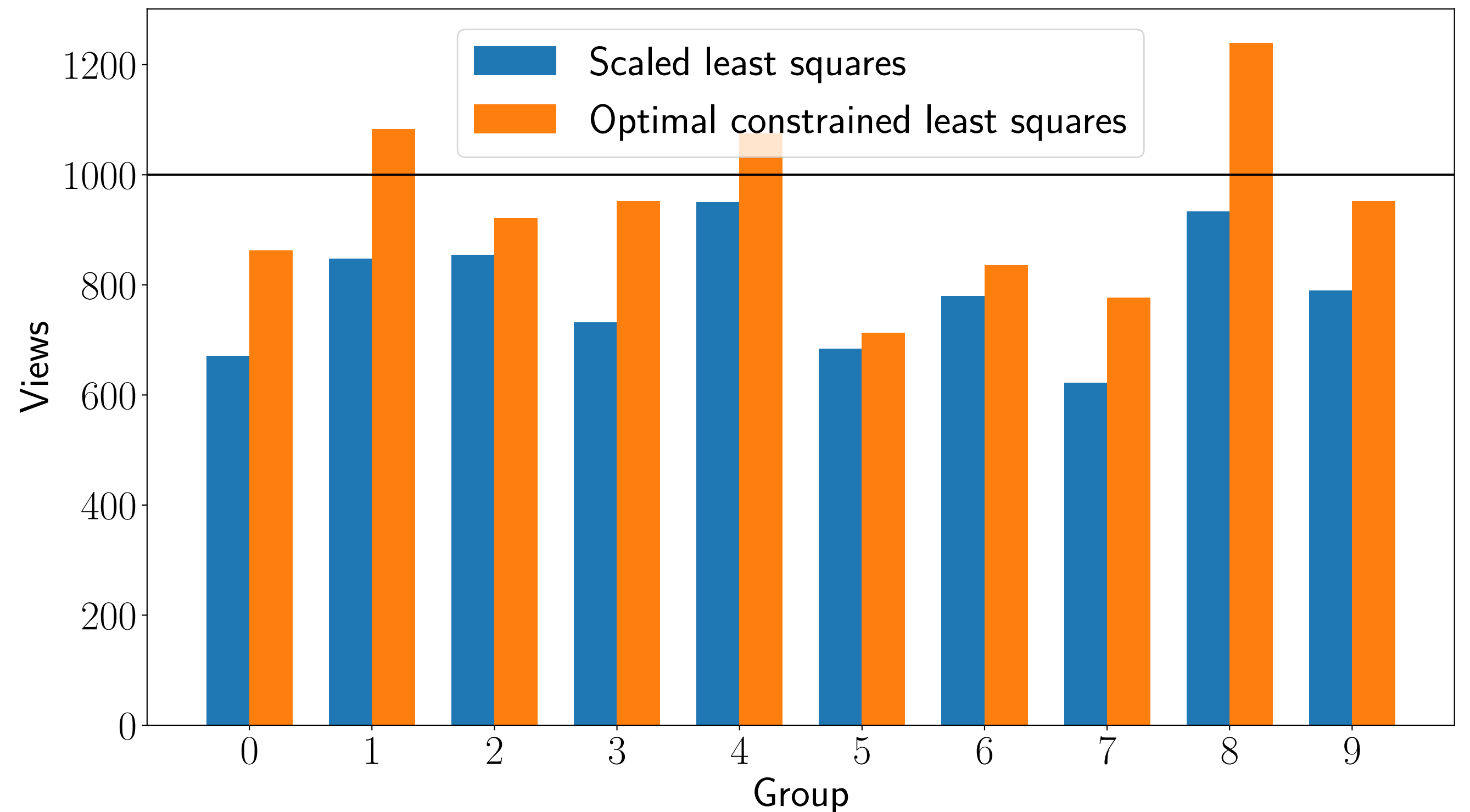
$m = 10$ groups, $n = 3$ channels

budget $B = 1284$

desired views vector $v^{\text{des}} = (10^3)\mathbf{1}$

minimize $\|As - v^{\text{des}}\|^2$

subject to $\mathbf{1}^T s = B$



optimal spending $s^* = (315, 110, 859)$

→ RMS 16.10%

rescaled least squares spending $s^* = (50, 80, 1154)$

→ RMS 23.85%

Least norm problem

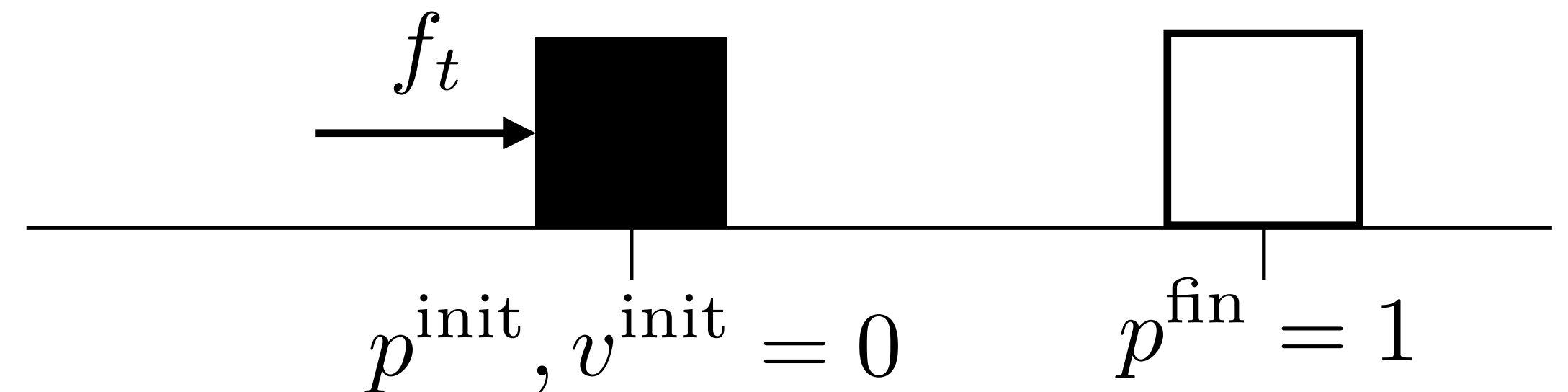
Special case of constrained least squares problem with $A = I$ and $b = 0$

$$\begin{array}{l} \text{minimize} \quad \|Ax - b\|^2 \\ \text{subject to} \quad Cx = d \end{array} \longrightarrow \begin{array}{l} \text{minimize} \quad \|x\|^2 \\ \text{subject to} \quad Cx = d \end{array}$$

Find the smallest vector that satisfies a set of linear equations

Force sequence

Unit mass on frictionless surface, initially at rest



10-vector f gives the forces applied for one second each

Final velocity and position (Newton's laws)

$$v^{\text{fin}} = f_1 + f_2 + \cdots + f_{10}$$

$$p^{\text{fin}} = (19/2)f_1 + (17/2)f_2 + \cdots + (1/2)f_{10}$$

Goal

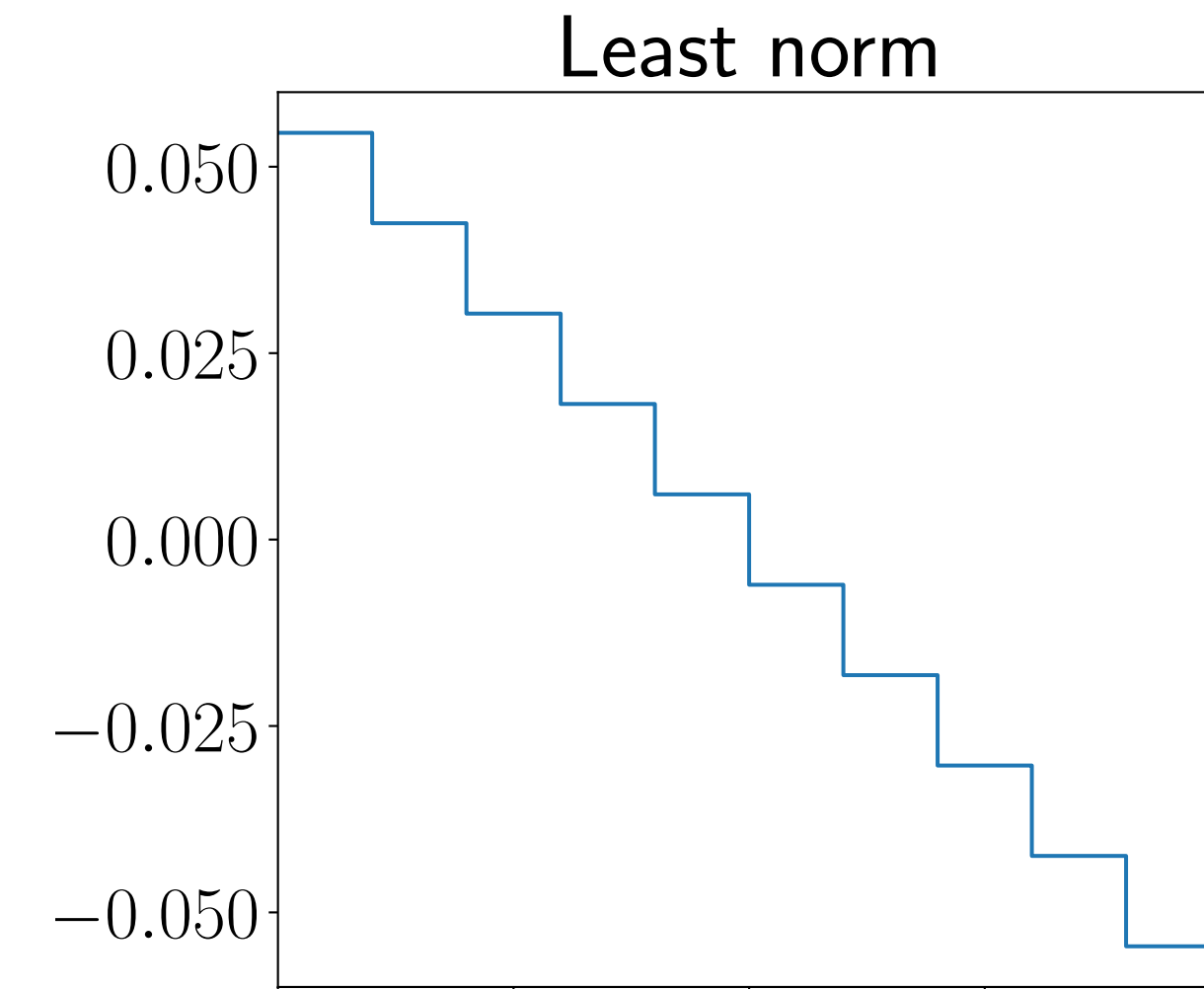
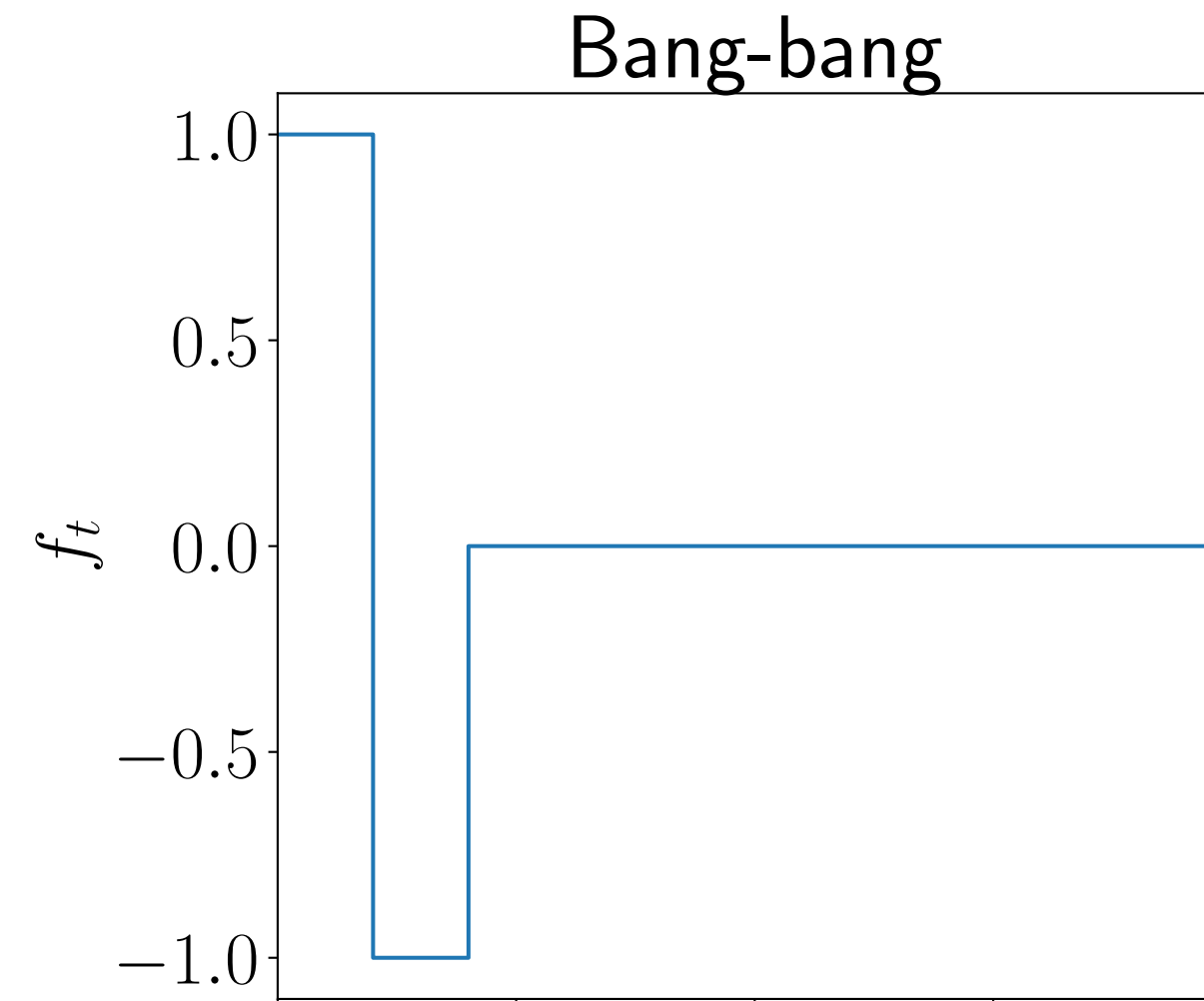
Let's find f such that $v^{\text{fin}} = 0$ and $p^{\text{fin}} = 1$

Least norm force sequence

Find f that brings to $p^{\text{fin}} = 1, v^{\text{fin}} = 0$

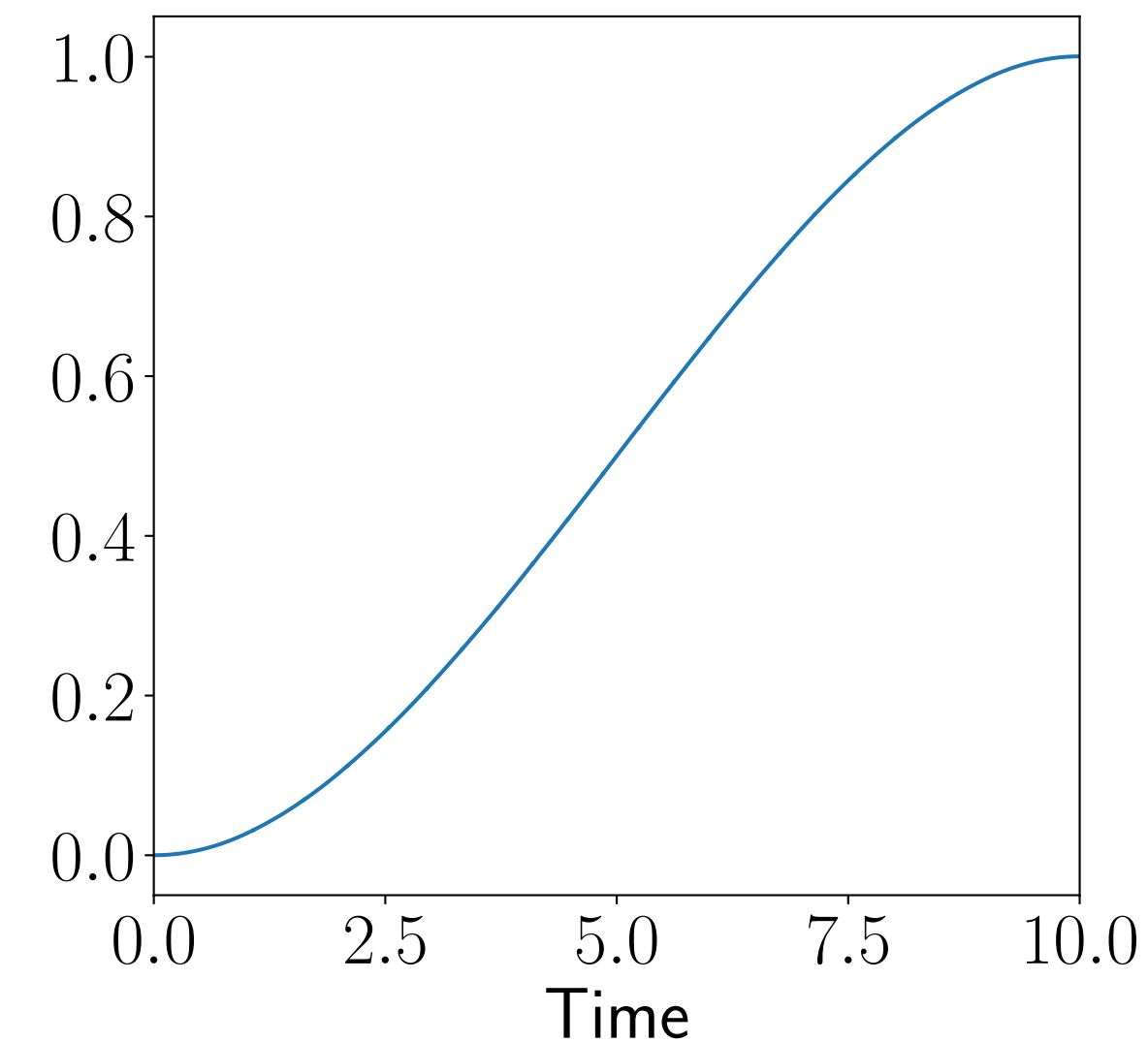
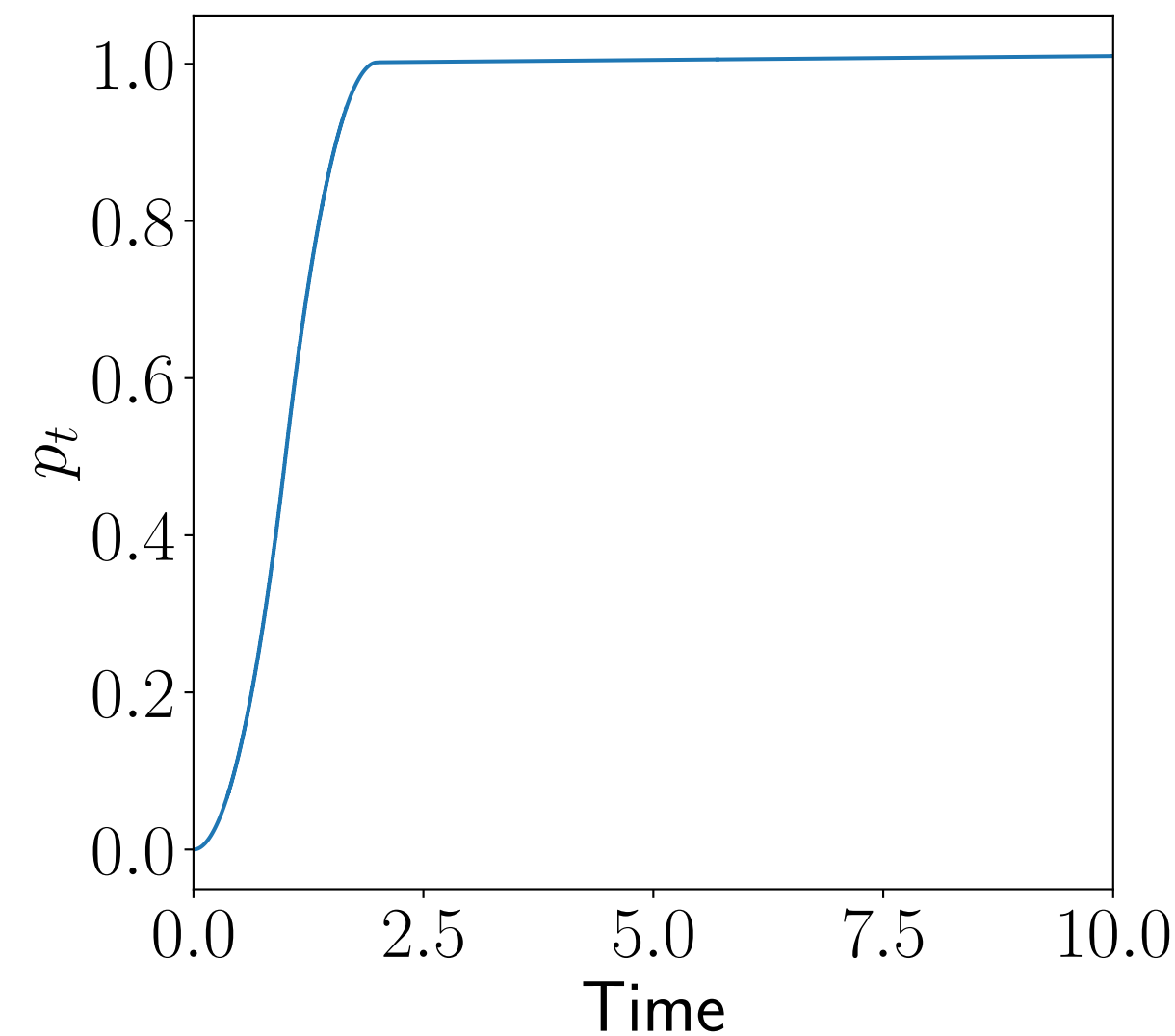
Bang-bang solution

$$f^{\text{bb}} = (1, -1, 0, \dots, 0) \quad \|f^{\text{bb}}\|^2 = 2$$



Least norm solution

minimize $\|f\|^2$
 subject to $\begin{bmatrix} 1 & 1 & \dots & 1 \\ 19/2 & 17/2 & \dots & 1/2 \end{bmatrix} f = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



$$\|f^{\text{ln}}\|^2 = 0.012$$

Much cheaper effort!

Solving the constrained least squares problem

Optimality conditions via calculus

$$\begin{array}{ll} \text{minimize} & f(x) = \|Ax - b\|^2 \\ \text{subject to} & Cx = d \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & f(x) = \|Ax - b\|^2 \\ \text{subject to} & c_i^T x = d_i, \quad i = 1, \dots, p \end{array}$$

Lagrangian function

$$L(x, z) = f(x) + z_1(c_1^T x - d_1) + \dots + z_p(c_p^T x - d_p)$$

Optimality conditions

$$\frac{\partial L}{\partial x_i}(x^*, z) = 0, \quad i = 1, \dots, n,$$

$$\frac{\partial L}{\partial z_i}(x^*, z) = 0, \quad i = 1, \dots, p$$

Optimality conditions via calculus

$$L(x, z) = x^T A^T A x - 2(A^T b)^T x + b^T b + z_1 (c_1^T x - d_1) + \cdots + z_p (c_p^T x - d_p)$$

Optimality conditions

Vector form

$$\begin{aligned} \frac{\partial L}{\partial z_i}(x^*, z) &= c_i^T x - d_i = 0 && \text{(we already knew)} && Cx = d \\ \frac{\partial L}{\partial x_i}(x^*, z) &= 2 \sum_{j=1}^n (A^T A)_{ij} x_j^* - 2(A^T b)_i + \sum_{j=1}^p z_j (c_j)_i = 0 && \longrightarrow && 2A^T A x^* - 2A^T b + C^T z = 0 \end{aligned}$$

Karush-Kuhn-Tucker (KKT) conditions

$$\begin{bmatrix} 2A^T A & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} x^* \\ z \end{bmatrix} = \begin{bmatrix} 2A^T b \\ d \end{bmatrix} \quad \text{(square set of } n + p \text{ linear equations)}$$

Note KKT equations are extension of normal equations to constrained least squares

Invertibility of KKT matrix

no longer positive definite in general \rightarrow
$$\begin{bmatrix} 2A^T A & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} x^* \\ z \end{bmatrix} = \begin{bmatrix} 2A^T b \\ d \end{bmatrix}$$

The KKT matrix is invertible if and only if

- C has linearly independent rows $\longrightarrow p \leq n$ (C is wide)
- $\begin{bmatrix} A \\ C \end{bmatrix}$ has linearly independent columns $\longrightarrow m + p \geq n$ ($\begin{bmatrix} A \\ C \end{bmatrix}$ is tall)
(true when A has linearly indep. cols)

Complexity (with $p \leq n \leq m$)

- Factor + solve: $2mn^2 + (2/3)(n + p)^3 + 2(n + p)^2 \approx 2mn^2$
- Solve given a new b (prefactored): $2mn + 2(n + p)^2 \approx 2mn$

**same as
unconstrained**

Optimality from KKT solution

For x^* and z^* such that

$$2A^T Ax^* + C^T z^* = 2A^T b, \quad Cx^* = d$$

Given a feasible x and z , we can write the objective (just as least squares)

$$\begin{aligned} \|Ax - b\|^2 &= \|(Ax - Ax^*) + (Ax^* - b)\|^2 \\ &= \|A(x - x^*)\|^2 + \|Ax^* - b\|^2 + 2(x - x^*)^T A^T (Ax^* - b) \end{aligned}$$

We can expand last term, using $2A^T (Ax^* - b) = -C^T z^*$ and $Cx = Cx^* = d$

$$2(x - x^*)^T A^T (Ax^* - b) = -(x - x^*)^T C^T z^* = -(C(x - x^*))^T z^* = 0$$

$$\|Ax - b\|^2 = \|A(x - x^*)\|^2 + \|Ax^* - b\|^2 \geq \|Ax^* - b\|^2$$

x^* is **optimal**

Portfolio optimization

Portfolio allocation weights

We want to invest V dollars in n different *assets* (stocks, bonds, ...)
over periods $t = 1, \dots, T$

Portfolio allocation weights

n -vector w gives the fraction of our total portfolio held in each asset

Properties

- Vw_j dollar value hold in asset j
- $\mathbf{1}^T w = 1$ (normalized)
- $w_j < 0$ means short positions (you borrow)
(must be returned at time T)
- Example: $w = (-0.2, 0.0, 1.2)$

Short position
of $0.2V$ on asset 1

Don't hold any
of asset 2

Hold $1.2V$
in asset 3

Leverage, long-only portfolios, and cash

Leverage

$$L = |w_1| + \cdots + |w_n| = \|w\|_1$$

$L = 1$ when all weights are nonnegative (“long only portfolio”)

Uniform portfolio

$$w = \mathbf{1}/n$$

Risk free asset

We often assume asset n is “risk-free” (e.g., cash)

if $w = e_n$, it means the portfolio is all cash

Return over a period

Asset returns

\tilde{r}_t is the (fractional) return of each asset over period t

example: $\tilde{r}_t = (0.01, -0.023, 0.02)$
(often expressed as percentage)

Portfolio return

$$r_t = \tilde{r}_t^T w$$

It is the (fractional) return for the entire portfolio over period t

Total portfolio value after a period

$$V_{t+1} = V_t + V_t \tilde{r}_t^T w = V_t (1 + r_t)$$

Return matrix

Hold constant portfolio
with weights w over T periods

Columns interpretation

Column j is time series
of asset j returns

Rows interpretation

Row t is \tilde{r}_t is the asset
return vector over period t

R is the $T \times n$ matrix of **asset returns**

R_{tj} is the return of asset j in period t

$$R = \begin{array}{cccc} & \text{AAPL} & \text{GOOG} & \text{MMM} & \text{US \$} \\ \left[\begin{array}{l} 0.00219 \\ 0.00744 \\ 0.01488 \end{array} \right. & & & & \\ & & & & \left. \begin{array}{l} \text{Mar 1, 2016} \\ \text{Mar 2, 2016} \\ \text{Mar 3, 2016} \end{array} \right] \end{array}$$

Note. If n th asset risk-free,
the last column of R is $\mu^{\text{rf}} \mathbf{1}$,
where μ^{rf} is the risk-free
per-period interest reate

Portfolio returns (time series)

$$r = Rw \quad (T\text{-vector})$$

Returns over multiple periods

r is time series T -vector of portfolio returns

average return
(or just return)

$$\mathbf{avg}(r) = \mathbf{1}^T r / T$$

risk
(standard deviation)

$$\mathbf{std}(r) = \|r - \mathbf{avg}(r)\mathbf{1}\| / \sqrt{T}$$

Total portfolio value

$$\begin{aligned} V_{T+1} &= V_1(1 + r_1) \cdots (1 + r_T) \\ &\approx V_1 + V_1(r_1 + \cdots + r_T) \\ &= V_1 + T\mathbf{avg}(r)V_1 \end{aligned}$$

(for $|r_t|$ small, e.g., ≤ 0.01
ignore higher order terms)

For high portfolio value we need large $\mathbf{avg}(r)$

Annualized return and risk

Mean return and risk are often expressed in **annualized form** (per year)

Given P trading periods per year (i.e., 250 days)

$$\text{annualized return} = P \text{avg}(r), \quad \text{annualized risk} = \sqrt{P} \text{std}(r)$$

Portfolio optimization

How shall we choose the portfolio weight vector w ?

Goals

High (mean) return
 $\text{avg}(r)$

Low risk
 $\text{std}(r)$

Data

- We know **realized asset returns** but not future ones
- **Optimization.** We choose w that would have worked well in the past
- **True goal.** Hope it will work well in the future (just like data fitting)

Portfolio optimization

Minimize risk given a target return

Chose n -vector w to solve

minimize $\text{std}(Rw)^2 = (1/T) \|Rw - \rho \mathbf{1}\|^2$

subject to $\mathbf{1}^T w = 1$

$\text{avg}(Rw) = \rho$

(past) target mean return

(past) portfolio returns time series

Solutions w are **Pareto optimal**

Our question

what would have been the best constant allocation w ,
had we known future returns?

Example allocations

Annual return 1% (risk-free asset has 1% return)

$$w = (0.00, 0.00, 0.00, \dots, 0.00, 0.00, 1.00)$$

Annual return 13%

$$w = (0.02, -0.07, -0.05, \dots, -0.03, 0.06, 0.56)$$

Annual return 25%

$$w = (0.05, -0.143, -0.09, \dots, -0.07, 0.12, 0.12)$$

Asking for higher annual returns yields

- More invested in risky, but high return assets
- Larger short positions ("leveraging")

Portfolio optimization

As constrained least squares

$$\begin{array}{ll} \text{minimize} & \|Rw - \rho \mathbf{1}\|^2 \\ \text{subject to} & \begin{bmatrix} \mathbf{1}^T \\ \mu^T \end{bmatrix} w = \begin{bmatrix} 1 \\ \rho \end{bmatrix} \end{array}$$

μ is the n -vector of average returns per asset

$$\begin{aligned} \text{avg}(r) &= (1/T) \mathbf{1}^T (Rw) \\ &= (1/T) (R^T \mathbf{1})^T w = \mu^T w \end{aligned}$$

Solution via KKT linear system

$$\begin{bmatrix} 2R^T R & \mathbf{1} & \mu \\ \mathbf{1}^T & 0 & 0 \\ \mu^T & 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 2\rho T \mu \\ 1 \\ \rho \end{bmatrix}$$

Optimal portfolios

Rewrite right-hand side

$$\begin{bmatrix} 2\rho T \mu \\ 1 \\ \rho \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \rho \begin{bmatrix} 2T \mu \\ 0 \\ 0 \end{bmatrix}$$

Two fund theorem

Optimal portfolio w is an affine function of ρ

$$\begin{bmatrix} w \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 2R^T R & \mathbf{1} & \mu \\ \mathbf{1}^T & 0 & 0 \\ \mu^T & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \rho \begin{bmatrix} 2R^T R & \mathbf{1} & \mu \\ \mathbf{1}^T & 0 & 0 \\ \mu^T & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 2T \mu \\ 0 \\ 1 \end{bmatrix}$$

We can rewrite the first n -components as the combination of two portfolios (funds)

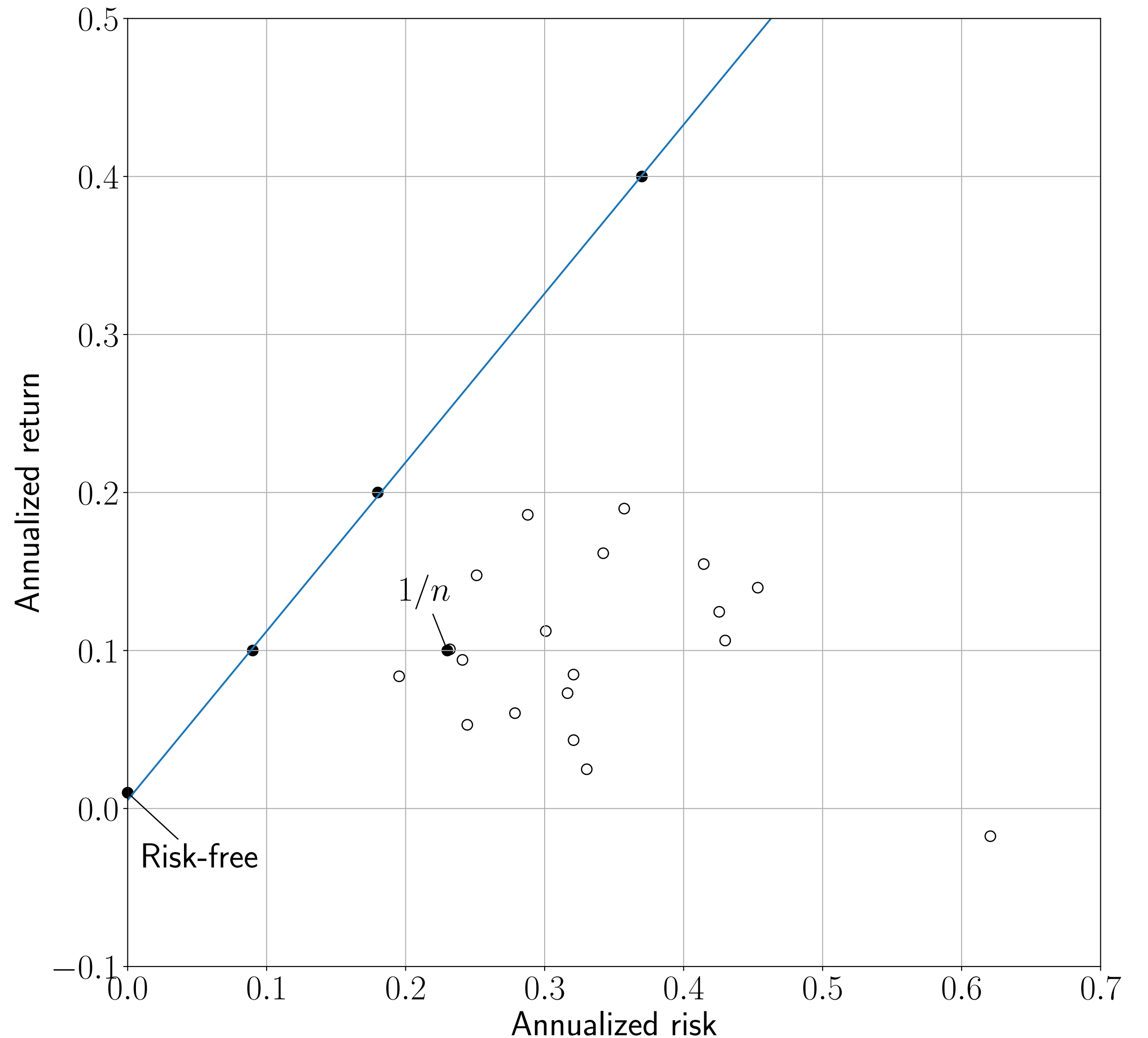
$$w = w_0 + \rho v$$

Risk-free $(\rho = 0)$ Other optimal portfolio

Example

20 assets over 2000 days (past)

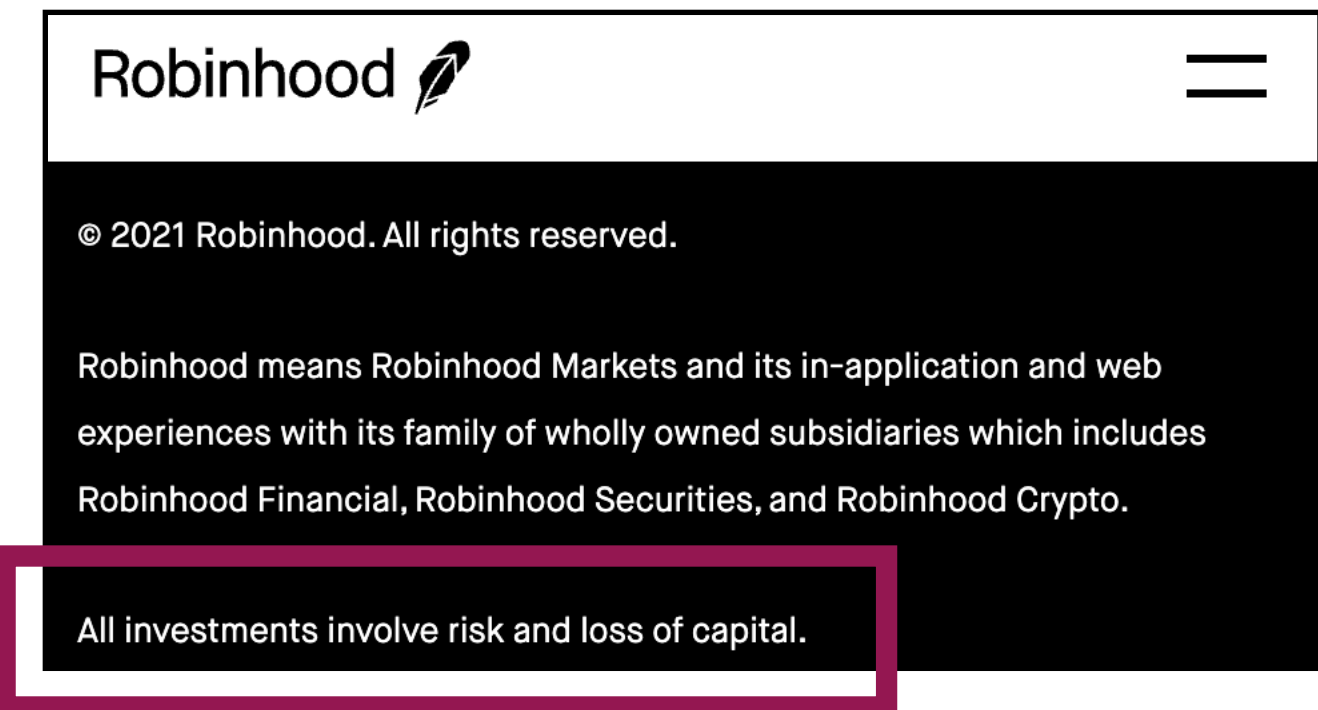
- Optimal portfolios on a **straight line**
- Line starts at risk-free portfolio ($\rho = 0$)
- $1/n$ much better than single portfolios



The big assumption

Future returns will look like past ones

- You are warned this is false, every time you invest
- It is often reasonable
- During crisis, market shifts, other big events not true



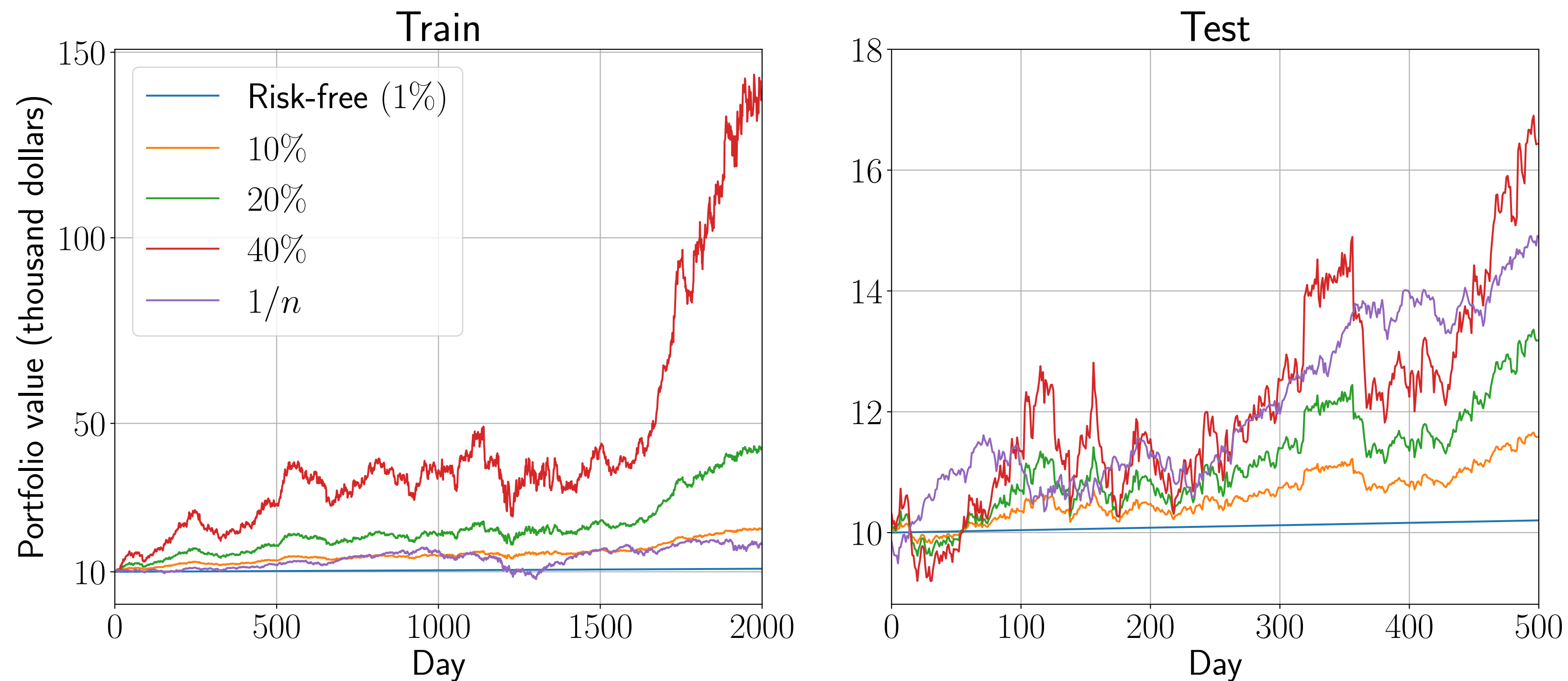
If assumption holds (even approximately), a good w on past returns leads to good future (unknown) returns

Example

- Pick w based on last 2 years of returns
- Use w during next 6 months

Total portfolio value

	Return		Risk		Leverage
	Train	Test	Train	Test	
Risk-free (1%)	0.01	0.01	0.00	0.00	1.00
10%	0.10	0.08	0.09	0.07	1.96
20%	0.20	0.15	0.18	0.15	3.03
40%	0.40	0.30	0.37	0.31	5.48
1/ <i>n</i>	0.10	0.21	0.23	0.13	1.00



Build your quantitative hedge fund

Rolling portfolio optimization

For each period t , find weight w_t using L past returns

$$r_{t-1}, \dots, r_{t-L}$$

Variations

- Update w every K periods (monthly, quarterly, ...)
- Add secondary objective $\lambda \|w_t - w_{t-1}\|^2$ to discourage turnover, reduce transaction cost
- Add logic to detect when the future is likely to not look like the past
- Add “signals” that predict future return of assets (Twitter sentiment analysis)

Constrained least squares

Today, we learned to:

- **Formulate** (linearly) and **solve** constrained least squares problems
- **Solve** portfolio allocations problems
- **Understand** the difference between past and future returns (be careful!)

References

- S. Boyd, L. Vandenberghe: Introduction to Applied Linear Algebra – Vectors, Matrices, and Least Squares
 - Chapter 16 and 17: constrained least squares

Next lecture

- Linear optimization