

ORF307 – Optimization

19. Linear optimization review

Today's lecture

Linear optimization review

- Formulations
- Piecewise linear optimization
- Duality
- Sensitivity analysis
- Simplex method
- Interior point methods

Formulations

Linear optimization

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \\ & Dx = f \end{array}$$

- Minimization
- Less-than ineq. constraints
- Equality constraints

x is **feasible** if it satisfies the constraints $Ax \leq b$ and $Dx = f$

The **feasible set** is the set of all feasible points

x^* is **optimal** if it is feasible and $c^T x^* \leq c^T x$ for all feasible x

The **optimal value** is $p^* = c^T x^*$

Unbounded problem: $c^T x$ is unbounded below on the feasible set ($p^* = -\infty$)

Infeasible problem: feasible set is empty ($p^* = +\infty$)

Feasibility problems

$$\begin{array}{ll} \text{find} & x \\ \text{subject to} & Ax \leq b \\ & Dx = f \end{array} \quad \longrightarrow \quad \begin{array}{ll} \text{minimize} & 0 \\ \text{subject to} & Ax \leq b \\ & Dx = f \end{array}$$

Possible results

- $p^* = 0$ if constraints are feasible (consistent).
(Every feasible x is optimal)
- $p^* = \infty$ otherwise

Standard form

Definition

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

- Minimization
- Equality constraints
- Nonnegative variables

Useful to

- develop **algorithms**
- **algebraic** manipulations

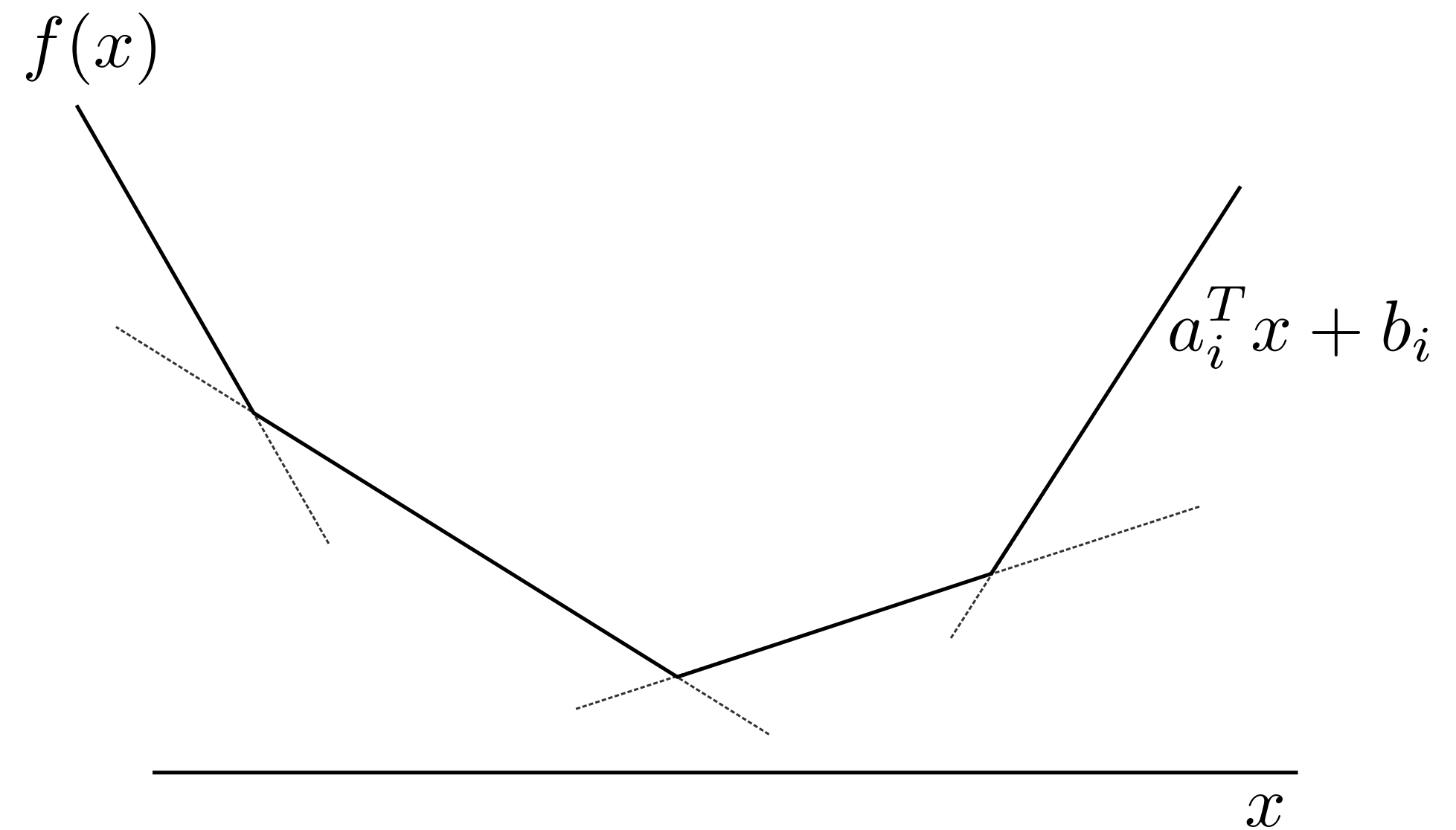
Piecewise linear optimization

Piecewise-linear minimization

$$\text{minimize } f(x) = \max_{i=1, \dots, m} (a_i^T x + b_i)$$



$$\begin{aligned} &\text{minimize } t \\ &\text{subject to } a_i^T x + b_i \leq t, \quad i = 1, \dots, m \end{aligned}$$



Matrix notation

$$\begin{aligned} &\text{minimize } \tilde{c}^T \tilde{x} \\ &\text{subject to } \tilde{A} \tilde{x} \leq \tilde{b} \end{aligned}$$

$$\tilde{x} = \begin{bmatrix} x \\ t \end{bmatrix}, \quad \tilde{c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} a_1^T & -1 \\ \vdots & \vdots \\ a_m^T & -1 \end{bmatrix}, \quad \tilde{b} = \begin{bmatrix} -b_1 \\ \vdots \\ -b_m \end{bmatrix}$$

1 and infinity norms reformulations

1-norm minimization:

$$\text{minimize } \|Ax - b\|_1 = \sum_i |(Ax - b)_i|$$

Equivalent to:

$$\text{minimize } \mathbf{1}^T u$$

$$\text{subject to } -u \leq Ax - b \leq u$$

Absolute value of every element $(Ax - b)_i$ is bounded by a component of the **vector** u

∞ -norm minimization:

$$\text{minimize } \|Ax - b\|_\infty = \max_i |(Ax - b)_i|$$

Equivalent to:

$$\text{minimize } t$$

$$\text{subject to } -t\mathbf{1} \leq Ax - b \leq t\mathbf{1}$$

Absolute value of every element $(Ax - b)_i$ is bounded by the same **scalar** t

Duality

Lagrangian and duality

Primal

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax \leq b \end{aligned}$$

Dual

$$\begin{aligned} &\text{maximize} && -b^T y \\ &\text{subject to} && A^T y + c = 0 \\ &&& y \geq 0 \end{aligned}$$

Dual function

$$\begin{aligned} g(y) &= \text{minimize}_x (c^T x + y^T (Ax - b)) \\ &= -b^T y + \text{minimize}_x (c + A^T y)^T x \\ &= \begin{cases} -b^T y & \text{if } c + A^T y = 0 \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

Lagrangian

$$L(x, y) = c^T x + y^T (Ax - b)$$

$$\nabla_x L(x, y) = c + A^T y = 0$$

Karush-Kuhn-Tucker conditions

Optimality conditions for linear optimization

Primal

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax \leq b \end{aligned}$$

Dual

$$\begin{aligned} &\text{maximize} && -b^T y \\ &\text{subject to} && A^T y + c = 0 \\ &&& y \geq 0 \end{aligned}$$

Primal feasibility

$$Ax \leq b$$

Dual feasibility

$$\nabla_x L(x, y) = A^T y + c = 0 \quad \text{and} \quad y \geq 0$$

Complementary slackness

$$y_i (Ax - b)_i = 0, \quad i = 1, \dots, m$$

General forms

Inequality form LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c = 0 \\ & y \geq 0 \end{array}$$

Standard form LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c \geq 0 \end{array}$$

LP with inequalities and equalities

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \\ & Dx = f \end{array}$$

$$\begin{array}{ll} \text{maximize} & -b^T y - f^T z \\ \text{subject to} & A^T y + D^T z + c = 0 \\ & y \geq 0 \end{array}$$

Weak duality

Theorem

If x, y satisfy:

- x is a feasible solution to the primal problem
 - y is a feasible solution to the dual problem
- $\longrightarrow -b^T y \leq c^T x$

Proof

We know that $Ax \leq b$, $A^T y + c = 0$ and $y \geq 0$. Therefore,

$$0 \leq y^T (b - Ax) = b^T y - y^T Ax = c^T x + b^T y \quad \blacksquare$$

Remark

- Any dual feasible y gives a **lower bound** on the primal optimal value
- Any primal feasible x gives an **upper bound** on the dual optimal value
- $c^T x + b^T y$ is the **duality gap**

Weak duality

Corollaries

Unboundedness vs feasibility

- Primal unbounded ($p^* = -\infty$) \Rightarrow dual infeasible ($d^* = -\infty$)
- Dual unbounded ($d^* = +\infty$) \Rightarrow primal infeasible ($p^* = +\infty$)

Optimality condition

If x, y satisfy:

- x is a feasible solution to the primal problem
- y is a feasible solution to the dual problem
- The duality gap is zero, *i.e.*, $c^T x + b^T y = 0$

Then x and y are **optimal solutions** to the primal and dual problem respectively

Strong duality

Primal

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax = b \\ &&& x \geq 0 \end{aligned}$$

Dual

$$\begin{aligned} &\text{maximize} && -b^T y \\ &\text{subject to} && A^T y + c \geq 0 \end{aligned}$$

Theorem

If a linear optimization problem has an optimal solution, then

- so does its dual
- the optimal values of the primal and dual are equal

Relationship between primal and dual

	$p^* = +\infty$	p^* finite	$p^* = -\infty$
$d^* = +\infty$	primal inf. dual unb.		
d^* finite		optimal values equal	
$d^* = -\infty$	exception		primal unb. dual inf

Complementary slackness

Primal

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax \leq b \end{aligned}$$

Dual

$$\begin{aligned} &\text{maximize} && -b^T y \\ &\text{subject to} && A^T y + c = 0 \\ &&& y \geq 0 \end{aligned}$$

Theorem

Primal, dual feasible x, y are optimal if and only if

$$y_i(b_i - a_i^T x) = 0, \quad i = 1, \dots, m$$

i.e., at optimum, $b - Ax$ and y have a **complementary sparsity** pattern:

$$\begin{aligned} y_i > 0 &\Rightarrow a_i^T x = b_i \\ a_i^T x < b_i &\Rightarrow y_i = 0 \end{aligned}$$

Complementary slackness

Primal

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax \leq b \end{aligned}$$

Dual

$$\begin{aligned} &\text{maximize} && -b^T y \\ &\text{subject to} && A^T y + c = 0 \\ &&& y \geq 0 \end{aligned}$$

Proof

The duality gap at primal feasible x and dual feasible y can be written as

$$c^T x + b^T y = (-A^T y)^T x + b^T y = (b - Ax)^T y = \sum_{i=1}^m y_i (b_i - a_i^T x) = 0$$

Since all the elements of the sum are nonnegative, they must all be 0 ■

For **feasible** x and y **complementary slackness = zero duality gap**

Example

$$\begin{array}{ll} \text{minimize} & -4x_1 - 5x_2 \\ \text{subject to} & \begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 3 \\ 0 \\ 3 \end{bmatrix} \end{array}$$

Let's **show** that feasible $x = (1, 1)$ is optimal

Second and fourth constraints are active at $x \longrightarrow y = (0, y_2, 0, y_4)$

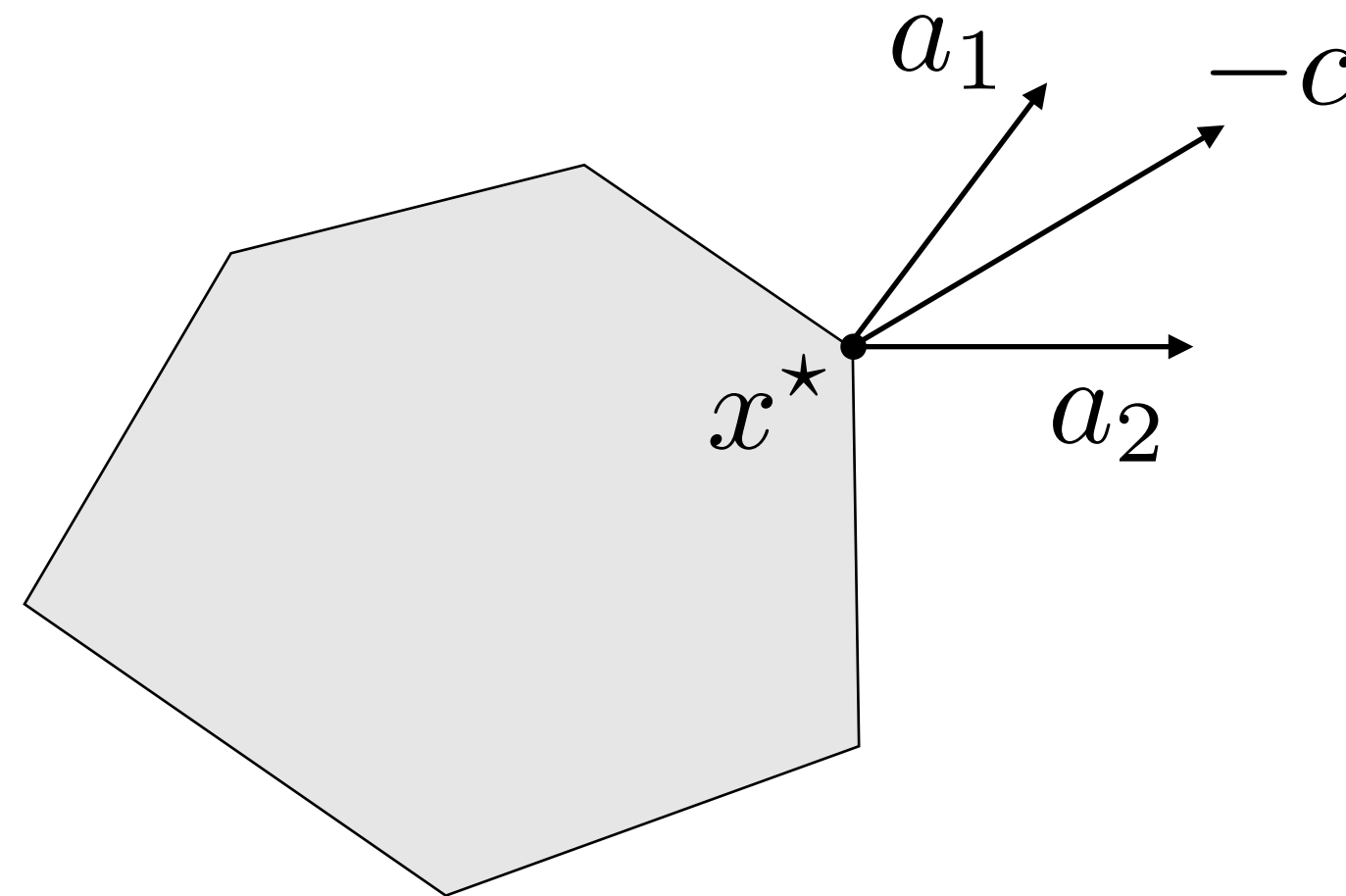
$$A^T y = -c \quad \Rightarrow \quad \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_2 \\ y_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \quad \text{and} \quad y_2 \geq 0, \quad y_4 \geq 0$$

$y = (0, 1, 0, 2)$ satisfies these conditions and proves that x is optimal

Complementary slackness is useful to recover y^* from x^*

Geometric interpretation

Example in \mathbb{R}^2



Two active constraints at optimum: $a_1^T x^* = b_1$, $a_2^T x^* = b_2$

Optimal dual solution y satisfies:

$$A^T y + c = 0, \quad y \geq 0, \quad y_i = 0 \text{ for } i \neq \{1, 2\}$$

In other words, $-c = a_1 y_1 + a_2 y_2$ with $y_1, y_2 \geq 0$

Sensitivity analysis

Changes in problem data

Goal: extract information from x^*, y^* about their sensitivity with respect to changes in problem data

Modified LP

minimize $c^T x$

subject to $Ax = b + u$

$x \geq 0$

Optimal value function

$$p^*(u) = \min\{c^T x \mid Ax = b + u, x \geq 0\}$$

Assumption: $p^*(0)$ is finite

Properties

- $p^*(u) > -\infty$ everywhere (from global lower bound)
- $p^*(u)$ is piecewise-linear on its domain

Global sensitivity

Dual of modified LP

$$\begin{aligned} &\text{maximize} && -(b + u)^T y \\ &\text{subject to} && A^T y + c \geq 0 \end{aligned}$$

Global lower bound

Given y^* a dual optimal solution for $u = 0$, then

$$\begin{aligned} p^*(u) &\geq -(b + u)^T y^* && \text{(from weak duality and} \\ &= p^*(0) - u^T y^* && \text{dual feasibility)} \end{aligned}$$

It holds for any u

Local sensitivity

u in neighborhood of the origin

Original LP

Optimal solution

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$



$$\begin{array}{ll} \text{Primal} & x_i = 0, \quad i \notin B \\ & x_B^* = A_B^{-1} b \\ \text{Dual} & y^* = -A_B^{-T} c_B \end{array}$$

Modified LP

Modified dual

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b + u \\ & x \geq 0 \end{array}$$

$$\begin{array}{ll} \text{maximize} & -(b + u)^T y \\ \text{subject to} & A^T y + c \geq 0 \end{array}$$

**Optimal basis
does not change**

Modified optimal solution

$$\begin{aligned} x_B^*(u) &= A_B^{-1} (b + u) = x_B^* + A_B^{-1} u \\ y^*(u) &= y^* \end{aligned}$$

Derivative of the optimal value function

Modified optimal solution

$$x_B^*(u) = A_B^{-1}(b + u) = x_B^* + A_B^{-1}u$$

$$y^*(u) = y^*$$

Optimal value function

$$p^*(u) = c^T x^*(u)$$

$$= c^T x^* + c_B^T A_B^{-1}u$$

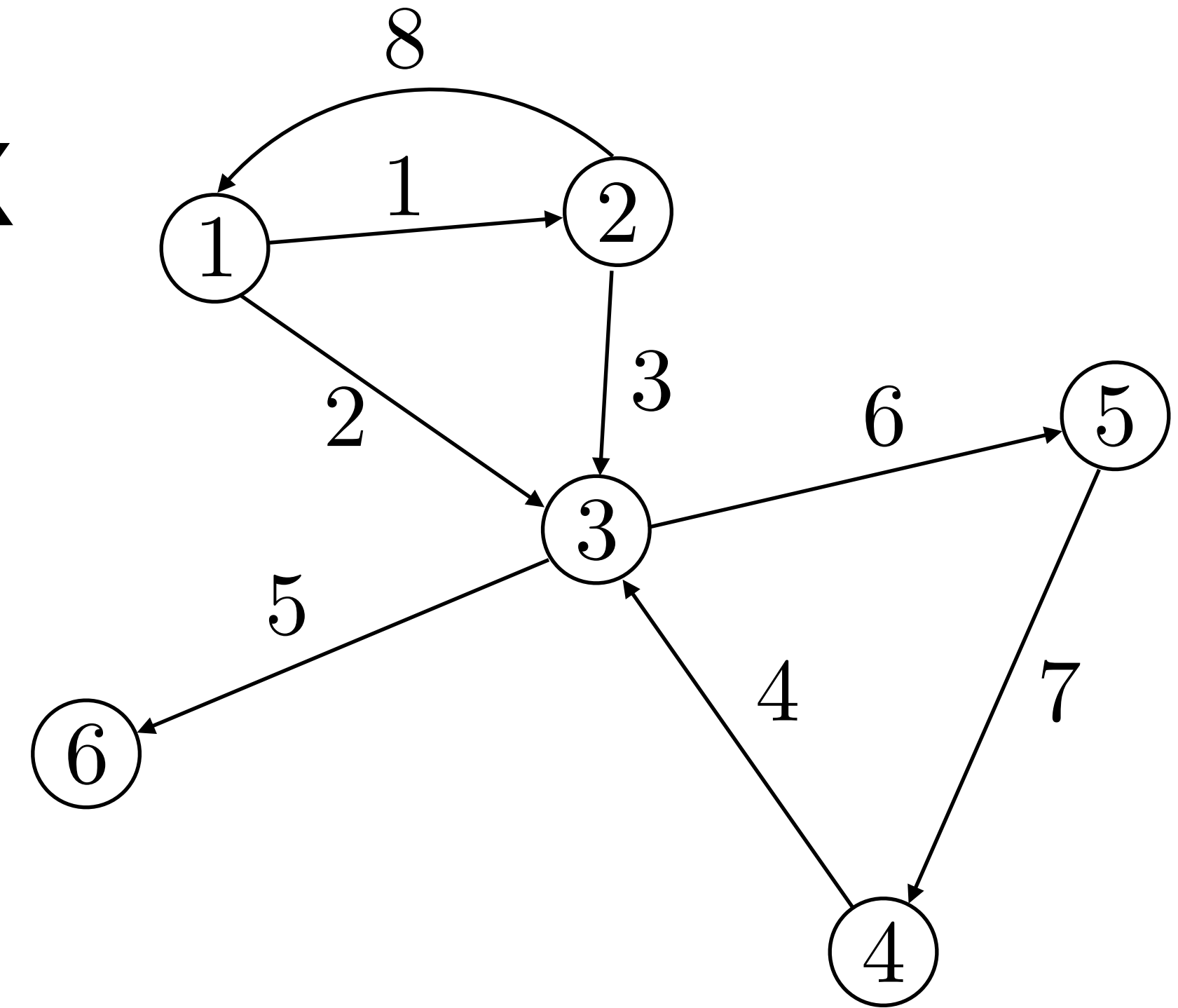
$$= p^*(0) - y^{*T}u \quad (\text{affine for small } u)$$

Local derivative

$$\nabla p^*(u) = -y^* \quad (y^* \text{ are the shadow prices})$$

Network flow optimization

Arc-node incidence matrix



$m \times n$ matrix A with entries

$$A_{ij} = \begin{cases} 1 & \text{if arc } j \text{ starts at node } i \\ -1 & \text{if arc } j \text{ ends at node } i \\ 0 & \text{otherwise} \end{cases}$$

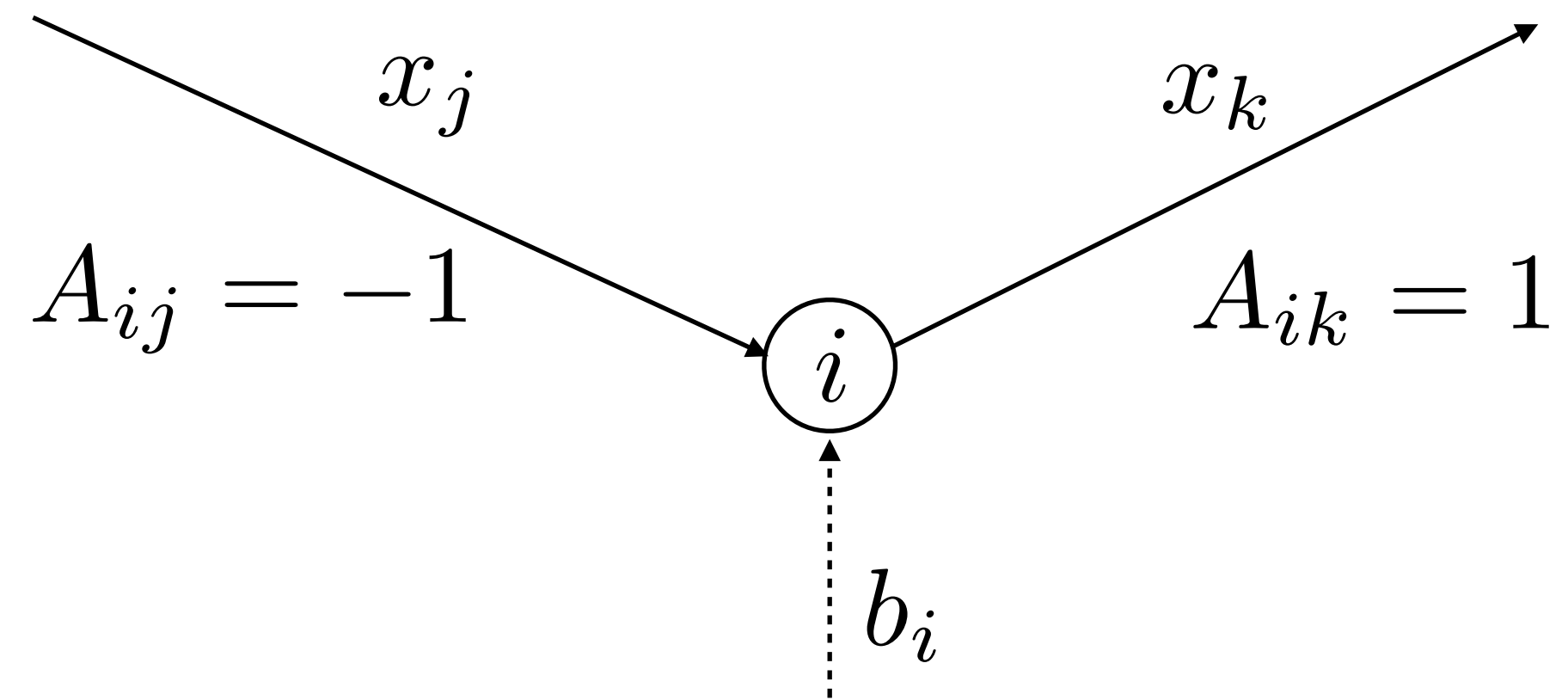
Note Each column has one -1 and one 1

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

External supply

supply vector $b \in \mathbb{R}^m$

- b_i is the external supply at node i (if $b_i < 0$, it represents demand)
- We must have $\mathbf{1}^T b = 0$ (total supply = total demand)



Balance equations

$$\sum_{j=1}^n A_{ij} x_j = (Ax)_i = b_i, \quad \text{for all } i$$

Total leaving
flow

Supply



$$Ax = b$$

Minimum cost network flow problem

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & 0 \leq x \leq u \end{array}$$

- c_i is unit cost of flow through arc i
- Flow x_i must be nonnegative
- u_i is the maximum flow capacity of arc i
- Many network optimization problems are just special cases

Integrality theorem

Given a polyhedron $P = \{x \in \mathbf{R}^n \mid Ax = b, \quad x \geq 0\}$

where

- A is totally unimodular
 - b is an integer vector
-
- all the extreme points of P are integer vectors.

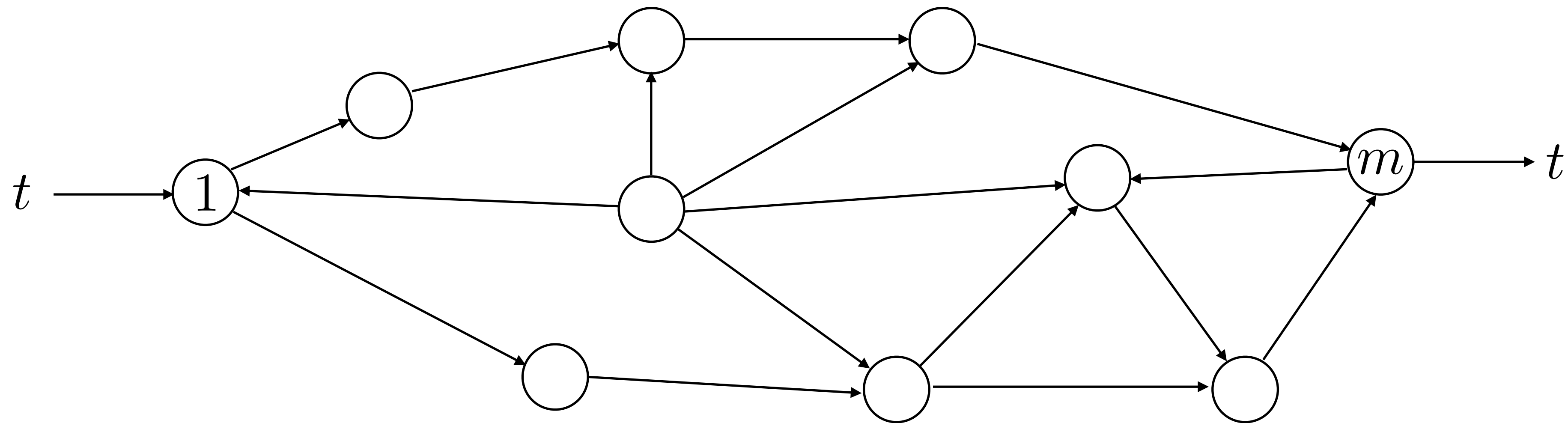
Proof

- All extreme points are basic feasible solutions with $x_B = A_B^{-1}b$ and $x_i = 0, i \neq B$
- A_B^{-1} has integer components because of total unimodularity of A
- b has also integer components
- Therefore, also x is integral



Maximum flow problem

Goal maximize flow from node 1 (source)
to node m (sink) through the network



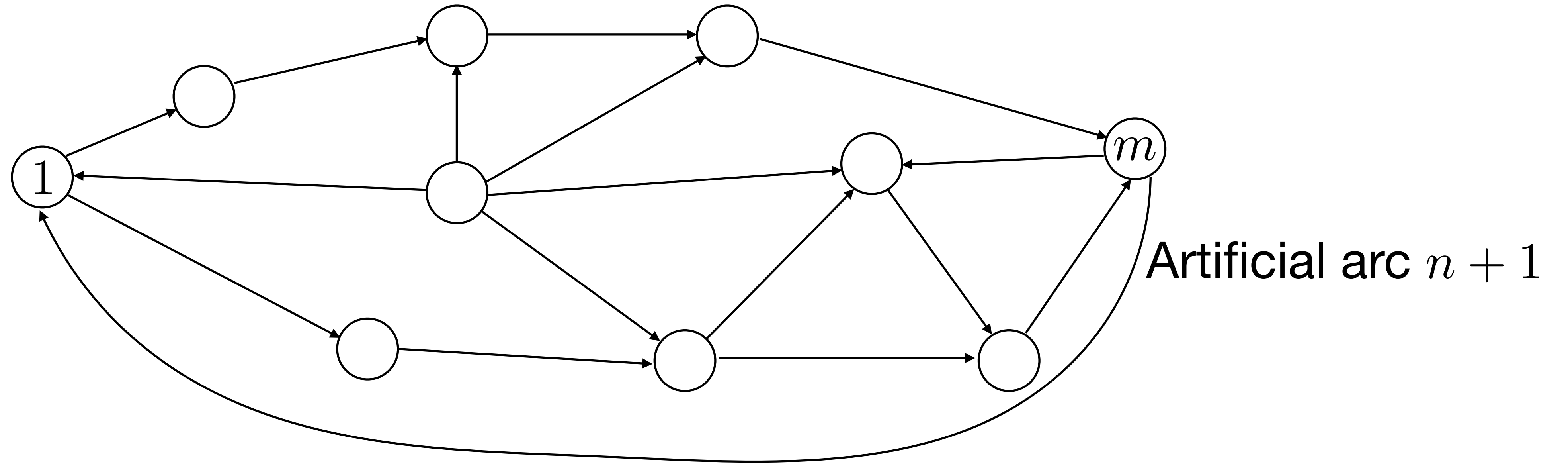
maximize t

subject to $Ax = te$

$0 \leq x \leq u$

$e = (1, 0, \dots, 0, -1)$

Maximum flow as minimum cost flow



minimize $-t$

subject to
$$\begin{bmatrix} A & -e \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} = 0$$

$$0 \leq \begin{bmatrix} x \\ t \end{bmatrix} \leq \begin{bmatrix} u \\ \infty \end{bmatrix}$$

Assignment problem

Goal match N persons to N tasks

- Each person assigned to one task, each task to one person
- C_{ij} Cost of matching person i to task j

LP formulation

minimize
$$\sum_{i,j=1}^N C_{ij} X_{ij}$$

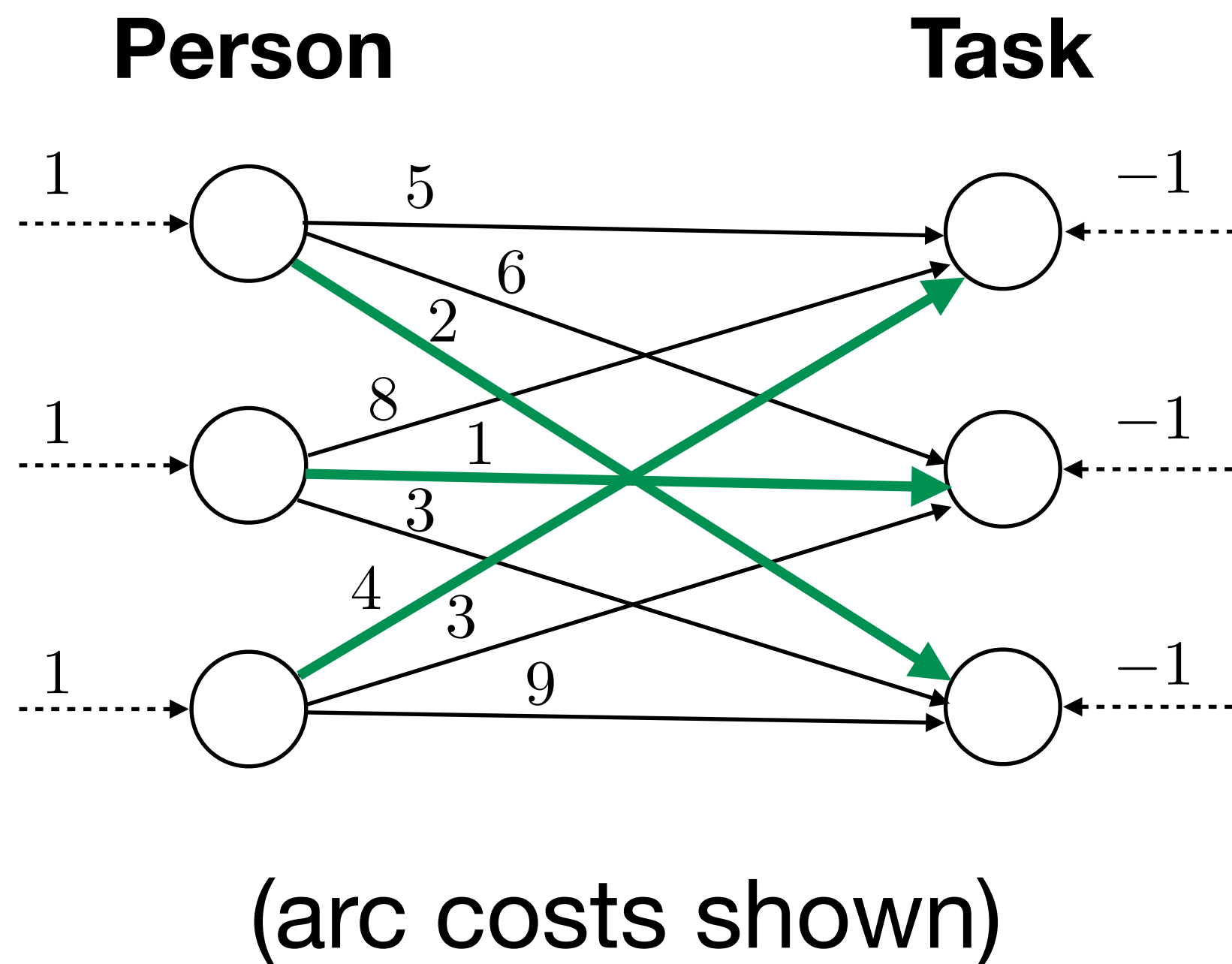
subject to
$$\sum_{i=1}^N X_{ij} = 1, \quad j = 1, \dots, N$$

$$\sum_{j=1}^N X_{ij} = 1, \quad i = 1, \dots, N$$

$$X_{ij} \in \{0, 1\}$$

**How do you define
the network?**

Task assignment as minimum cost network flow



$$c = (5, 6, 2, 8, 1, 3, 4, 3, 9)$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 \end{bmatrix}$$

$$b = (1, 1, 1, -1, -1, -1)$$

Minimum cost network flow

minimize $c^T x$

subject to $Ax = b$

Extreme points
satisfy $x_i \in \{0, 1\}$



$$0 \leq x \leq 1$$

Optimal solution

$$x^* = (0, 0, 1, 0, 1, 0, 0, 0, 1)$$

$$c^T x^* = 7$$

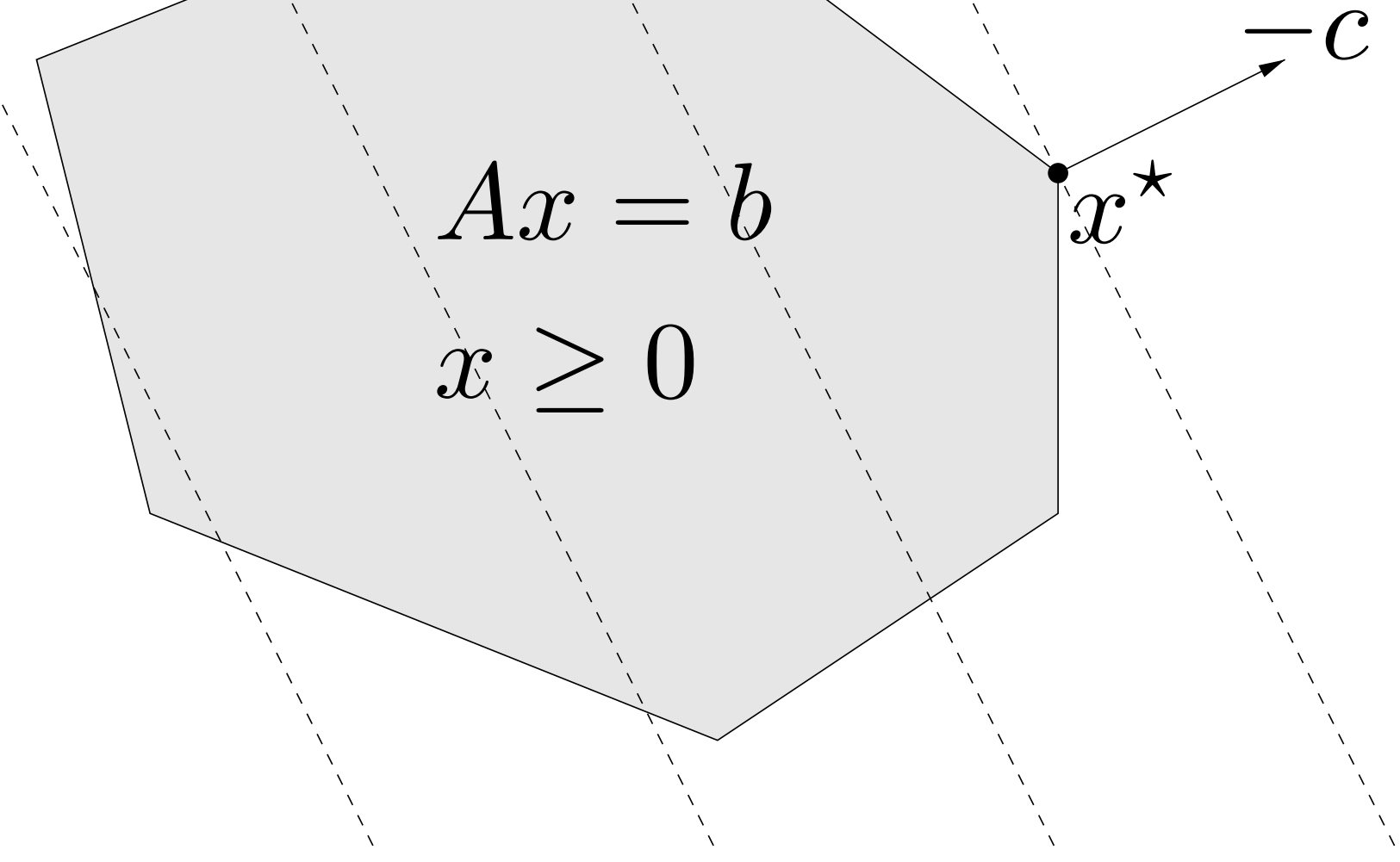
Simplex method

Optimality of extreme points

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

- If
- P has at least one extreme point
 - There exists an optimal solution x^*

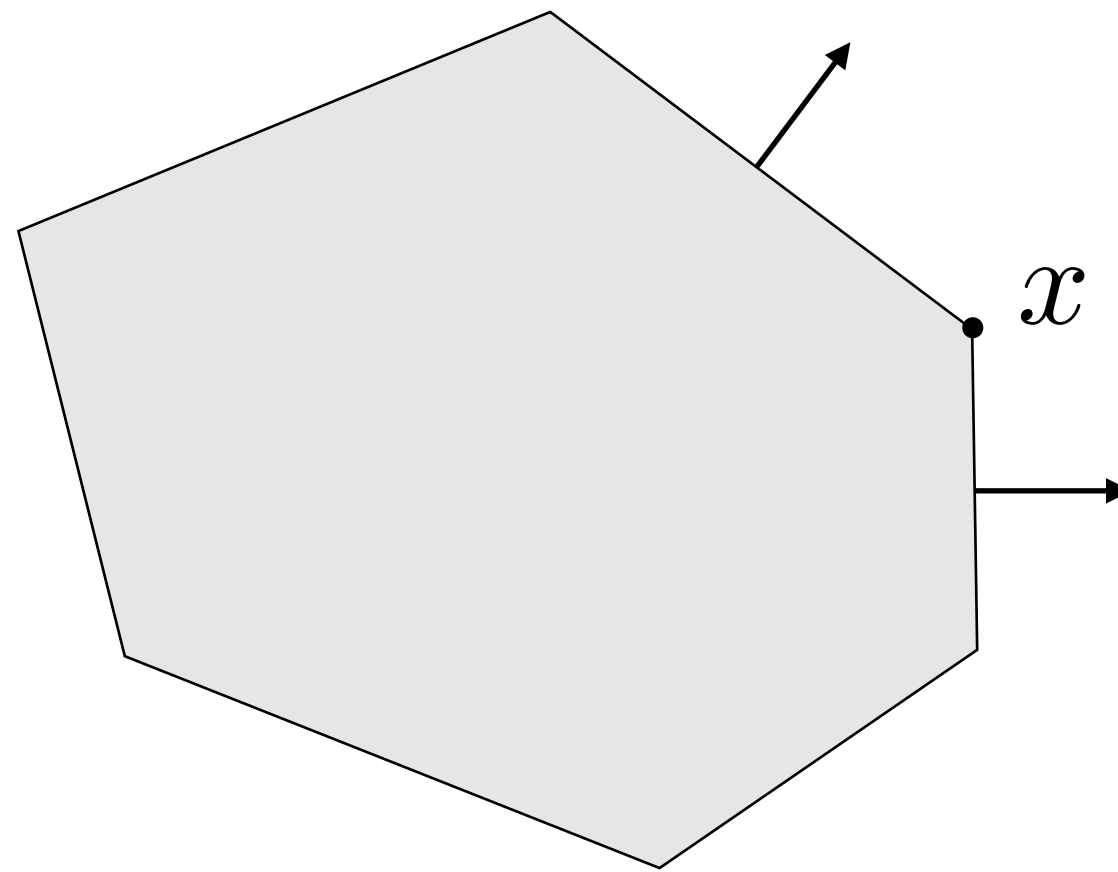
Then, there exists an optimal solution which is an **extreme point** of P



We only need to search between **extreme points**

Equivalence Theorem

Given a nonempty polyhedron $P = \{x \mid Ax = b, x \geq 0\}$



Let $x \in P$

x is a **vertex** \iff x is an **extreme point** \iff x is a **basic feasible solution**

Constructing basic solution

1. Choose any m independent columns of A : $A_{B(1)}, \dots, A_{B(m)}$
2. Let $x_i = 0$ for all $i \neq B(1), \dots, B(m)$
3. Solve $Ax = b$ for the remaining $x_{B(1)}, \dots, x_{B(m)}$

Basis matrix

Basis columns

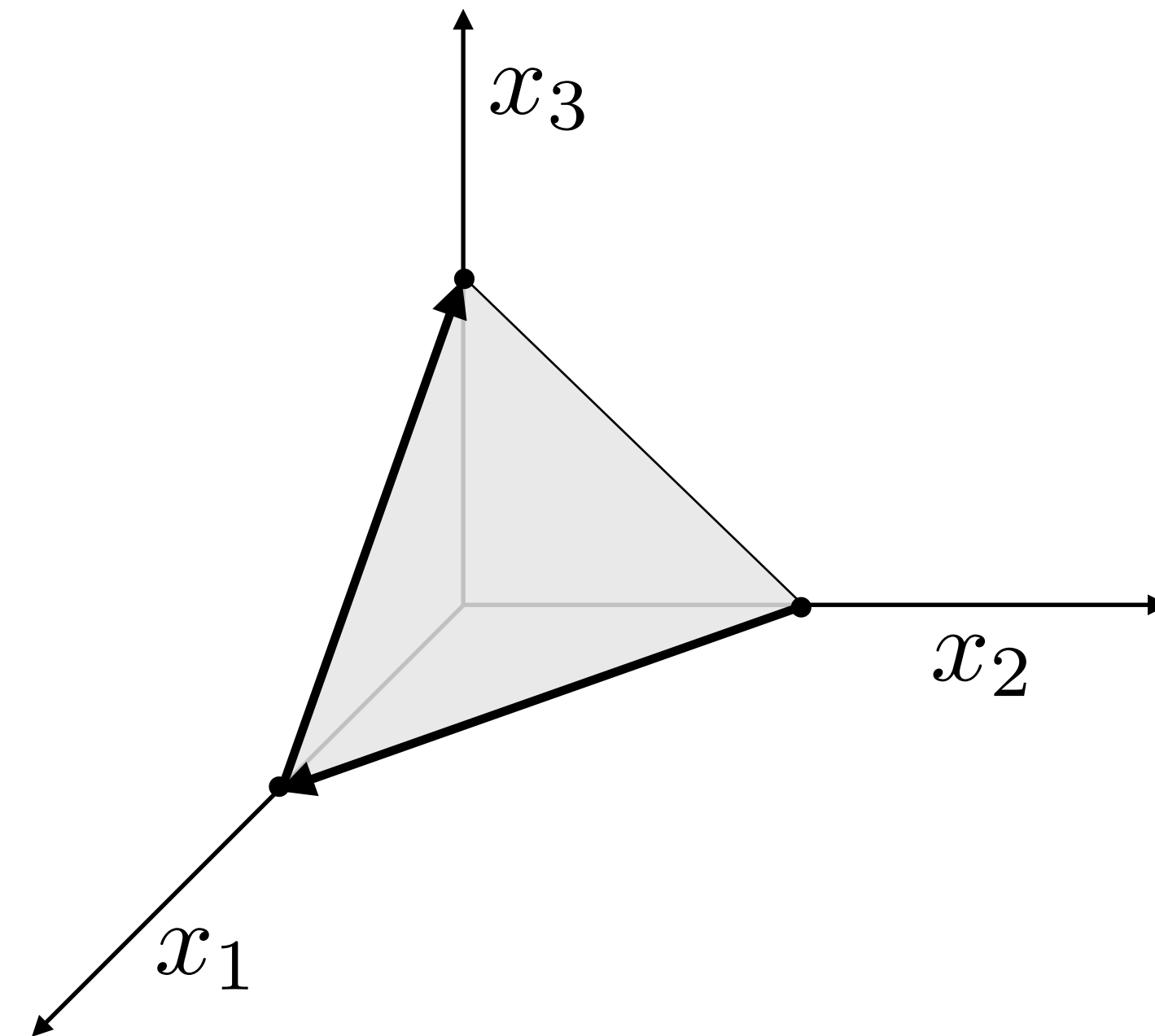
Basic variables

$$A_B = \begin{bmatrix} | & | & & | \\ A_{B(1)} & A_{B(2)} & \dots & A_{B(m)} \\ | & | & & | \end{bmatrix}, \quad x_B = \begin{bmatrix} x_{B(1)} \\ \vdots \\ x_{B(m)} \end{bmatrix} \longrightarrow \text{Solve } A_B x_B = b$$

If $x_B \geq 0$, then x is a **basic feasible solution**

Conceptual algorithm

- Start at corner
- Visit neighboring corner that improves the objective



How does the cost change?

Cost improvement

$$c^T(x + \theta d) - c^T x = \theta c^T d$$

New cost

Old cost

We call \bar{c}_j the **reduced cost** of (introducing) variable x_j in the basis

$$\bar{c}_j = c^T d = \sum_{i=1}^n c_i d_i = c_j + c_B^T d_B = c_j - c_B^T A_B^{-1} A_j$$

Optimality conditions

Theorem

Let x be a basic feasible solution associated with basis B

Let \bar{c} be the vector of reduced costs.

If $\bar{c} \geq 0$, then x is **optimal**

Remark

This is a **stopping criterion** for the simplex algorithm.

If the **neighboring solutions** do not improve the cost, we are done

Single simplex iteration

1. Compute the reduced costs \bar{c}
 - Solve $A_B^T p = c_B$
 - $\bar{c} = c - A^T p$
2. If $\bar{c} \geq 0$, x **optimal. break**
3. Choose j such that $\bar{c}_j < 0$
4. Compute search direction d with $d_j = 1$ and $A_B d_B = -A_j$
5. If $d_B \geq 0$, the problem is **unbounded** and the optimal value is $-\infty$. **break**
6. Compute step length $\theta^* = \min_{\{i \in B \mid d_i < 0\}} \left(-\frac{x_i}{d_i} \right)$
7. Define y such that $y = x + \theta^* d$
8. Get new basis \bar{B} (i exits and j enters)

Bottleneck
Two linear systems



Matrix inversion lemma trick
 $\approx n^2$ per iteration
(very cheap)

How many iterations do we need?

Complexity of the simplex method

We do not know any polynomial version of the simplex method, no matter which pivoting rule we pick.



Still open research question!

Worst-case

There are problem instances where the simplex method will run an **exponential number of iterations** in terms of the dimensions, e.g. 2^n

Good news: average-case

Practical performance is very good. On average, it stops in n iterations.

Interior point method

Optimality conditions

Primal

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax + s = b \\ &&& s \geq 0 \end{aligned}$$

Dual

$$\begin{aligned} &\text{maximize} && -b^T y \\ &\text{subject to} && A^T y + c = 0 \\ &&& y \geq 0 \end{aligned}$$

KKT conditions

$$\begin{aligned} Ax + s - b &= 0 \\ A^T y + c &= 0 \\ s_i y_i &= 0, \quad i = 1, \dots, m \\ s, y &\geq 0 \end{aligned}$$

$$S = \begin{bmatrix} s_1 & & & \\ & s_2 & & \\ & & \ddots & \\ & & & s_m \end{bmatrix} \quad Y = \begin{bmatrix} y_1 & & & \\ & y_2 & & \\ & & \ddots & \\ & & & y_m \end{bmatrix}$$

$$\implies SY\mathbf{1} = 0$$

Main idea

$$h(x, s, y) = \begin{bmatrix} Ax + s - b \\ A^T y + c \\ SY\mathbf{1} \end{bmatrix} = 0 \quad \begin{array}{l} S = \mathbf{diag}(s) \\ Y = \mathbf{diag}(y) \end{array}$$
$$s, y \geq 0$$

- Apply variants of Newton's method to solve $h(x, s, y) = 0$
- Enforce $s, y > 0$ (strictly) at every iteration
- **Motivation** avoid getting stuck in “corners”

Issue

Pure **Newton's step** does not allow significant progress towards

$$h(x, s, y) = 0 \text{ and } x, y \geq 0.$$

Smoothed optimality conditions

Optimality conditions

$$Ax + s - b = 0$$

$$A^T y + c = 0$$

$$s_i y_i = \tau \longleftarrow \text{Same } \tau \text{ for every pair}$$

$$s, y \geq 0$$

Same optimality conditions for a “smoothed” version of our problem

Central path

$$\begin{aligned} &\text{minimize} && c^T x - \tau \sum_{i=1}^m \log(s_i) \\ &\text{subject to} && Ax + s = b \end{aligned}$$

Set of points $(x^*(\tau), s^*(\tau), y^*(\tau))$
with $\tau > 0$ such that

$$Ax + s - b = 0$$

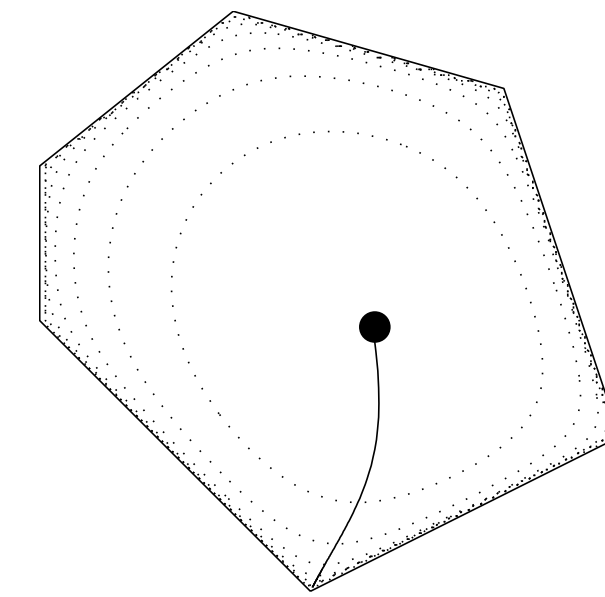
$$A^T y + c = 0$$

$$s_i y_i = \tau$$

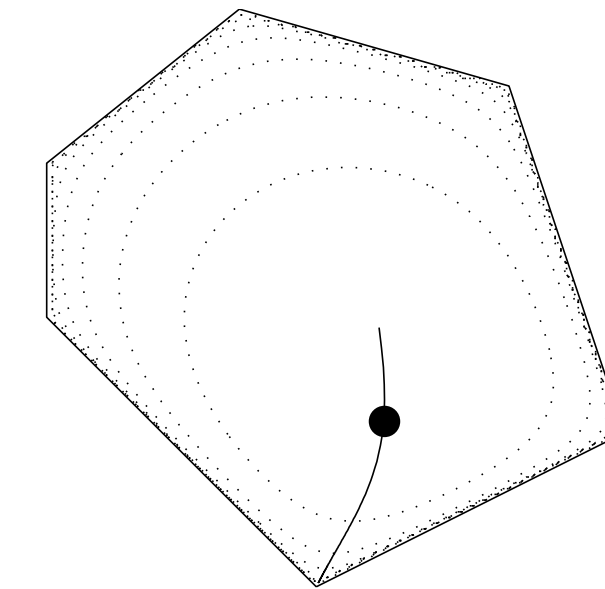
$$s, y \geq 0$$

**Analytic
Center**

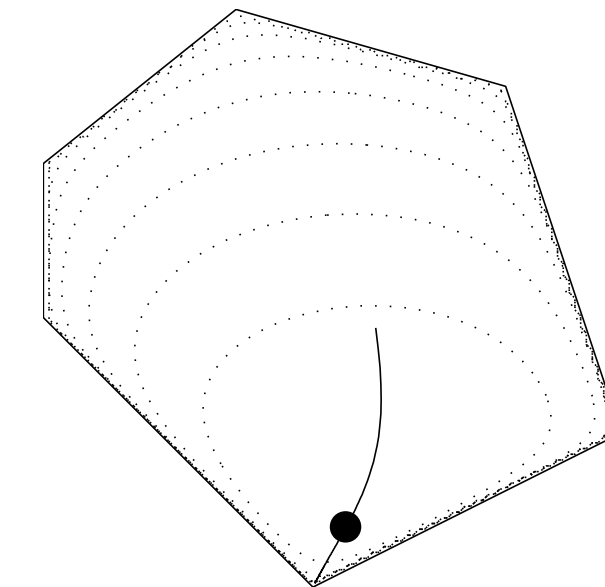
$$\tau \rightarrow \infty$$



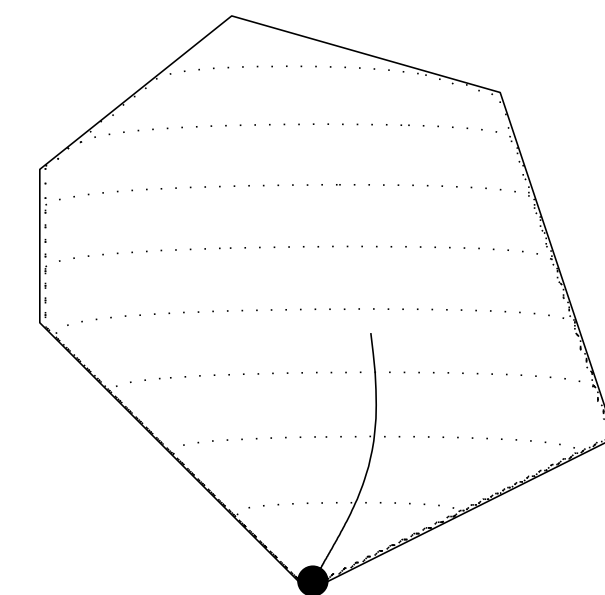
1000



1



1/5



1/100

τ

49

Main idea

Follow central path as $\tau \rightarrow 0$

Newton's method for smoothed optimality conditions

Smoothed optimality conditions

$$h_\tau(x, s, y) = \begin{bmatrix} Ax + s - b \\ A^T y + c \\ SY\mathbf{1} - \tau\mathbf{1} \end{bmatrix} = 0$$
$$s, y \geq 0$$

Linear system

$$\begin{bmatrix} 0 & A & I \\ A^T & 0 & 0 \\ S & 0 & Y \end{bmatrix} \begin{bmatrix} \Delta y \\ \Delta x \\ \Delta s \end{bmatrix} = \begin{bmatrix} -r_p \\ -r_d \\ -SY + \tau\mathbf{1} \end{bmatrix}$$

Line search to enforce $x, s > 0$

$$(x, s, y) \leftarrow (x, s, y) + \alpha(\Delta x, \Delta s, \Delta y)$$

Algorithm step

Linear system

$$\begin{bmatrix} 0 & A & I \\ A^T & 0 & 0 \\ S & 0 & Y \end{bmatrix} \begin{bmatrix} \Delta y \\ \Delta x \\ \Delta s \end{bmatrix} = \begin{bmatrix} -r_p \\ -r_d \\ -SY\mathbf{1} + \sigma\mu\mathbf{1} \end{bmatrix}$$

Duality measure

$$\mu = \frac{s^T y}{m}$$

Centering parameter

$$\sigma \in [0, 1]$$

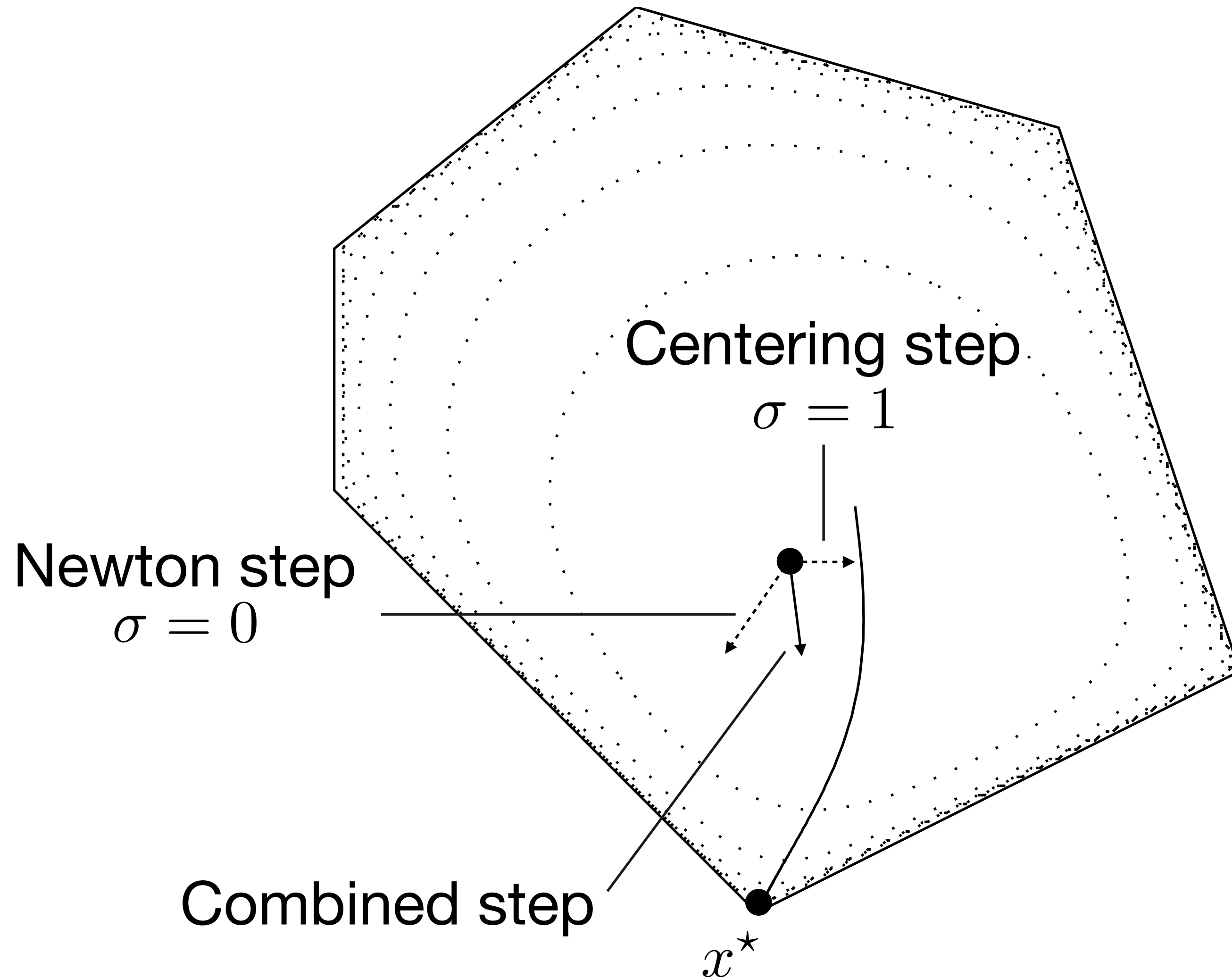
$\sigma = 0 \Rightarrow$ Newton step

$\sigma = 1 \Rightarrow$ Centering step towards $(x^*(\mu), s^*(\mu), y^*(\mu))$

Line search to enforce $x, s > 0$

$$(x, s, y) \leftarrow (x, s, y) + \alpha(\Delta x, \Delta s, \Delta y)$$

Path-following algorithm idea



Centering step

Moves towards the **central path** and is usually biased towards $s, y > 0$.
No progress on duality measure μ

Newton step

Moves towards the **zero duality measure** μ . Quickly violates $s, y > 0$.

Combined step

Best of both, with longer steps.

Choosing the centering parameter

Newton direction

$$(\Delta x_a, \Delta s_a, \Delta y_a)$$

- The Newton step might quickly hit nonnegativity.
- The centering step might not reduce complementary condition.

Maximum step-size

$$\alpha_p = \max\{\alpha \in [0, 1] \mid s + \alpha \Delta s_a \geq 0\}$$

$$\alpha_d = \max\{\alpha \in [0, 1] \mid y + \alpha \Delta y_a \geq 0\}$$

Centering parameter heuristic (after a Newton step)

$$\mu_a = \frac{(s + \alpha_p \Delta s_a)^T (y + \alpha_d \Delta y_a)}{m} \implies \sigma = \left(\frac{\mu_a}{\mu} \right)^3$$

Mehrotra predictor-corrector algorithm

Initialization

Given (x, s, y) such that $s, y > 0$

1. Termination conditions

$$r_p = Ax + s - b, \quad r_d = A^T y + c, \quad \mu = (s^T y)/m$$

If $\|r_p\|, \|r_d\|, \mu$ are small, **break** Optimal solution (x^*, s^*, y^*)

2. Newton step (affine scaling)

$$\begin{bmatrix} 0 & A & I \\ A^T & 0 & 0 \\ S & 0 & Y \end{bmatrix} \begin{bmatrix} \Delta y_a \\ \Delta x_a \\ \Delta s_a \end{bmatrix} = \begin{bmatrix} -r_p \\ -r_d \\ -SY\mathbf{1} \end{bmatrix}$$

Mehrotra predictor-corrector algorithm

3. Centering parameter

$$\alpha_p = \max\{\alpha \in [0, 1] \mid s + \alpha\Delta s_a \geq 0\}$$

$$\alpha_d = \max\{\alpha \in [0, 1] \mid y + \alpha\Delta y_a \geq 0\}$$

$$\mu_a = \frac{(s + \alpha_p\Delta s_a)^T (y + \alpha_d\Delta y_a)}{m}$$

$$\sigma = \left(\frac{\mu_a}{\mu}\right)^3$$

4. Corrected direction

$$\begin{bmatrix} 0 & A & I \\ A^T & 0 & 0 \\ S & 0 & Y \end{bmatrix} \begin{bmatrix} \Delta y \\ \Delta x \\ \Delta s \end{bmatrix} = \begin{bmatrix} -r_p \\ -r_d \\ -SY\mathbf{1} - \Delta S_a \Delta Y_a \mathbf{1} + \sigma\mu\mathbf{1} \end{bmatrix}$$

Linearization correction

Centering heuristic

Mehrotra predictor-corrector algorithm

5. Update iterates

$$\alpha_p = \max\{\alpha \geq 0 \mid s + \alpha\Delta s \geq 0\}$$

$$\alpha_d = \max\{\alpha \geq 0 \mid y + \alpha\Delta y \geq 0\}$$

$$(x, s) = (x, s) + \min\{1, \eta\alpha_p\}(\Delta x, \Delta s)$$

$$y = y + \min\{1, \eta\alpha_d\}\Delta y$$

Avoid corners

$$\eta = 1 - \epsilon \approx 0.99$$

Convergence

Mehrotra's algorithm

No convergence theory \longrightarrow Examples where it **diverges** (rare!)

Fantastic convergence **in practice** \longrightarrow Fewer than 30 iterations

Theoretical iteration complexity

Alternative versions (slower than Mehrotra) converge in $O(\sqrt{n})$ iterations



Operations

$$O(n^{3.5})$$

Average iteration complexity

Average iterations complexity is $O(\log n)$




$$O(n^3 \log n)$$


Interior-point vs simplex

Comparison between interior-point method and simplex


Primal simplex

- Primal feasibility
- 
- Zero duality gap
 - Dual feasibility

Dual simplex

- Dual feasibility
- 
- Zero duality gap
 - Primal feasibility

Primal-dual interior-point

- Interior condition
- 
- Primal feasibility
 - Dual feasibility
 - Zero duality gap

Exponential worst-case complexity

Requires feasible point

Can be warm-started

Polynomial worst-case complexity

Allows infeasible start

Cannot be warm-started

Which algorithm should I use?

Dual simplex

- Small-to-medium problems
- Repeated solves with varying constraints

Interior-point (barrier)

- Medium-to-large problems
- Sparse structured problems

How do solvers with multiple options decide?

Concurrent Optimization

Why not both? (crossover)

Interior-point \longrightarrow Few simplex steps

Average simplex complexity

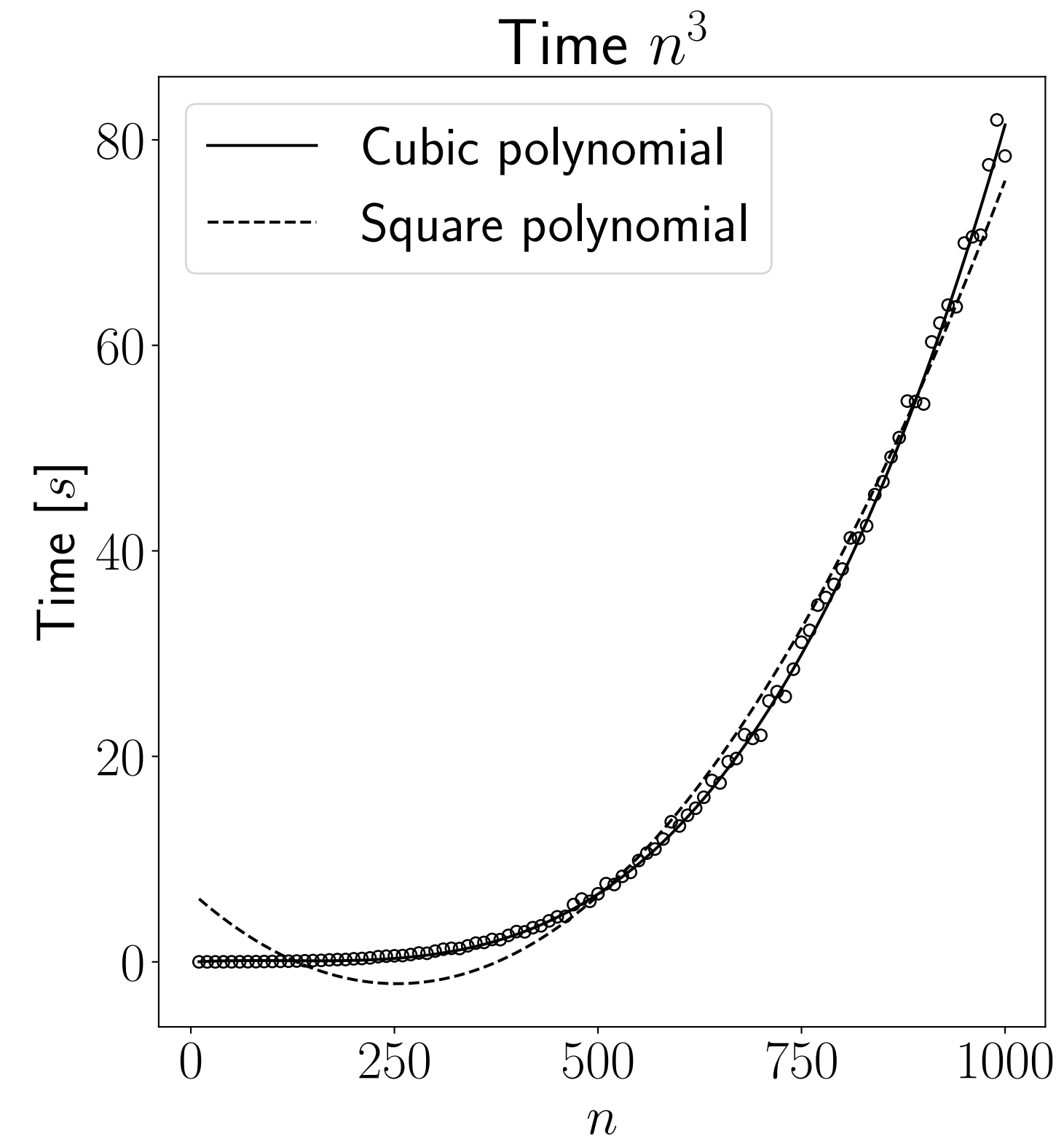
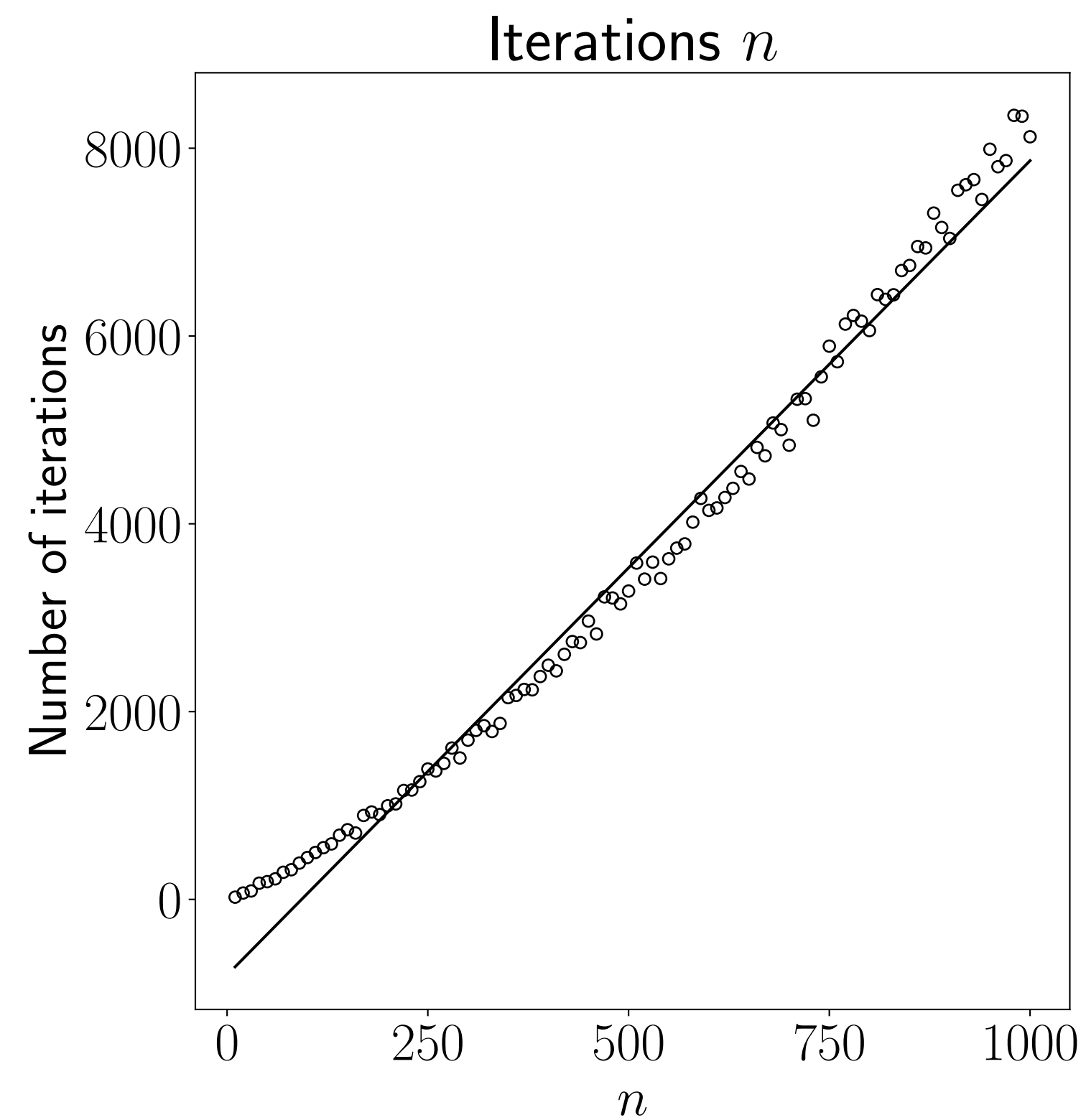
Random LPs

minimize $c^T x$

n variables

subject to $Ax \leq b$

$3n$ constraints



Average interior-point complexity

Random LPs

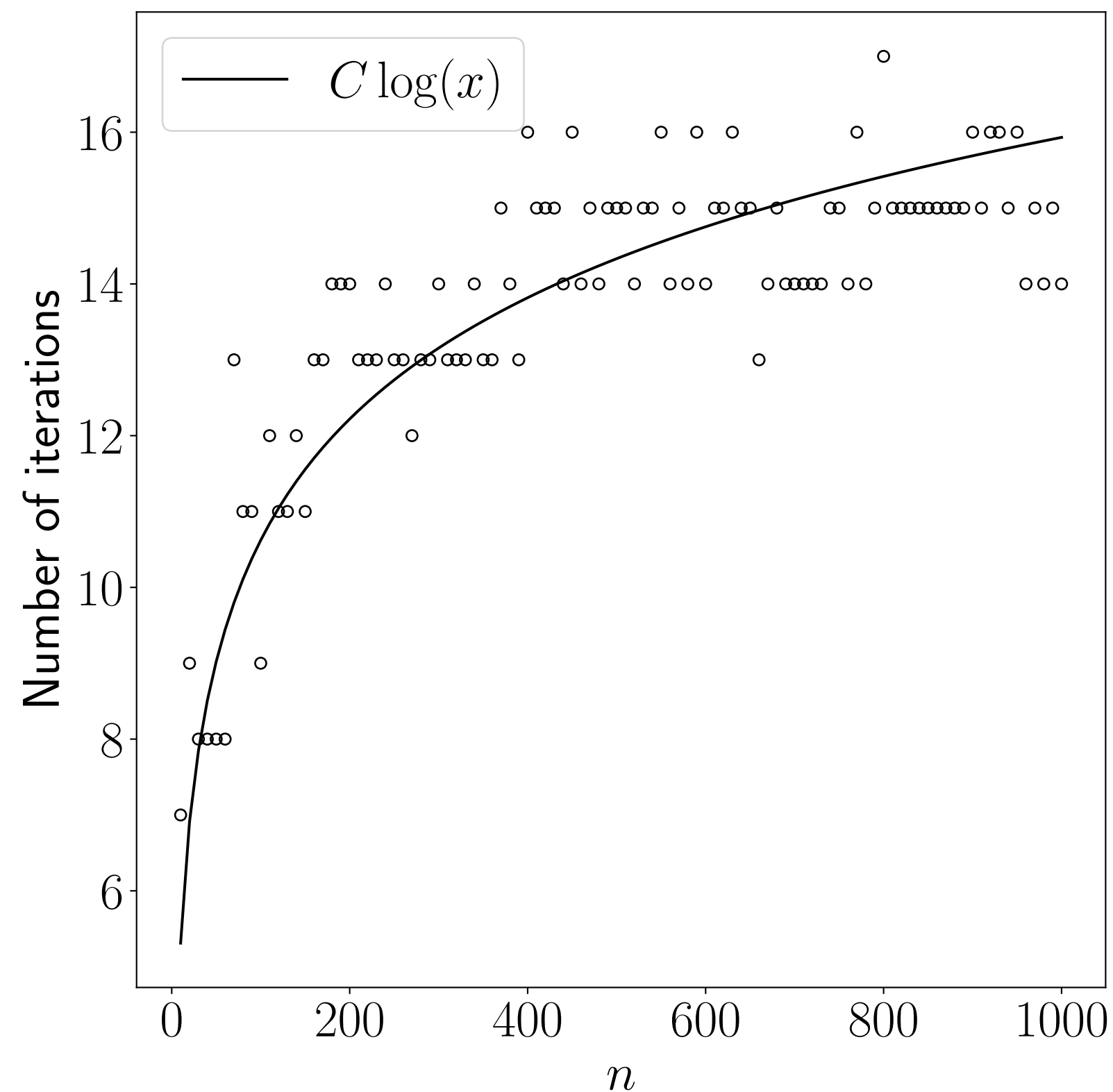
minimize $c^T x$

n variables

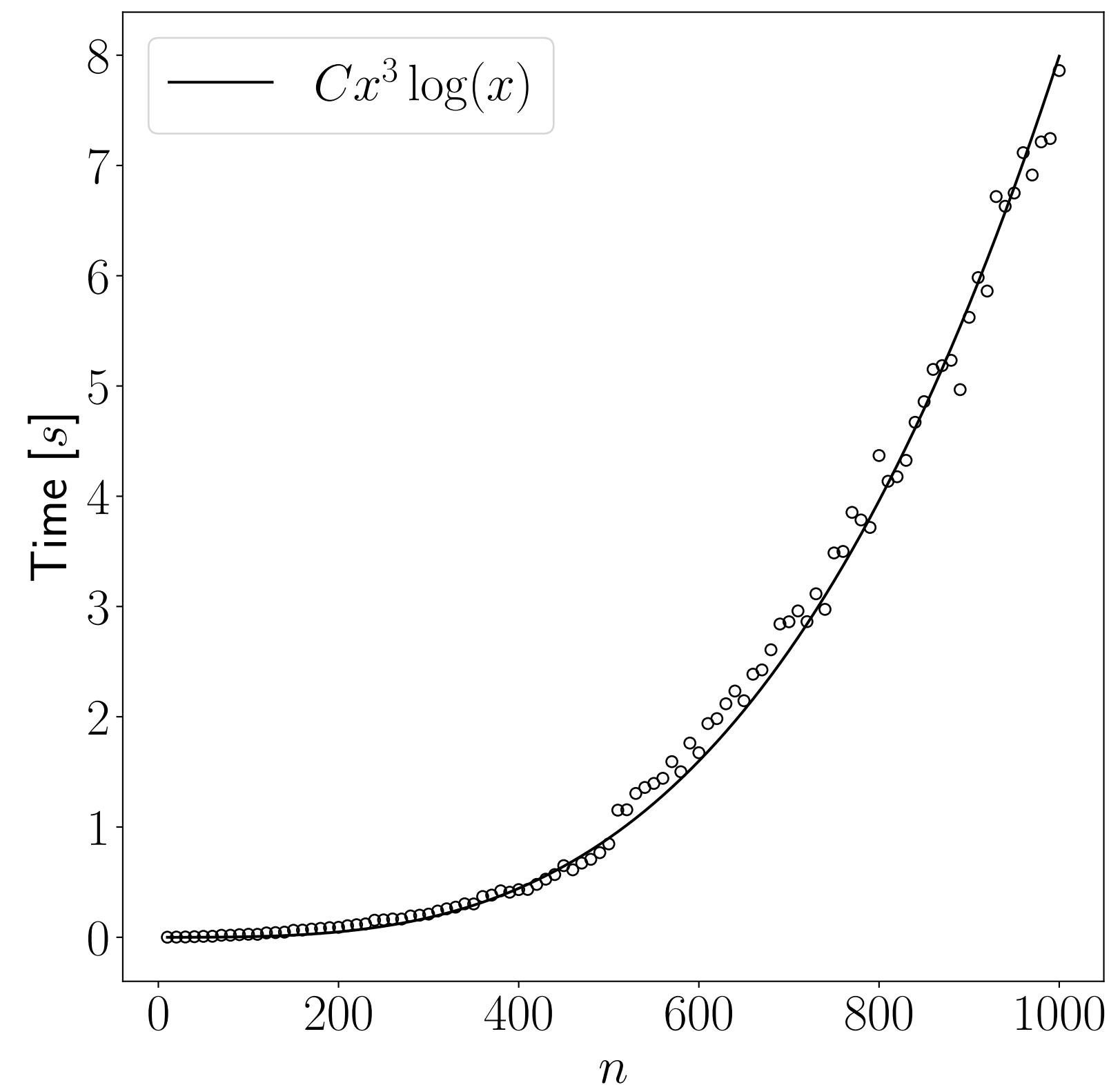
subject to $Ax \leq b$

$3n$ constraints

Iterations: $O(\log n)$



Time: $O(n^3 \log n)$



Questions

Next lecture

- Integer optimization