

# **ORF307 – Optimization**

## **20. Integer optimization**

# Announcements

- Last precepts next week
- Last homework out Thursday next week

# Today's lecture

## Mixed-integer optimization

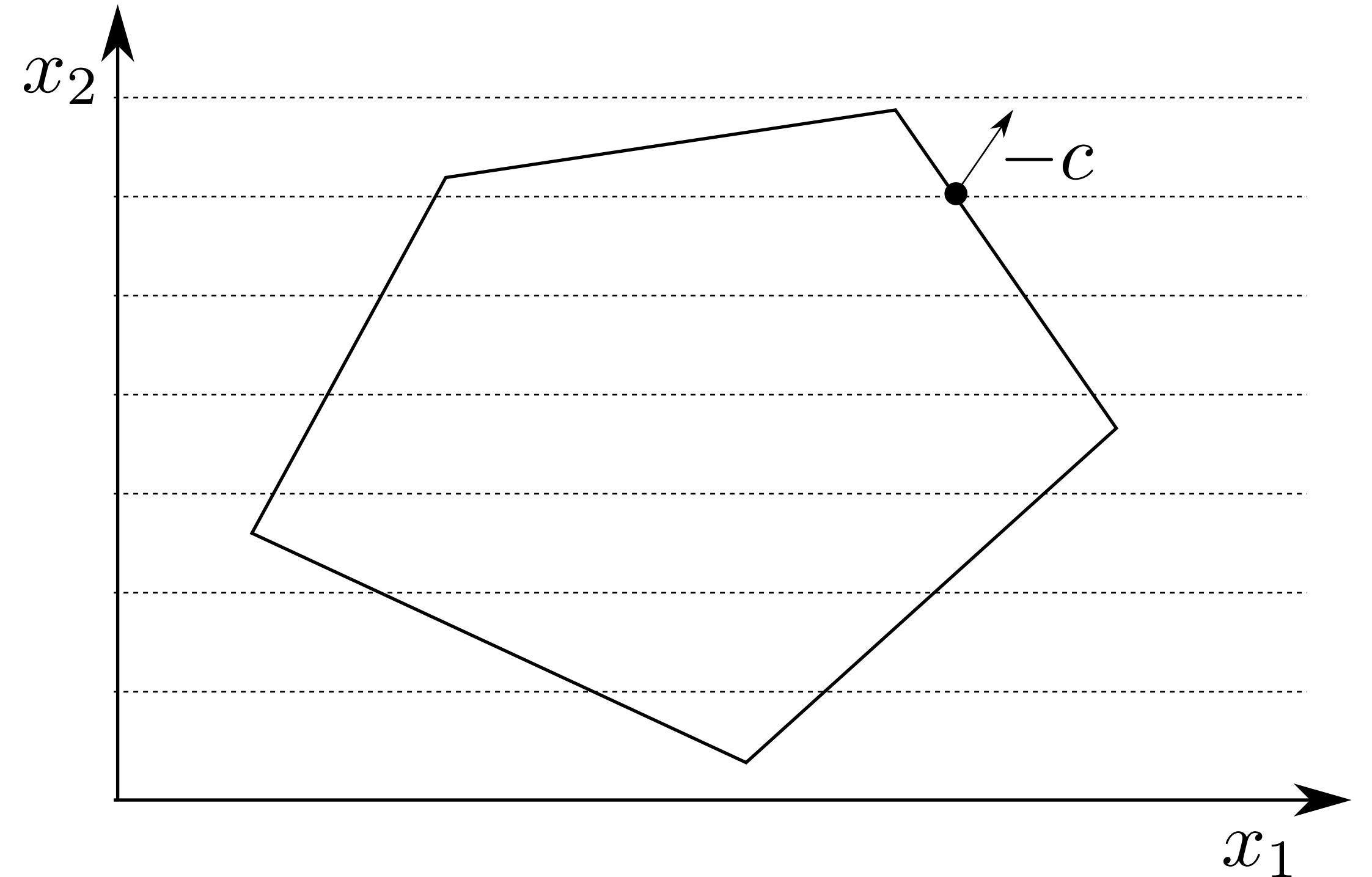
- Mixed-integer programs
- Modeling techniques
- Formulations
- Ideal formulations

# Mixed-integer optimization

# Mixed-integer program

Optimization problem where some variables are restricted to be integer

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \\ & x_i \in \mathbf{Z}, \quad i \in \mathcal{I} \end{array}$$



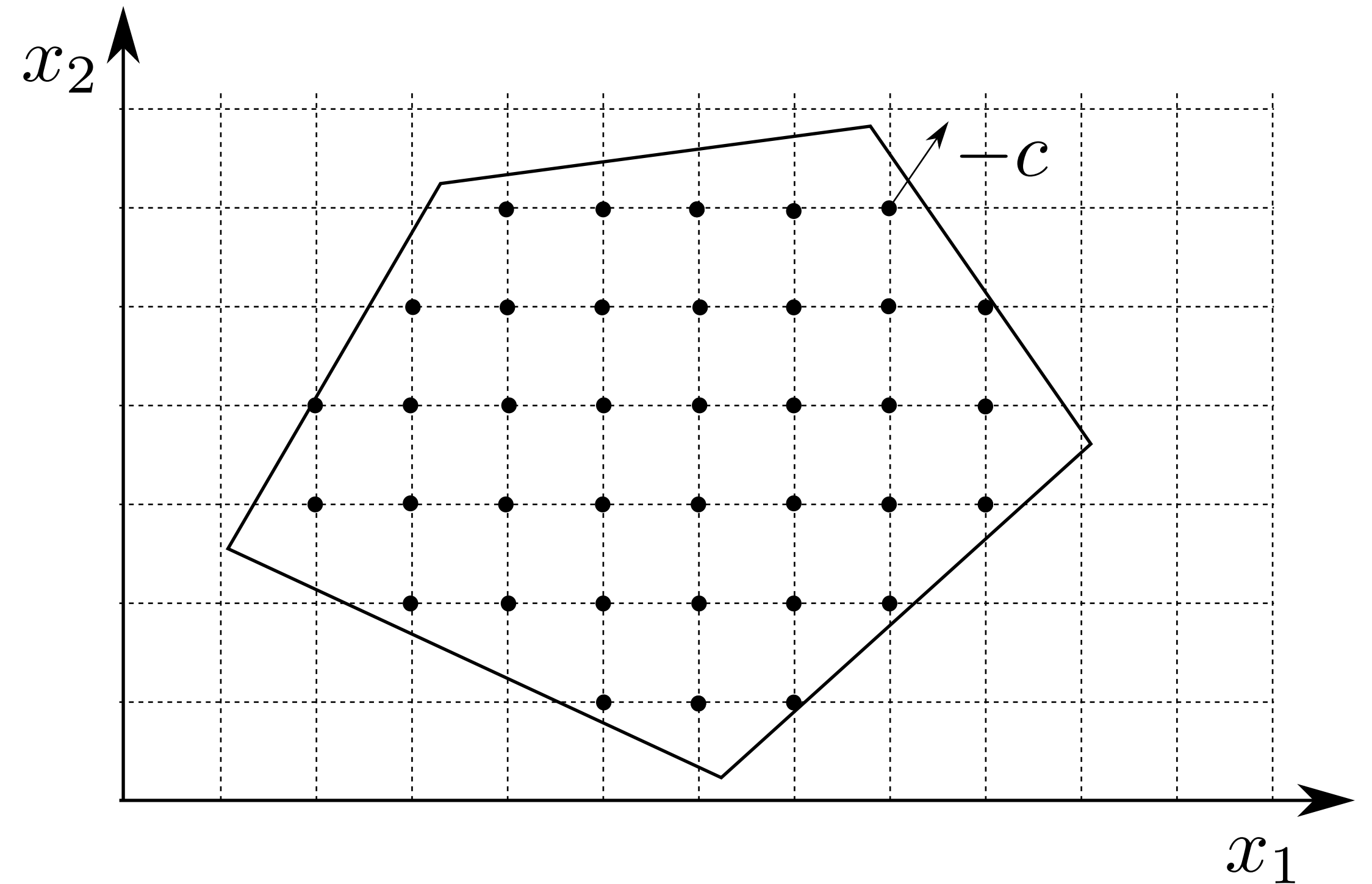
# Mixed-integer program

## Special cases

### Integer linear program

$$\mathcal{I} = \{1, \dots, n\}$$

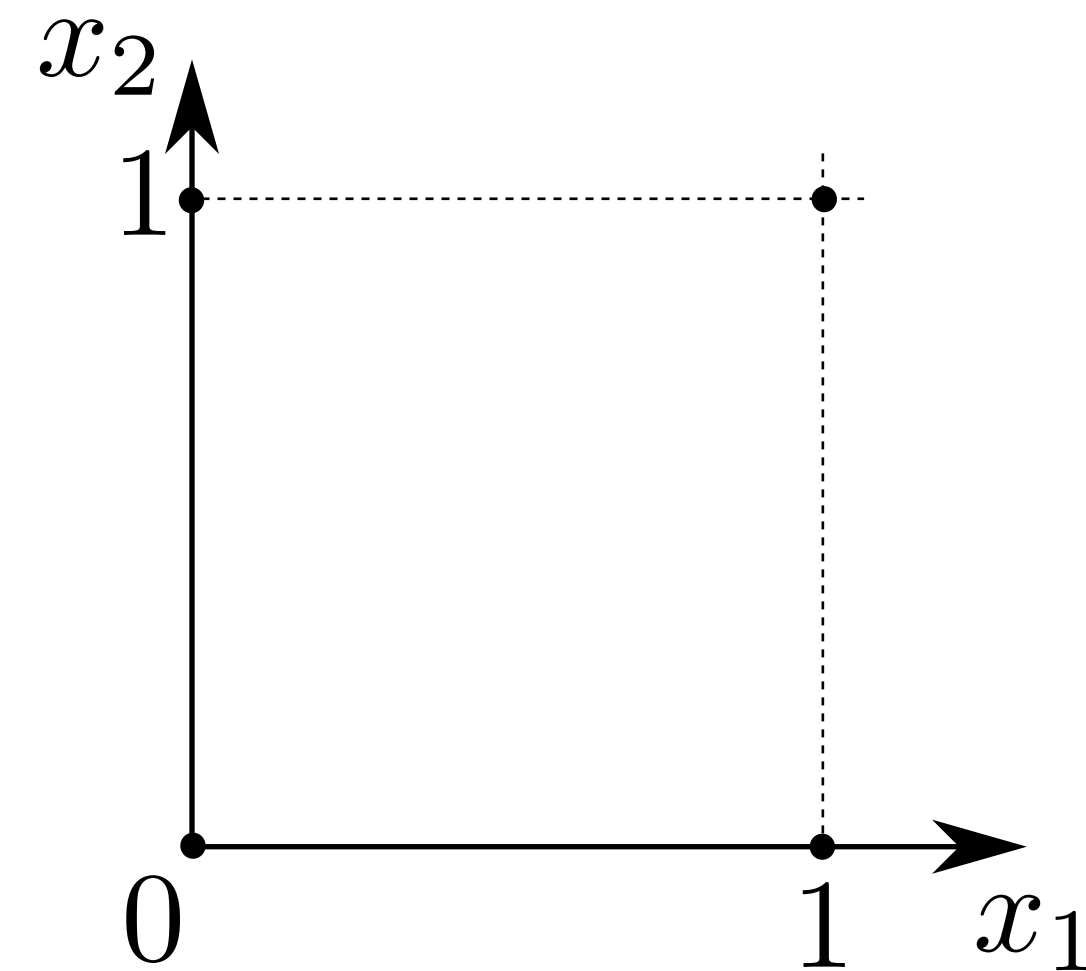
(all variables are integer)



### Boolean linear program

$$x_i \in \{0, 1\}, \quad i \in \mathcal{I}$$

(integer variables take values 0 or 1)



# Modeling techniques

# Binary choice

$$x_i = \begin{cases} 1 & \text{event occurs} \\ 0 & \text{otherwise} \end{cases} \longrightarrow x \in \{0, 1\}^n$$

## Examples

- Perform an financial transaction
- Select an arc in a graph
- Open a store



# Knapsack problem



**Goal** decide between  $n$  items to put into knapsack

- Maximum total weight:  $b$
- Weight of item  $i$ :  $a_i$
- Value of item  $i$ :  $c_i$

## Formulation

maximize  $c^T x$

subject to  $a^T x \leq b$

$x_i \in \{0, 1\}, \quad i = 1, \dots, n$

# Logical relations

$$x \in \{0, 1\}^n$$

**At most one event occurs**

$$\mathbf{1}^T x \leq 1$$

**Neither or both events occur**

$$x_1 = x_2$$

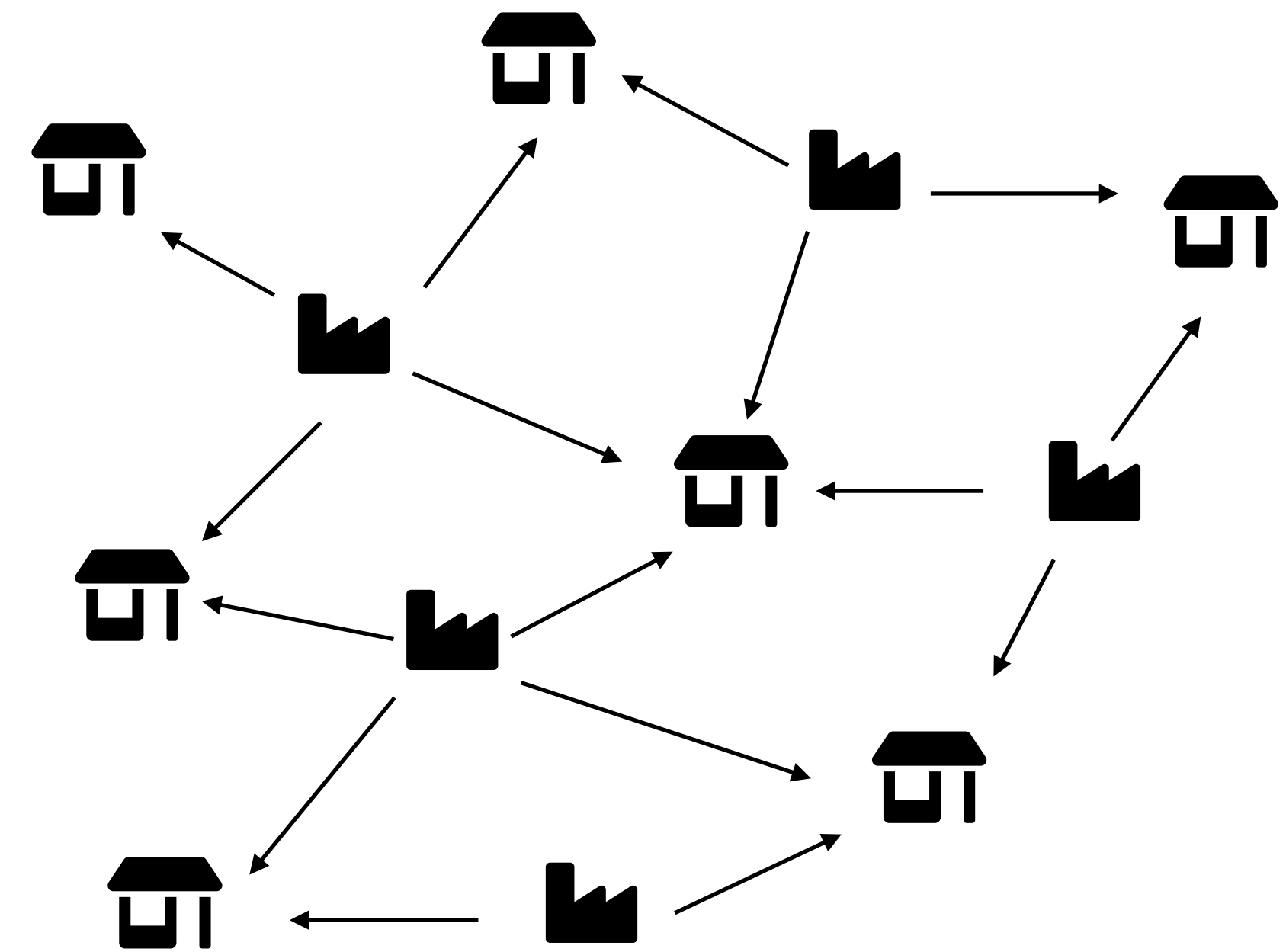
**If  $x_2 = 0$  (does not occur), then  $x_1 = 0$  (does not occur)**

$$x_1 \leq x_2$$

# Facility location problem

## Data

- $n$  potential facility locations,  $m$  clients
- $c_j$  cost of opening facility at location  $j$
- $d_{ij}$  cost of serving client  $i$  from location  $j$



## Variables

$$y_j = \begin{cases} 1 & \text{location } j \text{ is selected} \\ 0 & \text{otherwise} \end{cases}$$
$$x_{ij} = \begin{cases} 1 & \text{location } j \text{ serves client } i \\ 0 & \text{otherwise} \end{cases}$$

## Problem

minimize  $\sum_{j=1}^n c_j y_j + \sum_{i=1}^m \sum_{j=1}^n d_{ij} x_{ij}$

subject to  $\sum_{j=1}^n x_{ij} = 1, \quad i = 1, \dots, m$

$$x_{ij} \leq y_j, \quad i = 1, \dots, m, \quad j = 1, \dots, n$$
$$x_{ij}, y_j \in \{0, 1\}$$

# Mixed-logical relations (big-M formulations)

$$x \in \mathbf{R}, y \in \{0, 1\}$$

**If  $y = 0$ , then  $x = 0$ . Otherwise,  $x$  unconstrained.**

$$0 \leq x \leq yM$$

## Disjunctive constraints

either  $a^T x \leq b$  or  $d^T x \leq f$  is valid

$$a^T x \leq b + yM$$

$$d^T x \leq f + (1 - y)M$$

# Cardinality

$$x \in \mathbf{R}^n, y \in \{0, 1\}^n$$

## Cardinality (0-norm)

number of nonzero elements

$$\text{card } x = \|x\|_0 = \sum \{i \mid x_i \neq 0\}$$

## Cardinality constraint

$$\text{card } x \leq k$$



$$\sum_{i=1}^m y_i \leq k$$

$$-My_i \leq x_i \leq My_i, \quad i = 1, \dots, n$$

$$y_i \in \{0, 1\}$$

# Restricted range of values

We want to restrict variable  $x \in \mathbb{R}$  to take values  $\{a_1, \dots, a_d\}$

Introduce  $d$  binary variables  $z_i \in \{0, 1\}$

$$x = \sum_{j=1}^d a_j z_j$$

$$\sum_{j=1}^d z_j = 1$$

$$z_j \in \{0, 1\}$$



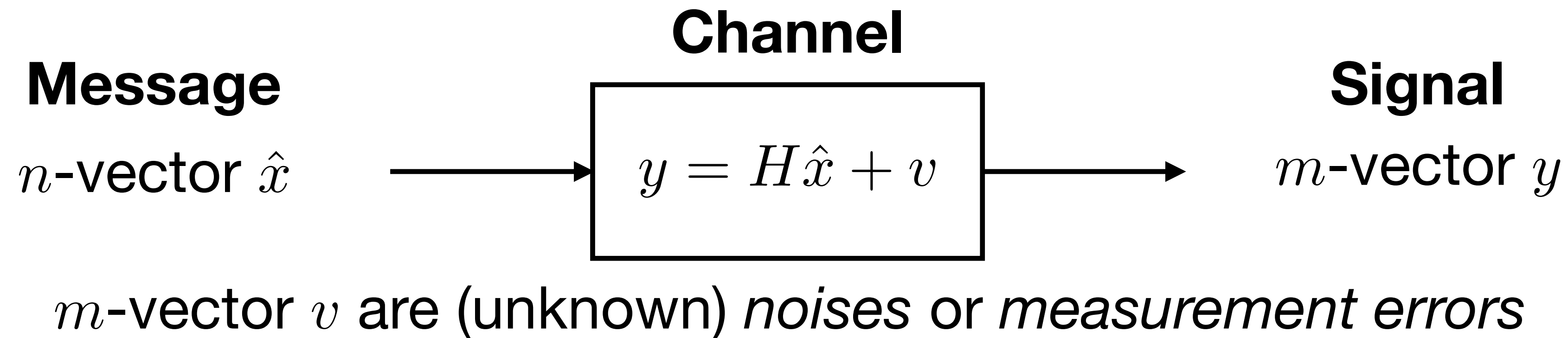
## Vector form

$$x = a^T z$$

$$\mathbf{1}^T z = 1$$

$$z \in \{0, 1\}^d$$

# Signal decoding



**Goal** recover message  $\hat{x}$

## Signal constellation

At every time  $k$ ,  $x_k$  can take only values  $\{a_1, \dots, a_d\}$

## Signal decoding problem

minimize  $\|Hx - y\|_1$   
subject to  $x_k \in \{a_1, \dots, a_d\}, \quad k = 1, \dots, n$

# Signal decoding as mixed-integer optimization

## Signal decoding problem

$$\begin{aligned} &\text{minimize} && \|Hx - y\|_1 \\ &\text{subject to} && x_k \in \{a_1, \dots, a_d\}, \quad k = 1, \dots, n \end{aligned}$$

## Mixed-integer optimization

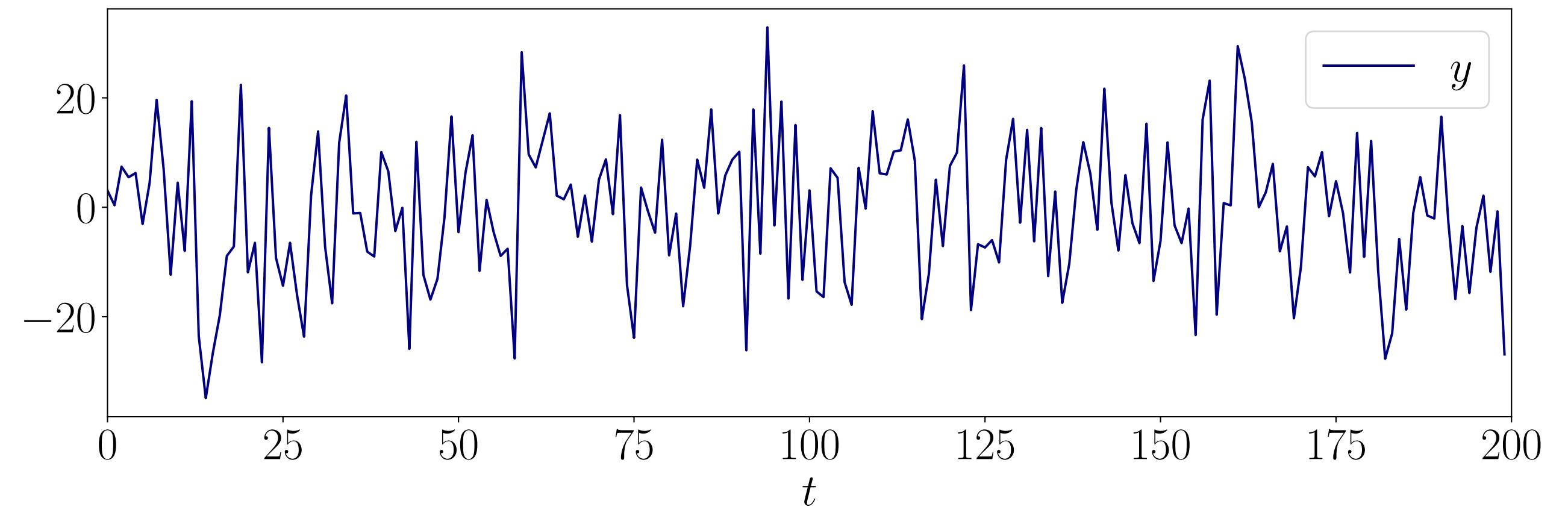
$$\begin{aligned} &\text{minimize} && \mathbf{1}^T u \\ &\text{subject to} && -u \leq Hx - y \leq u \\ &&& x_k = a^T z_k, \quad k = 1, \dots, n \\ &&& \mathbf{1}^T z_k = 1, \quad k = 1, \dots, n \\ &&& z_k \in \{0, 1\}^d \end{aligned}$$



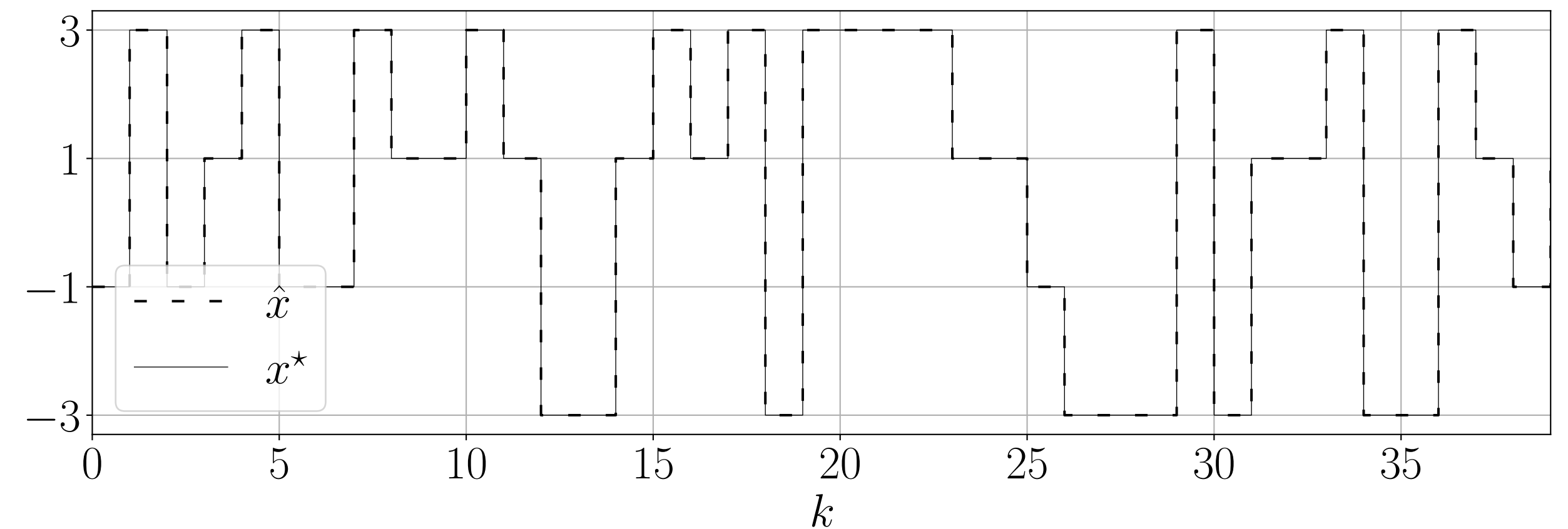
# Signal decoding example

Exact message  $\hat{x} \in \{-3, -1, 1, 3\}^{40}$

Noisy signal  $y = H\hat{x} + v \in \mathbf{R}^{200}$



**Exact message decoded!**



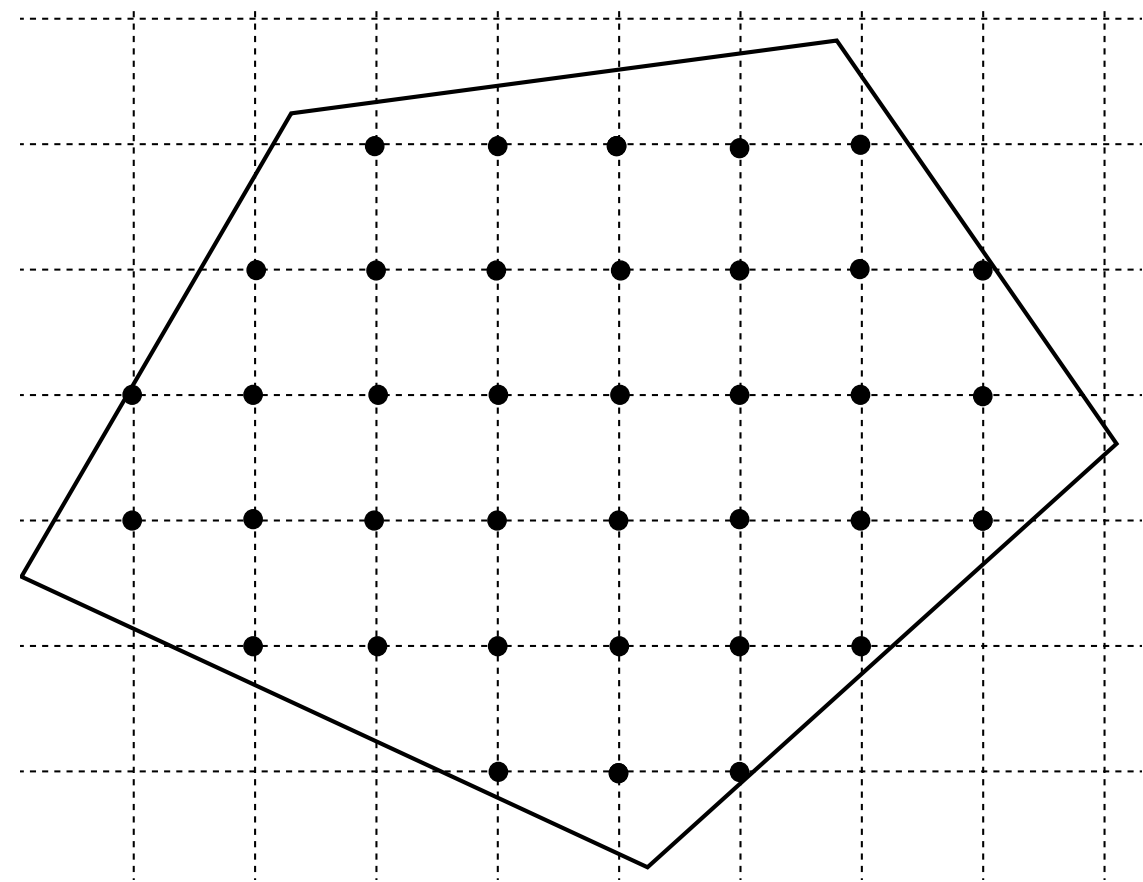
# Relaxations

# Relaxations

Remove integrality constraints

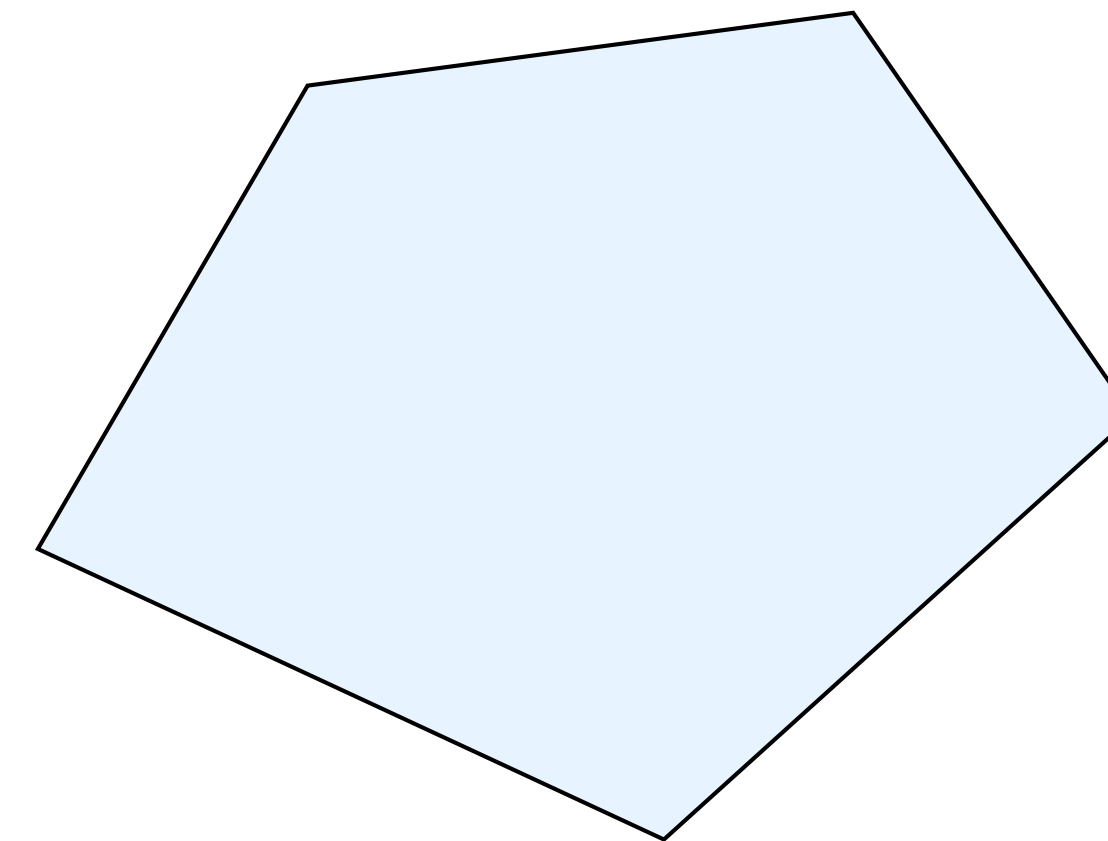
minimize  $c^T x$   
subject to  $Ax \leq b$   
 $x_i \in \mathbf{Z}, \quad i \in \mathcal{I}$

$P_{\text{ip}}$   $\longrightarrow$



minimize  $c^T x$   
subject to  $Ax \leq b$

$P_{\text{rel}}$   $\longleftarrow$



$P_{\text{ip}} \subset P_{\text{rel}}$



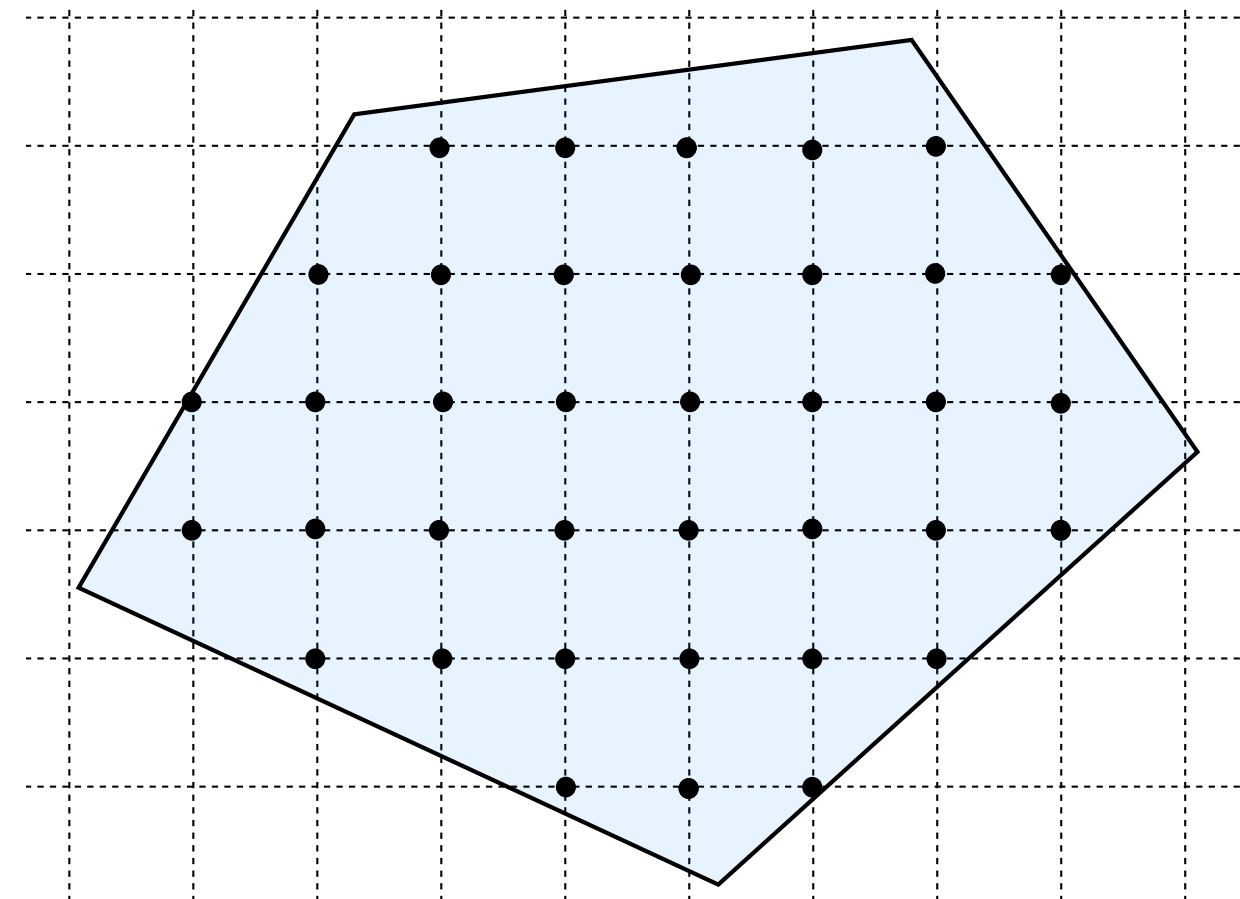
Relaxations provide  
**lower bounds** to  $p_{\text{ip}}^*$   
 $p_{\text{rel}}^* \leq p_{\text{ip}}^*$

# Multiple formulations exist

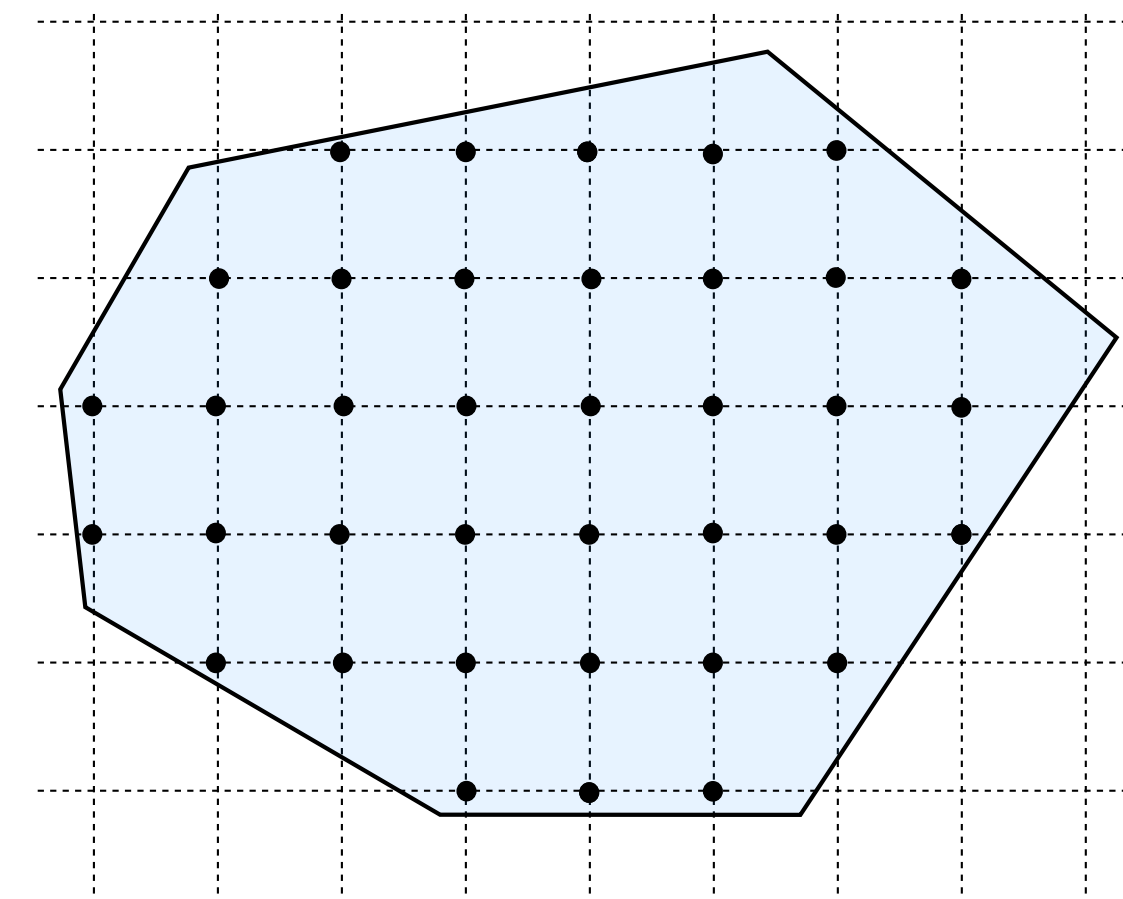
$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax \leq b \\ &&& x_i \in \mathbf{Z}, \quad i \in \mathcal{I} \end{aligned}$$

**Equivalent formulations**  
(same feasible points)  
**with different relaxations**

**Formulation 1**



**Formulation 2**



**Which one is better?**

$$p_{\text{rel1}}^* \begin{matrix} \leq \\ \geq \\ = \end{matrix} p_{\text{rel2}}^* ?$$

# Facility location problem

## Multiple formulations

### Formulation 1

$$\text{minimize } \sum_{j=1}^n c_j y_j + \sum_{i=1}^m \sum_{j=1}^n d_{ij} x_{ij}$$

$$\text{subject to } \sum_{j=1}^n x_{ij} = 1, \quad i = 1, \dots, m$$

$$x_{ij} \leq y_j, \quad i = 1, \dots, m, \quad j = 1, \dots, n$$

$$x_{ij}, y_j \in \{0, 1\}$$

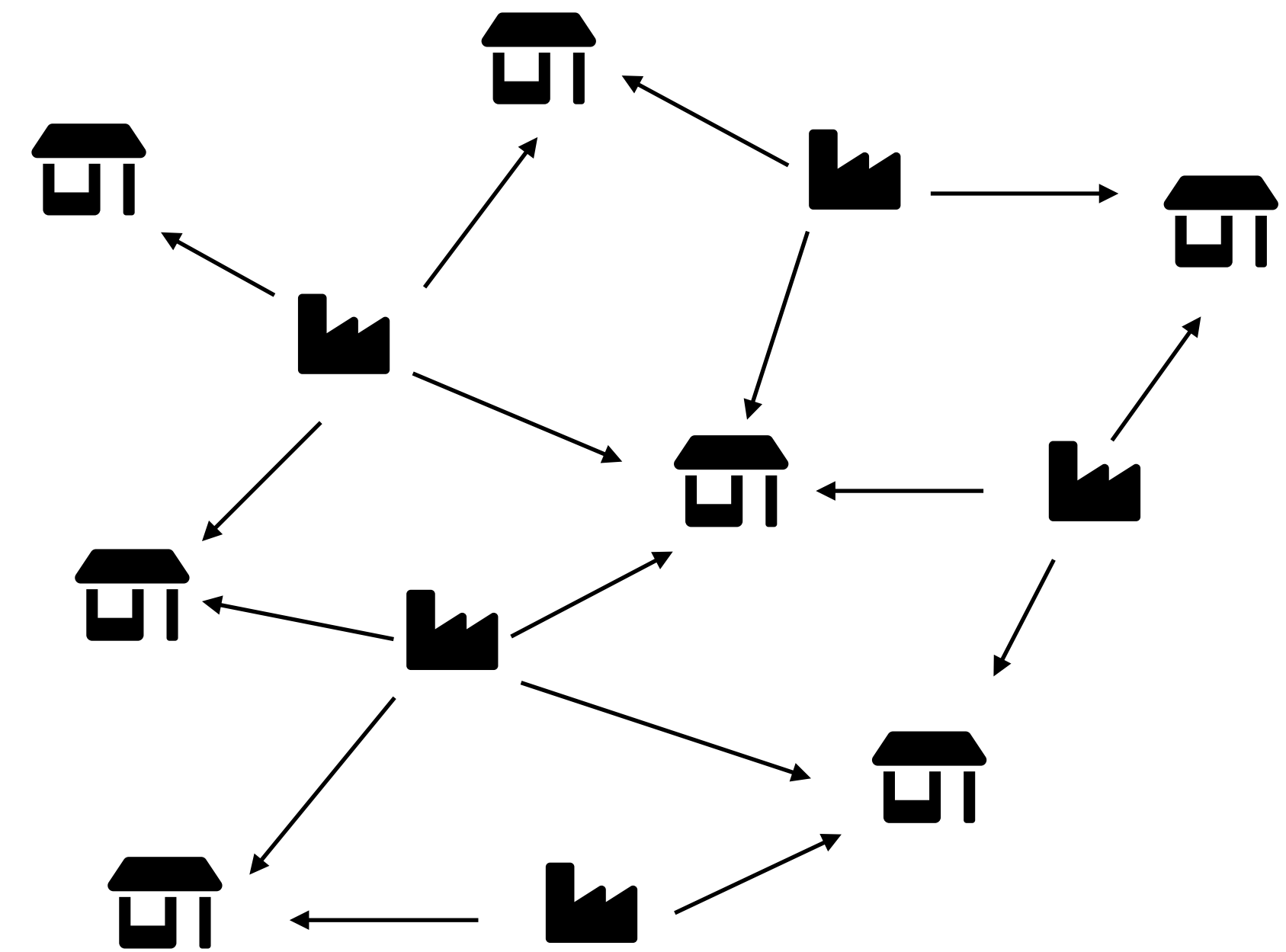
### Formulation 2 (fewer constraints)

$$\text{minimize } \sum_{j=1}^n c_j y_j + \sum_{i=1}^m \sum_{j=1}^n d_{ij} x_{ij}$$

$$\text{subject to } \sum_{j=1}^n x_{ij} = 1, \quad i = 1, \dots, m$$

$$\sum_{i=1}^m x_{ij} \leq m y_j, \quad j = 1, \dots, n$$

$$x_{ij}, y_j \in \{0, 1\}$$



**Are they both valid?**

**Which one is better?**

# Facility location problem

## Multiple formulations

### Formulation 1

$$P_{\text{rel1}} = \left\{ \sum_{j=1}^n x_{ij} = 1, \quad x_{ij} \leq y_j, \quad x_{ij}, y_j \in [0, 1] \right\}$$

### Formulation 2

$$P_{\text{rel2}} = \left\{ \sum_{j=1}^n x_{ij} = 1, \quad \sum_{i=1}^m x_{ij} \leq m y_j, \quad x_{ij}, y_j \in [0, 1] \right\}$$

### Relationship

$$P_{\text{rel1}} \subset P_{\text{rel2}} \implies p_{\text{rel2}}^* \leq p_{\text{rel1}}^* \leq p^* = p_1^* = p_2^*$$

**Formulation 1  
is better**

# Facility location problem

Multiple formulations proof  $P_{\text{rel1}} \subset P_{\text{rel2}}$

Formulation 1:  $P_{\text{rel1}}$

$$x_{ij} \leq y_j, \forall i, j \iff \max_i x_{ij} \leq y_j$$

Maximum less than  $y_j$   
implies average less than  $y_j$

Average less than  $y_j$   
doesn't imply maximum less than  $y_j$

- $(x_{1j}, x_{2j}, x_{3j}) = (0.3, 0.4, 0.5)$
- $y_j = 0.45$

Formulation 2:  $P_{\text{rel2}}$

$$\sum_{i=1}^m x_{ij} \leq my_j, \forall j \iff \text{avg}_i x_{ij} \leq y_j$$



$$P_{\text{rel1}} \subseteq P_{\text{rel2}}$$



$$P_{\text{rel1}} \neq P_{\text{rel2}}$$



**Ideal formulations**



# What's the best possible formulation?

## Problem

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \\ & x_i \in \mathbf{Z}, \quad i \in \mathcal{I} \end{array}$$

## Relaxation

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

**What happens if the relaxation solution is integer feasible point?**

We found an optimal solution!

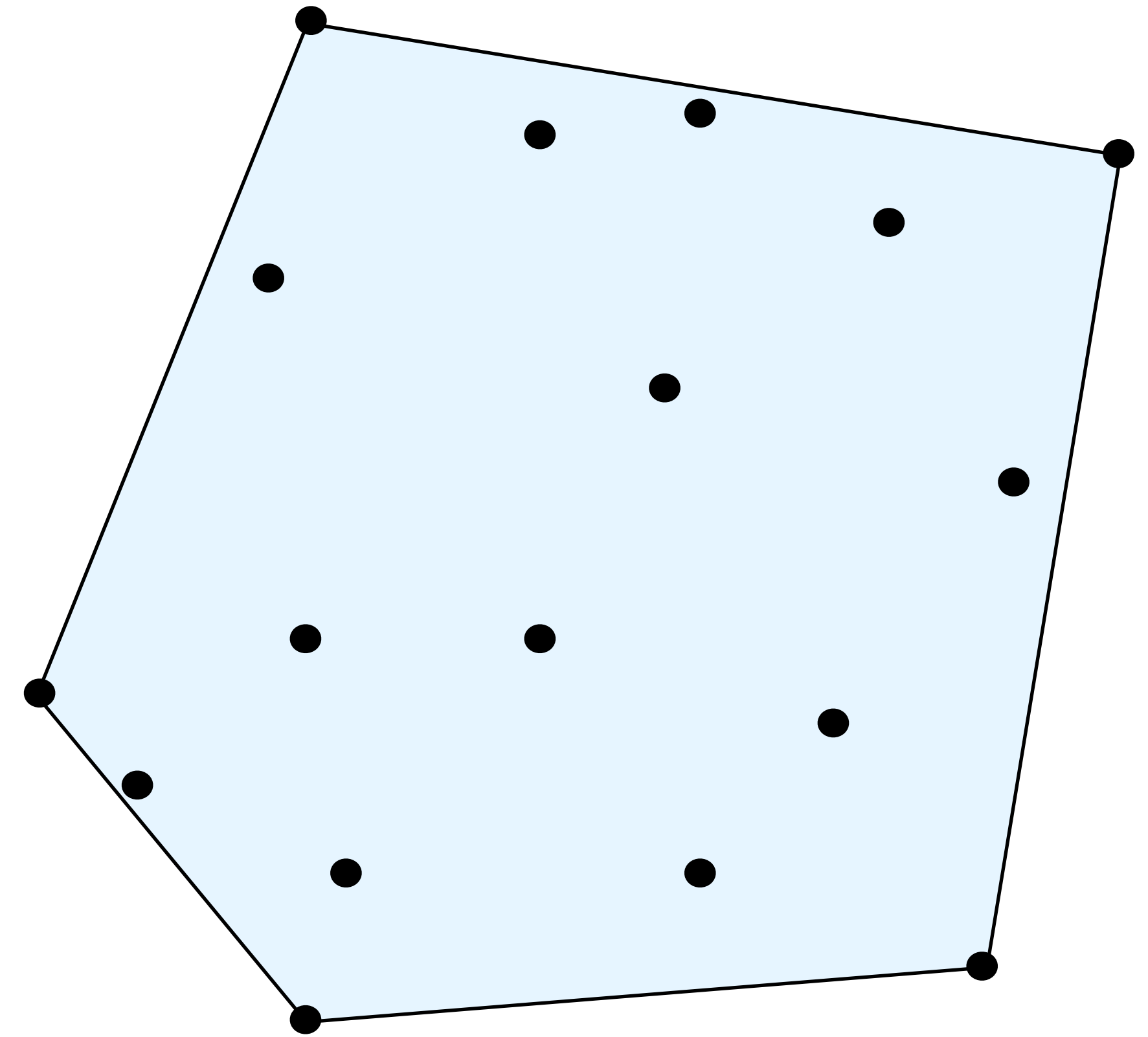
**Does this formulation always exist?**

# Convex hull

## Recap

The **convex hull** is the set of all possible convex combinations of the points.

$$\text{conv } C = \left\{ \sum_{i=1}^n \alpha_i x_i \mid \alpha \geq 0, \quad \mathbf{1}^T \alpha = 1 \right\}$$



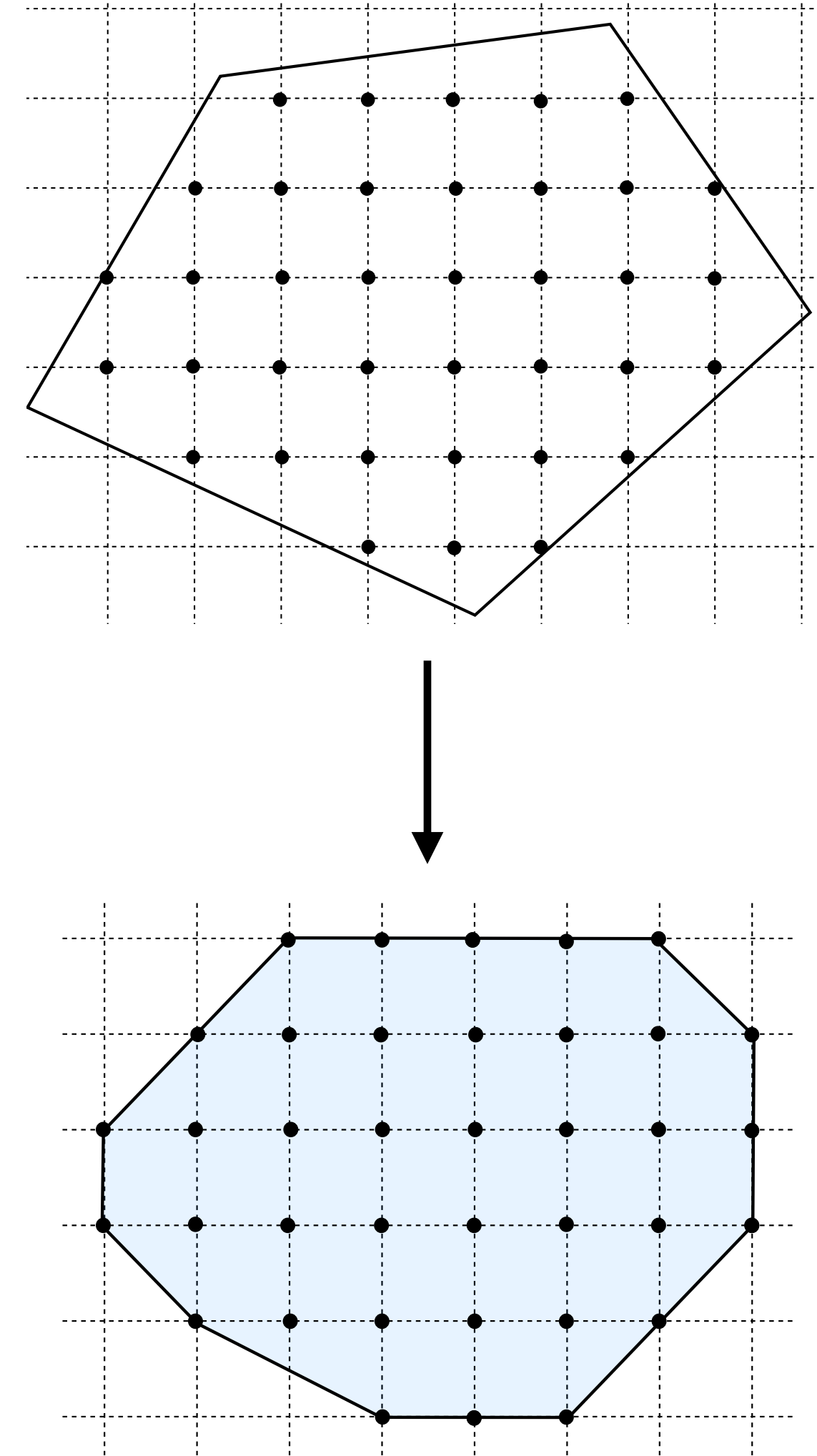
**What is the convex hull of an integer optimization problem?**

# Convex hull of integer optimization

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \\ & x_i \in \mathbf{Z}, \quad i \in \mathcal{I} \end{array}$$

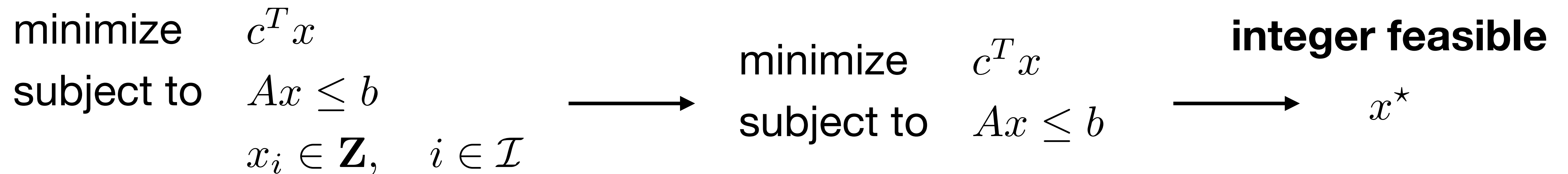
**The convex hull has  
integer feasible extreme points**

$$\text{conv } P = \text{conv}\{x \mid Ax \leq b, \quad x_i \in \mathbf{Z}, \quad i \in \mathcal{I}\}$$

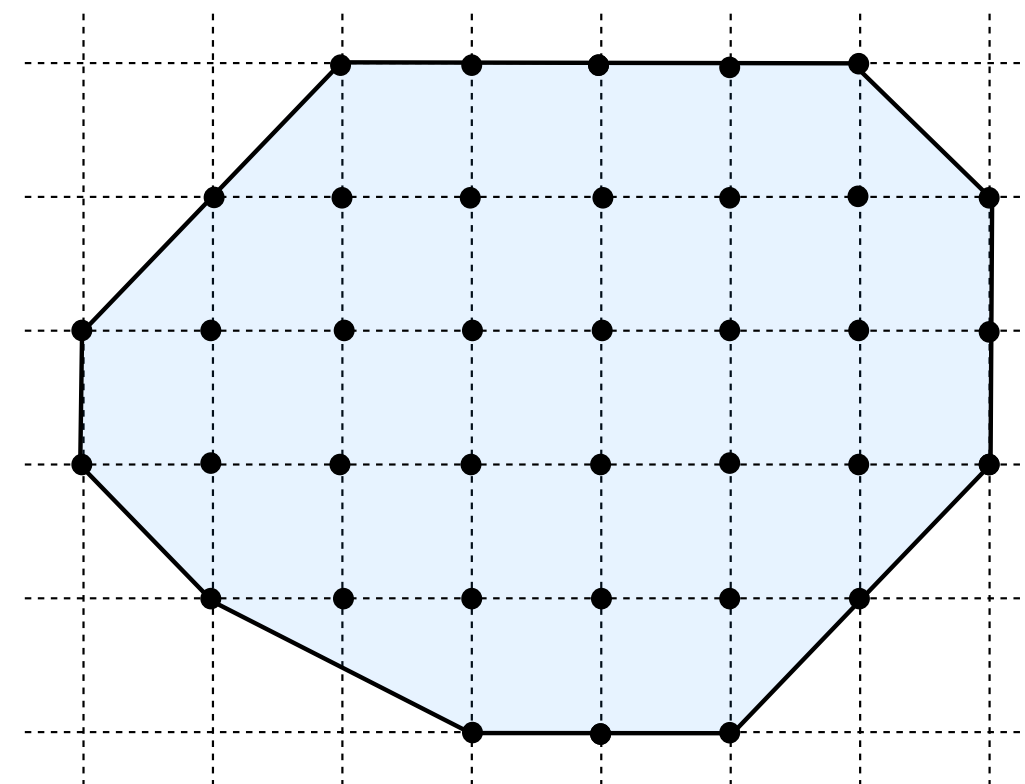


# Ideal formulations

A formulation is ideal if solving its relaxation gives an integer feasible point



This happens if  
 $\text{conv } P = \{Ax \leq b\}$



**It is very hard to construct ideal formulations!**

# Facility location problem

## Formulation 1

$$\begin{aligned} &\text{minimize} && \sum_{j=1}^n c_j y_j + \sum_{i=1}^m \sum_{j=1}^n d_{ij} x_{ij} \\ &\text{subject to} && \sum_{j=1}^n x_{ij} = 1, \quad i = 1, \dots, m \\ &&& x_{ij} \leq y_j, \quad i = 1, \dots, m, \quad j = 1, \dots, n \\ &&& x_{ij}, y_j \in \{0, 1\} \end{aligned}$$

## Formulation 2 (fewer constraints)

$$\begin{aligned} &\text{minimize} && \sum_{j=1}^n c_j y_j + \sum_{i=1}^m \sum_{j=1}^n d_{ij} x_{ij} \\ &\text{subject to} && \sum_{j=1}^n x_{ij} = 1, \quad i = 1, \dots, m \\ &&& \sum_{i=1}^m x_{ij} \leq m y_j, \quad j = 1, \dots, n \\ &&& x_{ij}, y_j \in \{0, 1\} \end{aligned}$$

## Ranking relaxations

$$\text{conv } P \subseteq P_{\text{rel1}} \subseteq P_{\text{rel2}}$$

# Judging formulations

## Size of feasible region

Goal:  $\text{conv } P \approx \{Ax \leq b\}$

## Objective function value

Goal:  $p_{\text{rel}}^* \approx p_{\text{ip}}^*$

## Problem size

Goal: keep moderate LP relaxation size  
(unfortunately, better formulations  
tend to have more  
variables/constraints)

## Problem formulation

minimize  $c^T x$   
subject to  $Ax \leq b$   
 $x_i \in \mathbf{Z}, \quad i \in \mathcal{I}$

# Minimum cost network flow

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & 0 \leq x \leq u \end{array}$$



**Integrality theorem**  
If  $A$  totally unimodular  
(e.g., graph arc-node incidence)  
 $b$  and  $u$  are integral  
solutions  $x^*$  are integral

## Formulation is ideal

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & 0 \leq x \leq u \\ & x \in \mathbf{Z}^n \end{array}$$

**Very easy  
special case!**

# How do we solve integer optimization problems?

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax \leq b \\ &&& x_i \in \mathbf{Z}, \quad i \in \mathcal{I} \end{aligned}$$

**Idea:** Refine the feasible set until the relaxation gives integer feasible solutions!



# Mixed-integer optimization

Today, we learned to:

- **Define** mixed-integer optimization problems
- **Model** logical relationships with integer variables and constraints
- **Analyze** relaxations and formulations

# References

- D. Bertsimas & J. Tsitsiklis “Introduction to Linear Optimization”
  - Chapter 10: integer programming formulations
- R. Vanderbei “Linear Programming”
  - Chapter 23: Integer programming

# Next lecture

- Integer optimization algorithms