ORF307 – Optimization

18. Interior-point methods II

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Ed Forum

- **2nd Midterm: April 18**
  Time: 11:00am — 12:20pm
  Students with extensions please reach out to me
  Location: Same room as lecture
  Topics: linear optimization
  Material allowed: Single sheet of paper. Double sided. Hand-written or typed.
  Exercises to prepare: past midterm + extra exercises on canvas

- **Questions**

  - How are tau, sigma, and mu related?

  - I was still a little confused by SY1. Why do we need to include it in the matrix?

Recap

(Sparse) Cholesky factorization

Every positive definite matrix A can be factored as

$$A = PLL^TP^T \longrightarrow P^TAP = LL^T$$

P permutation, L lower triangular

Permutations

- Reorder rows/cols of A with P to (heuristically) get **sparser** L
- P depends only on sparsity pattern of A (unlike LU factorization)
- If A is dense, we can set $P = I$

Cost

- If A dense, typically $O(n^3)$ but usually much less
- It depends on the number of nonzeros in A, sparsity pattern, etc.
- Typically 50% faster than LU (need to find only one matrix)

Linear optimization as a root finding problem

Optimality conditions

Primal | **Dual**

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax \leq b
\end{array}
\longrightarrow
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax + s = b \\
& s \geq 0
\end{array}
\qquad
\begin{array}{ll}
\text{maximize} & -b^T y \\
\text{subject to} & A^T y + c = 0 \\
& y \geq 0
\end{array}
$$

KKT conditions

$$
\begin{aligned}
Ax + s - b &= 0 \\
A^T y + c &= 0 \\
s_i y_i &= 0, \quad i = 1, \dots, m \\
s, y &\geq 0
\end{aligned}
$$

Linear optimization as a root finding problem

$$
\begin{aligned}
Ax + s - b &= 0\\
A^T y + c &= 0\\
s_i y_i &= 0, \quad i = 1, \dots, m\\
s, y &\geq 0
\end{aligned}
$$

Diagonalize complementary slackness

$$S = \mathbf{diag}(s) = \begin{bmatrix} s_1 & & & \\ & s_2 & & \\ & & \ddots & \\ & & & s_m \end{bmatrix}$$

$$Y = \mathbf{diag}(y) = \begin{bmatrix} y_1 & & & \\ & y_2 & & \\ & & \ddots & \\ & & & y_m \end{bmatrix}$$

$$SY\mathbf{1} = \mathbf{diag}(s)\mathbf{diag}(y)\mathbf{1} = \begin{bmatrix} s_1y_1 & & & \\ & s_2y_2 & & \\ & & \ddots & \\ & & & s_my_m \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} s_1y_1 \\ s_2y_2 \\ \vdots \\ s_my_m \end{bmatrix}$$

$$\downarrow$$

$$s_i y_i = 0, \quad i = 1, \dots, m \quad \iff \quad SY\mathbf{1} = 0$$

Main idea

Optimality conditions

$$h(y, x, s) = \begin{bmatrix} Ax + s - b \\ A^T y + c \\ SY\mathbf{1} \end{bmatrix} = \begin{bmatrix} r_p \\ r_d \\ SY\mathbf{1} \end{bmatrix} = 0$$

$$s, y \geq 0$$

$$S = \mathbf{diag}(s)$$
$$Y = \mathbf{diag}(y)$$

- Apply variants of Newton's method to solve $h(x, s, y) = 0$
- Enforce $s, y > 0$ (strictly) at every iteration
- **Motivation** avoid getting stuck in “corners”

Smoothed optimality conditions

Optimality conditions

$$
\begin{aligned}
Ax + s - b &= 0 \\
A^T y + c &= 0 \\
s_i y_i &= \boxed{\tau} \\
s, y &\geq 0
\end{aligned}
$$

Same τ for every pair

Same optimality conditions for a “smoothed” version of our problem

Duality gap

$$s^T y = (b - Ax)^T y = b^T x - x^T A^T y = b^T y + c^T x$$

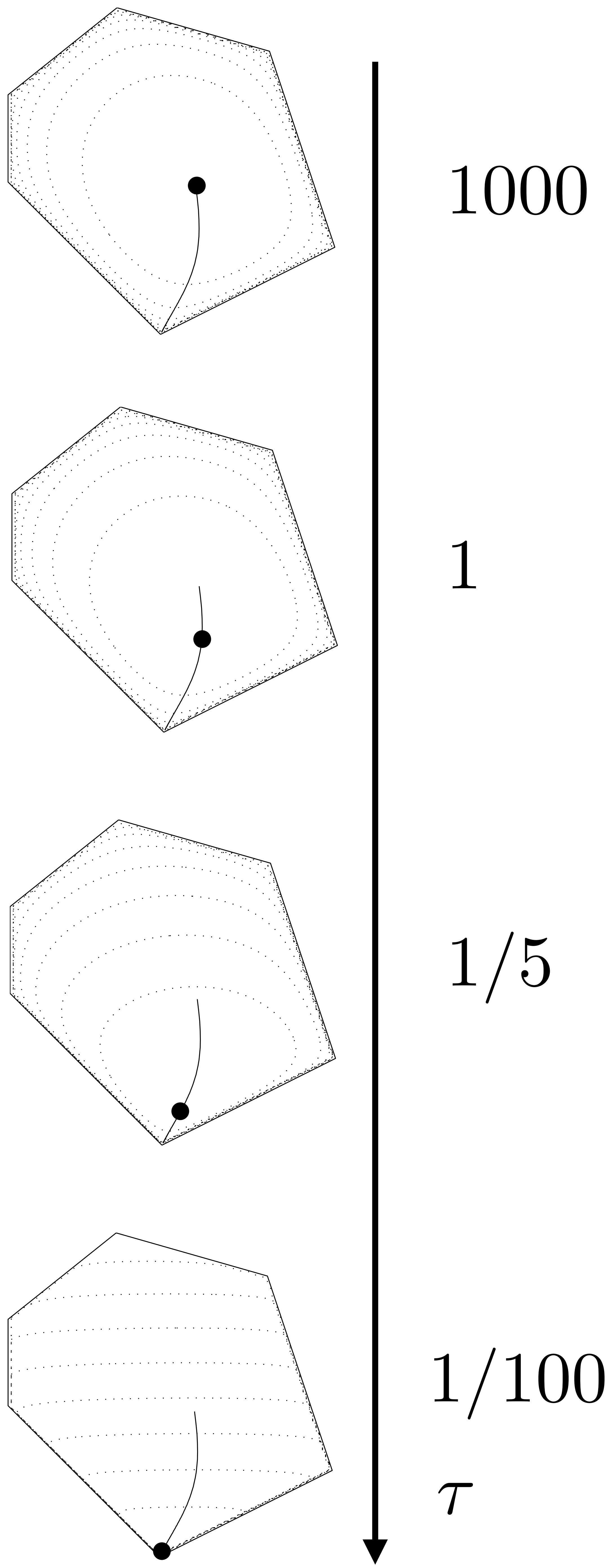
Central path

minimize $\quad c^T x - \tau \sum_{i=1}^{m} \log(s_i)$
subject to $\quad Ax + s = b$

Set of points $(x^\star(\tau), s^\star(\tau), y^\star(\tau))$
with $\tau > 0$ such that

$$
\begin{aligned}
Ax + s - b &= 0 \\
A^T y + c &= 0 \\
s_i y_i &= \tau \\
s, y &\geq 0
\end{aligned}
$$

Analytic Center
$\tau \to \infty$



Main idea

Follow central path as $\tau \to 0$

The path parameter

Duality measure

$$\mu = \frac{s^T y}{m}$$ (average value of the pairs $s_i y_i$)

Linear system

$$\begin{bmatrix} 0 & A & I \\ A^T & 0 & 0 \\ S & 0 & Y \end{bmatrix} \begin{bmatrix} \Delta y \\ \Delta x \\ \Delta s \end{bmatrix} = \begin{bmatrix} -r_p \\ -r_d \\ -SY\mathbf{1} + \sigma\mu\mathbf{1} \end{bmatrix}$$

Centering parameter

$\sigma \in [0, 1]$

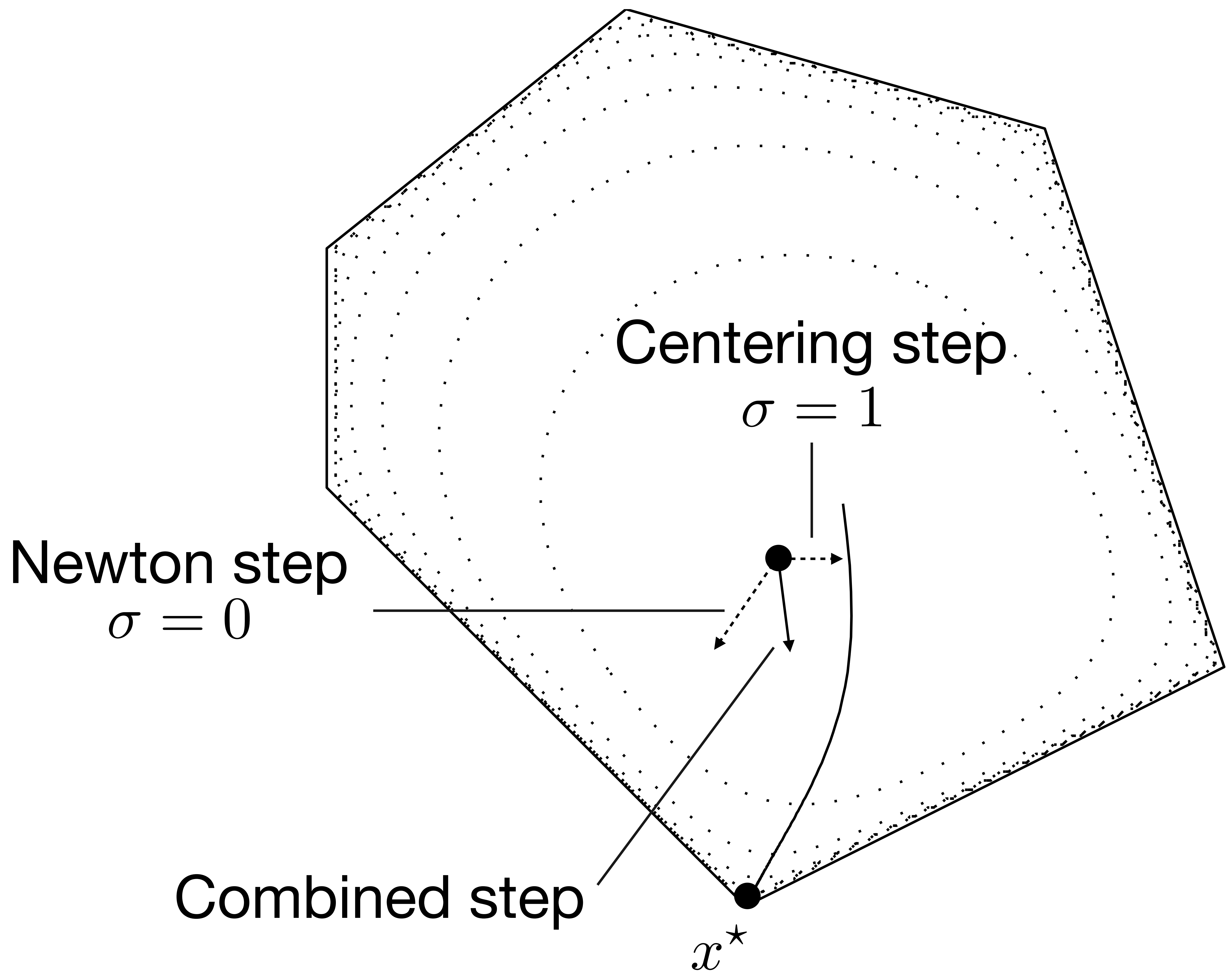
$\sigma = 0 \quad \Rightarrow$ Newton step

$\sigma = 1 \quad \Rightarrow$ Centering step towards $(y^\star(\mu), x^\star(\mu), s^\star(\mu))$

Line search to enforce $s, y > 0$

$$(y, x, s) \leftarrow (y, x, s) + \alpha(\Delta y, \Delta x, \Delta s)$$

Path-following algorithm idea

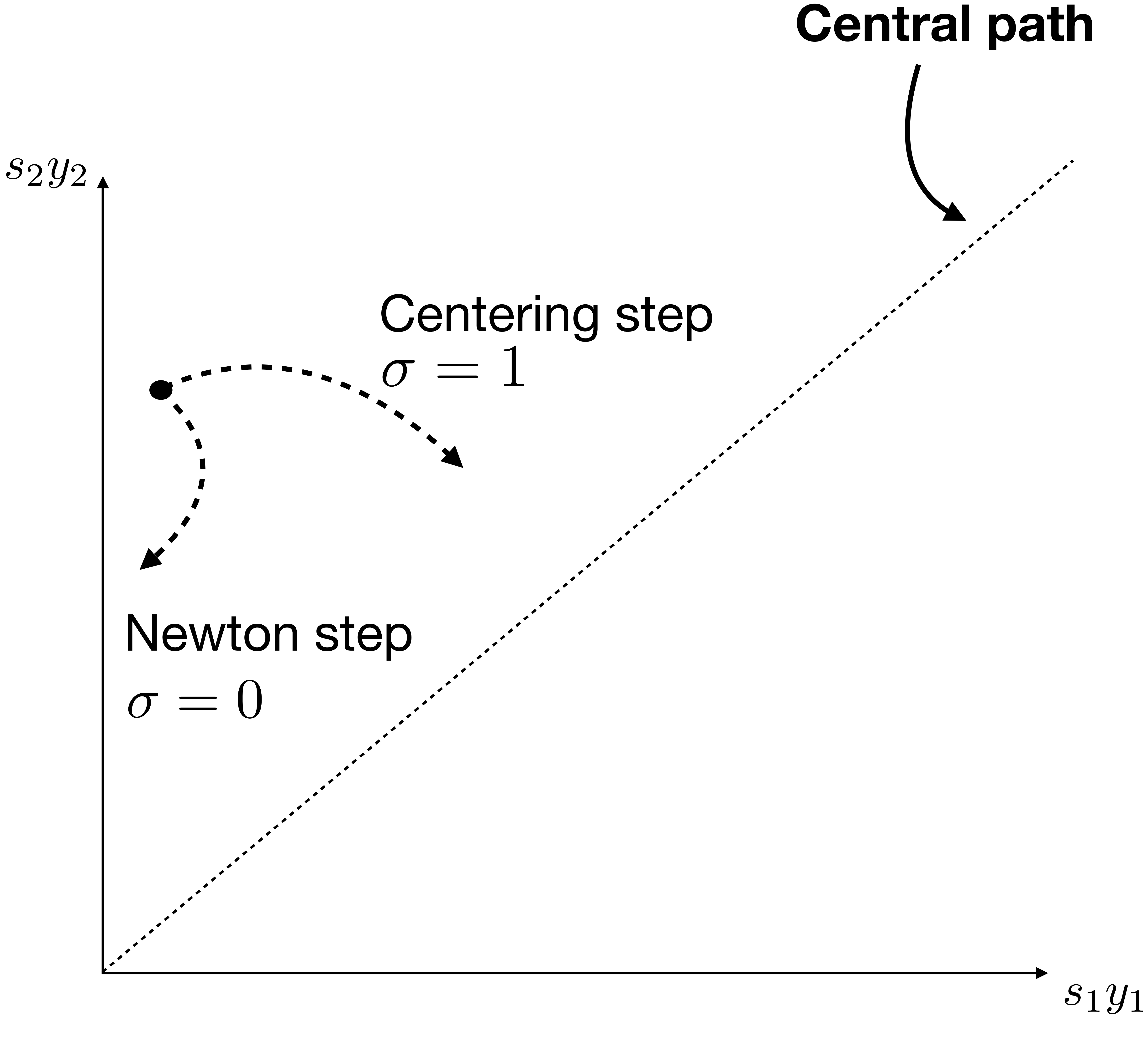


Centering step
It brings towards the **central path** and is usually biased towards $s, y > 0$. **No progress** on duality measure μ

Newton step
It brings towards the **zero duality measure** μ. Quickly violates $s, y > 0$.

Combined step
Best of both worlds with longer steps

Path-following algorithm idea



Centering step
It brings towards the **central path** and is usually biased towards $s, y > 0$. **No progress** on duality measure μ

Newton step
It brings towards the **zero duality measure** μ. Quickly violates $s, y > 0$.

Combined step
Best of both worlds with longer steps

Primal-dual path-following algorithm

Initialization

1. Given (x_0, s_0, y_0) such that $s_0, y_0 > 0$

Iterations

1. Choose $\sigma \in [0, 1]$
2. Solve $\begin{bmatrix} 0 & A & I \\ A^T & 0 & 0 \\ S & 0 & Y \end{bmatrix} \begin{bmatrix} \Delta y \\ \Delta x \\ \Delta s \end{bmatrix} = \begin{bmatrix} -r_p \\ -r_d \\ -SY\mathbf{1} + \sigma\mu\mathbf{1} \end{bmatrix}$ where $\mu = s^T y / m$
3. Find maximum α such that $y + \alpha\Delta y > 0$ and $s + \alpha\Delta s > 0$
4. Update $(y, x, s) \leftarrow (y, x, s) + \alpha(\Delta y, \Delta x, \Delta s)$

Today's lecture

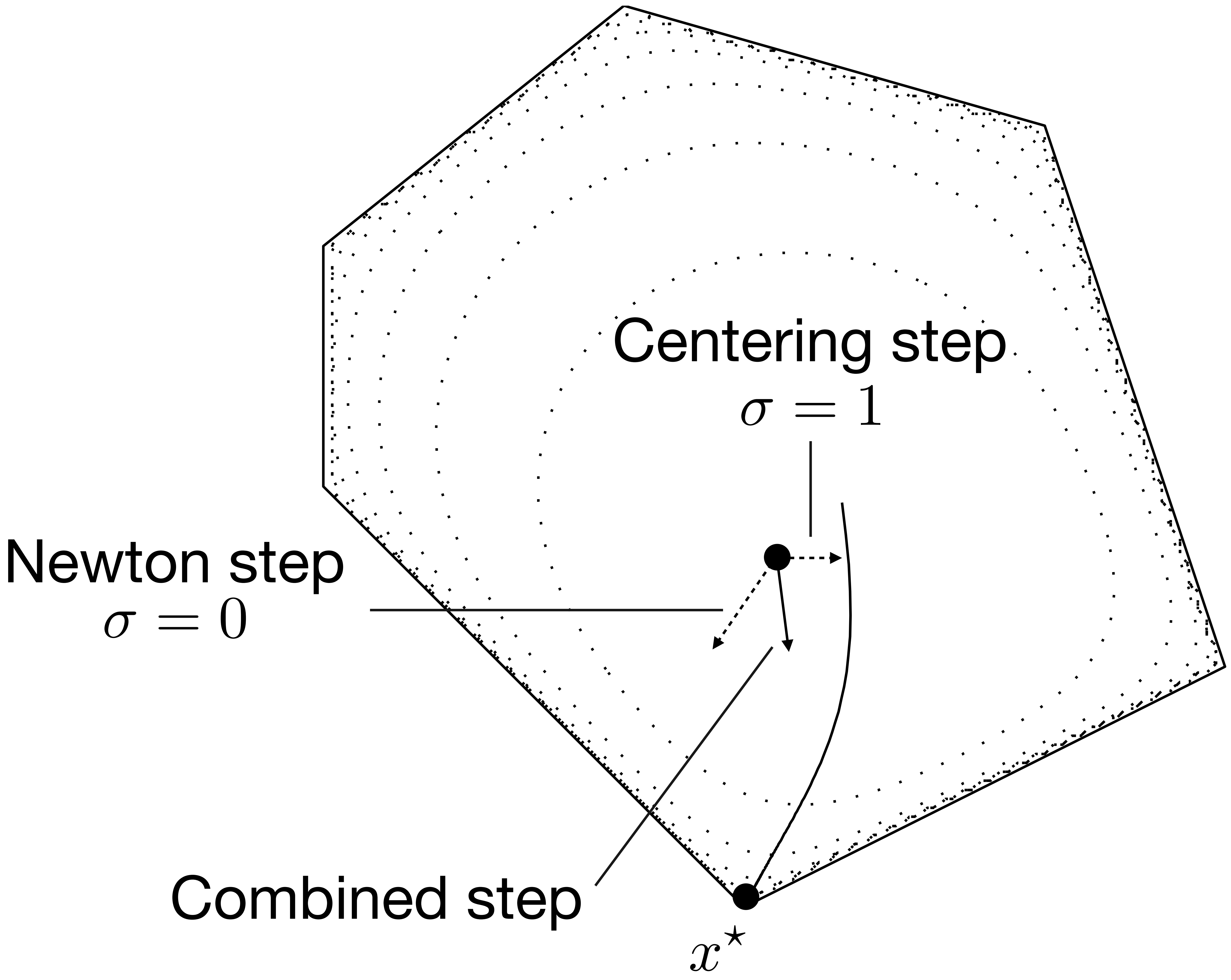
Interior-point methods II

- Mehrotra predictor-corrector algorithm
- Implementation and linear algebra
- Interior-point vs simplex

Predictor-corrector algorithm

Main idea

Predict and select centering parameter



Predict

Compute Newton direction

Estimate

How good is the Newton step?
(how much can μ decrease?)

Select centering parameter

Very roughly:
Pick $\sigma \approx 0$ if Newton step is good
Pick $\sigma \approx 1$ if Newton step is bad

How good is the Newton step?

Newton step

$$(\Delta x_a, \Delta s_a, \Delta y_a)$$

Maximum step-size

$$\alpha_p = \max\{\alpha \in [0,1] \mid s + \alpha \Delta s_a \geq 0\}$$
$$\alpha_d = \max\{\alpha \in [0,1] \mid y + \alpha \Delta y_a \geq 0\}$$

Two issues

- The new points will not produce much improvement:
  $(s + \alpha_p \Delta s_a)_i (y + \alpha_d \Delta y_a)_i$ much larger than 0
- The complementarity error depends on step lengths α_p and α_d

Choosing a centering parameter to make good improvement

Newton step

$$(\Delta x_a, \Delta s_a, \Delta y_a)$$

Maximum step-size

$$\alpha_p = \max\{\alpha \in [0,1] \mid s + \alpha \Delta s_a \geq 0\}$$
$$\alpha_d = \max\{\alpha \in [0,1] \mid y + \alpha \Delta y_a \geq 0\}$$

Duality measure candidate
(after Newton step)

$$\mu_a = \frac{(s + \alpha_p \Delta s_a)^T (y + \alpha_d \Delta y_a)}{m}$$

$\longrightarrow$

Centering parameter heuristic σ

$$\sigma = \left(\frac{\mu_a}{\mu}\right)^3$$

Correcting for complementary error

Newton step

$$\begin{bmatrix} 0 & A & I \\ A^T & 0 & 0 \\ S & 0 & Y \end{bmatrix} \begin{bmatrix} \Delta y_a \\ \Delta x_a \\ \Delta s_a \end{bmatrix} = \begin{bmatrix} -r_p \\ -r_d \\ -SY\mathbf{1} \end{bmatrix} \longrightarrow \quad s_i(\Delta y_a)_i + y_i(\Delta s_a)_i + s_i y_i = 0$$

Complementarity error

$$(s_i + (\Delta s_a)_i)(y_i + (\Delta y_a)_i) = (\Delta s_a)_i(\Delta y_a)_i \neq 0$$

Complementarity violation
depends on step length

Corrected direction

$$\begin{bmatrix} 0 & A & I \\ A^T & 0 & 0 \\ S & 0 & Y \end{bmatrix} \begin{bmatrix} \Delta y \\ \Delta x \\ \Delta s \end{bmatrix} = \begin{bmatrix} -r_p \\ -r_d \\ -SY\mathbf{1} - \Delta S_a \Delta Y_a \mathbf{1} + \sigma\mu\mathbf{1} \end{bmatrix}$$

$$\Delta S_a = \mathbf{diag}(\Delta s_a)$$
$$\Delta Y_a = \mathbf{diag}(\Delta y_a)$$

Mehrotra predictor-corrector algorithm

Initialization

Given (x, s, y) such that $s, y > 0$

1. **Termination conditions**

   $$r_p = Ax + s - b, \quad r_d = A^T y + c, \quad \mu = (s^T y)/m$$

   If $\|r_p\|, \|r_d\|, \mu$ are small, **break** Optimal solution $(x^\star, s^\star, y^\star)$

2. **Newton step** (affine scaling)

   $$\begin{bmatrix} 0 & A & I \\ A^T & 0 & 0 \\ S & 0 & Y \end{bmatrix} \begin{bmatrix} \Delta y_a \\ \Delta x_a \\ \Delta s_a \end{bmatrix} = \begin{bmatrix} -r_p \\ -r_d \\ -SY\mathbf{1} \end{bmatrix}$$

Mehrotra predictor-corrector algorithm

3. **Barrier parameter**

$$\alpha_p = \max\{\alpha \in [0,1] \mid s + \alpha \Delta s_a \geq 0\}$$

$$\alpha_d = \max\{\alpha \in [0,1] \mid y + \alpha \Delta y_a \geq 0\}$$

$$\mu_a = \frac{(s + \alpha_p \Delta s_a)^T (y + \alpha_d \Delta y_a)}{m}$$

$$\sigma = \left(\frac{\mu_a}{\mu}\right)^3$$

4. **Corrected direction**

$$\begin{bmatrix} 0 & A & I \\ A^T & 0 & 0 \\ S & 0 & Y \end{bmatrix} \begin{bmatrix} \Delta y \\ \Delta x \\ \Delta s \end{bmatrix} = \begin{bmatrix} -r_p \\ -r_d \\ -SY\mathbf{1} - \Delta S_a \Delta Y_a \mathbf{1} + \sigma\mu\mathbf{1} \end{bmatrix}$$

Mehrotra predictor-corrector algorithm

5. **Update iterates**

$$\alpha_p = \max\{\alpha \geq 0 \mid s + \alpha \Delta s \geq 0\}$$
$$\alpha_d = \max\{\alpha \geq 0 \mid y + \alpha \Delta y \geq 0\}$$

$$(x, s) = (x, s) + \min\{1, \eta\alpha_p\}(\Delta x, \Delta s)$$
$$y = y + \min\{1, \eta\alpha_d\}\Delta y$$

Avoid corners

$$\eta = 1 - \epsilon \approx 0.99$$

Implementation and linear algebra

Search equations

Step 2 (**Newton**) and 4 (**Corrected direction**) solve equations of the form

$$\begin{bmatrix} 0 & A & I \\ A^T & 0 & 0 \\ S & 0 & Y \end{bmatrix} \begin{bmatrix} \Delta y \\ \Delta x \\ \Delta s \end{bmatrix} = \begin{bmatrix} b_y \\ b_x \\ b_s \end{bmatrix}$$

The **Newton** step right hand side: $$\begin{bmatrix} b_y \\ b_x \\ b_s \end{bmatrix} = \begin{bmatrix} -r_p \\ -r_d \\ -SY\mathbf{1} \end{bmatrix}$$

The **corrector** step right hand side: $$\begin{bmatrix} b_y \\ b_x \\ b_s \end{bmatrix} = \begin{bmatrix} -r_p \\ -r_d \\ -SY\mathbf{1} - \Delta S_a \Delta Y_a \mathbf{1} + \sigma\mu\mathbf{1} \end{bmatrix}$$

Solving the search equations

Our linear system is not symmetric

$$\begin{bmatrix} 0 & A & I \\ A^T & 0 & 0 \\ S & 0 & Y \end{bmatrix} \begin{bmatrix} \Delta y \\ \Delta x \\ \Delta s \end{bmatrix} = \begin{bmatrix} b_y \\ b_x \\ b_s \end{bmatrix}$$

Substitute last equation, $\Delta s = Y^{-1}(b_s - S\Delta y)$, into first

$$\begin{bmatrix} -Y^{-1}S & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} \Delta y \\ \Delta x \end{bmatrix} = \begin{bmatrix} b_y - Y^{-1}b_s \\ b_x \end{bmatrix}$$

Solving the search equations

Our reduced system is symmetric but not positive definite

$$\begin{bmatrix} -Y^{-1}S & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} \Delta y \\ \Delta x \end{bmatrix} = \begin{bmatrix} b_y - Y^{-1}b_s \\ b_x \end{bmatrix}$$

Substitute first equation, $\Delta y = S^{-1}Y(A\Delta x - b_y + Y^{-1}b_s)$, into second

$$A^T S^{-1} Y A \Delta x = b_x + A^T S^{-1} Y b_y - A^T S^{-1} b_s$$

Reduced linear system

Coefficient matrix

$$B = A^T S^{-1} Y A$$

Characteristics

- A is **large** and **sparse**
- $S^{-1}Y$ is **positive** and **diagonal**, different at each iteration
- B is **positive definite** if $\mathrm{rank}(A) = n$
- Sparsity pattern of B is the **pattern** of $A^T A$ (independent of $S^{-1}Y$)

Reduced linear system

Coefficient matrix

$$B = A^T S^{-1} Y A$$

Cholesky factorizations

$$B = PLL^T P^T$$

- Reordering only once to get P
- One numerical factorizaton per interior-point iteration $O(n^3)$ $\longrightarrow$ **Per-iteration complexity** $O(n^3)$
- Forward/backward substitution twice per iteration $O(n^2)$

Convergence

Mehrotra's algorithm

No convergence theory $\longrightarrow$ Examples where it **diverges** (rare!)

Fantastic convergence **in practice** $\longrightarrow$ Less than 30 iterations

Floating point operations

Theoretical iteration complexity

Alternative versions (slower than Mehrotra) converge in $O(\sqrt{n})$ iterations $\longrightarrow$ $O(n^{3.5})$

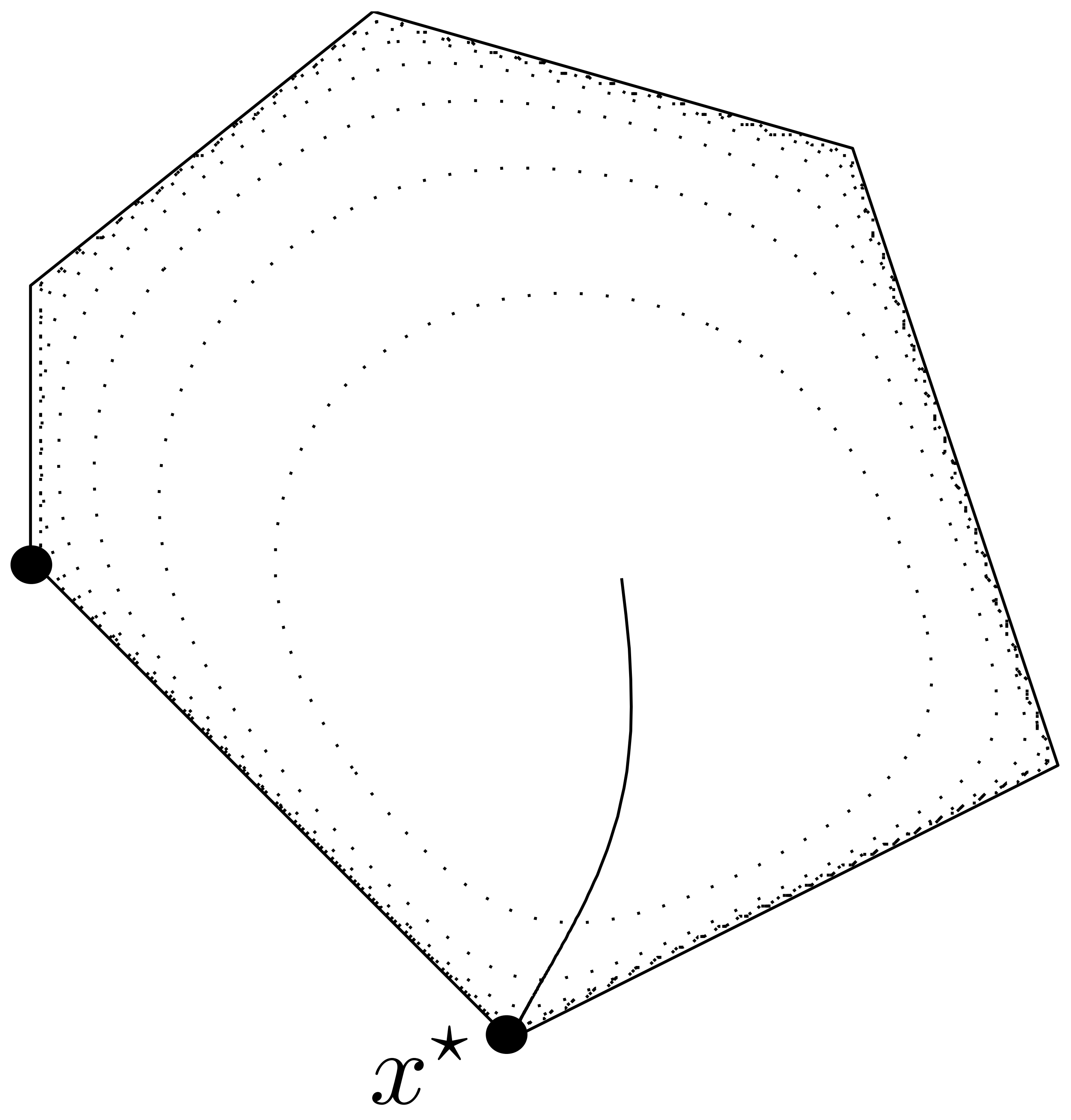
Average iteration complexity

Average iterations complexity is $O(\log n)$ $\longrightarrow$ $O(n^3 \log n)$

Warm-starting

Interior-point methods are **difficult to warm-start**

Previous solution



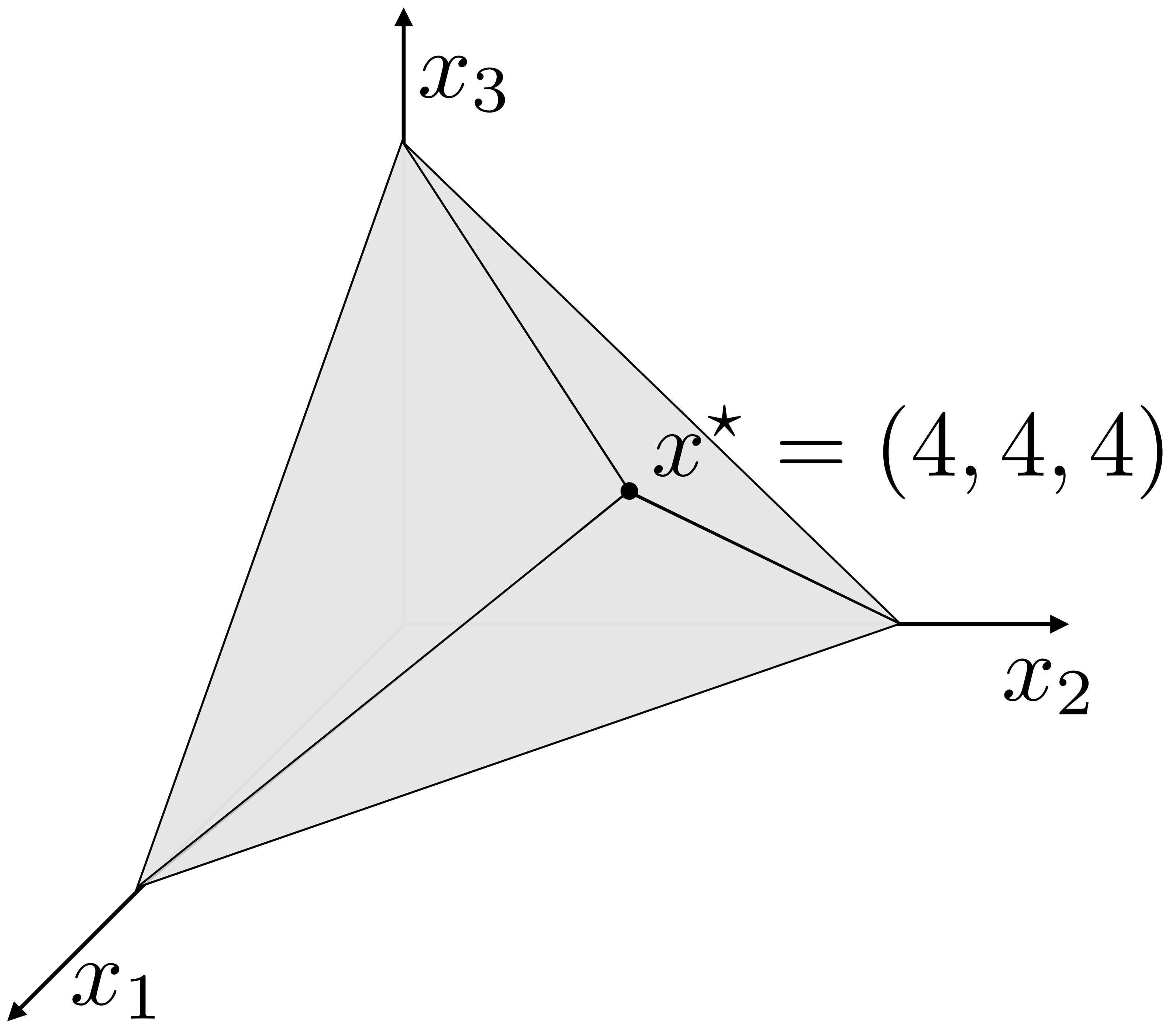
Badly centered
initial point

Hard to make progress
with long steps

Interior-point vs simplex

Example

$$\begin{array}{ll} \text{minimize} & -10x_1 - 12x_2 - 12x_3 \\ \text{subject to} & x_1 + 2x_2 + 2x_3 \leq 20 \\ & 2x_1 + x_2 + x_3 \leq 20 \\ & 2x_1 + 2x_2 + x_3 \leq 20 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$



$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \\ & x \geq 0 \end{array}$$

$$c = (-10, -12, -12)$$

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$b = (20, 20, 20)$$

Example with real solver

CVXOPT (open-source)

Code

```python
import numpy as np
import cvxpy as cp

c = np.array([-10, -12, -12])
A = np.array([[1, 2, 2],
              [2, 1, 2],
              [2, 2, 1]])
b = np.array([20, 20, 20])
n = len(c)

x = cp.Variable(n)
problem = cp.Problem(cp.Minimize(c @ x),
                     [A @ x <= b, x >= 0])
problem.solve(solver=cp.CVXOPT, verbose=True)
```

Output

```
     pcost       dcost       gap    pres   dres   k/t
 0: -1.3077e+02 -2.3692e+02  2e+01  1e-16  6e-01  1e+00
 1: -1.3522e+02 -1.4089e+02  1e+00  2e-16  3e-02  4e-02
 2: -1.3599e+02 -1.3605e+02  1e-02  2e-16  3e-04  4e-04
 3: -1.3600e+02 -1.3600e+02  1e-04  1e-16  3e-06  4e-06
 4: -1.3600e+02 -1.3600e+02  1e-06  1e-16  3e-08  4e-08
Optimal solution found.
```

Solution

```
In [3]: x.value
Out[3]: array([3.99999999, 4.        , 4.        ])
```

[The CVXOPT linear and quadratic cone program solvers, L. Vandenberghe 2010]

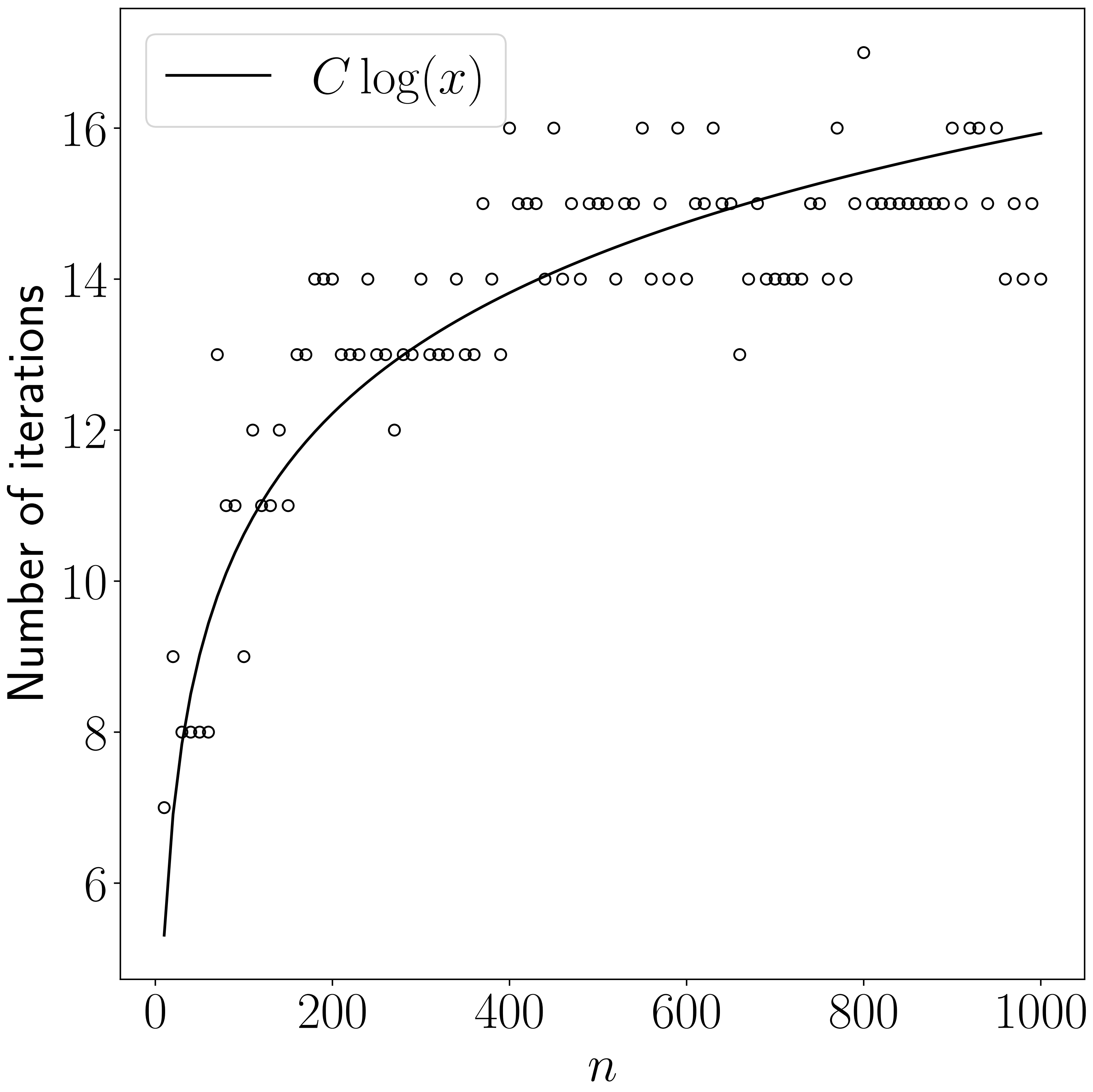
Average interior-point complexity

Random LPs

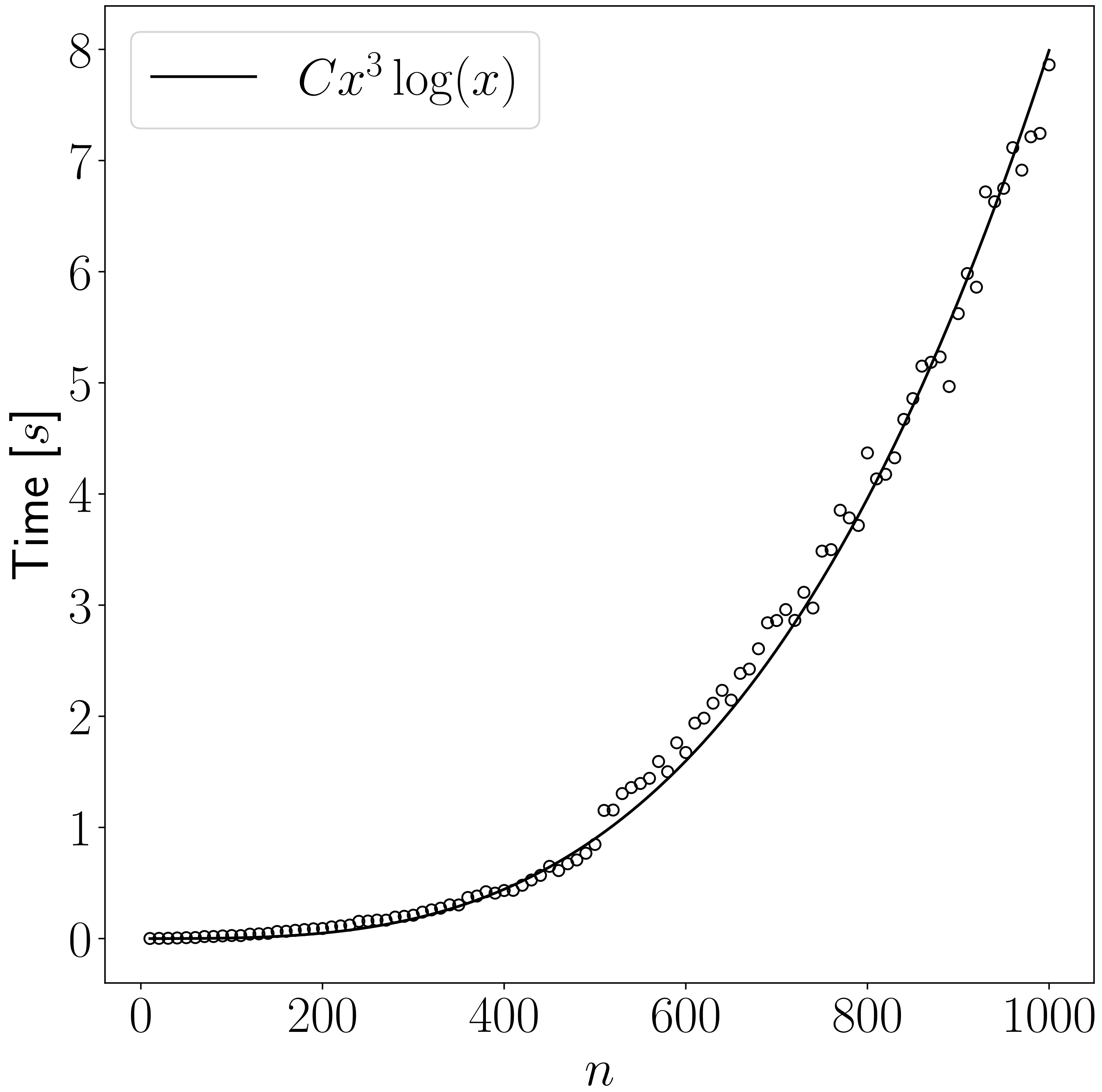
$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

n variables
$3n$ constraints

Iterations: $O(\log n)$



Time: $O(n^3 \log n)$



Comparison between interior-point method and simplex

Primal simplex

- Primal feasibility
- Zero duality gap

↓

Dual feasibility

Dual simplex

- Dual feasibility
- Zero duality gap

↓

Primal feasibility

Primal-dual interior-point

- Interior condition

↓

- Primal feasibility
- Dual feasibility
- Zero duality gap

Simplex	Primal-dual interior-point
Exponential worst-case complexity	**Polynomial worst-case complexity**
Requires feasible point	**Allows infeasible start**
Can be warm-started	**Cannot be warm-started**

Which algorithm should I use?

Dual simplex

- Small-to-medium problems
- Repeated solves with varying data

Interior-point (barrier)

- Medium-to-large problems
- Sparse structured problems

How do solvers with multiple options decide?

<u>Concurrent Optimization</u>

Why not both? (crossover)

Interior-point $\longrightarrow$ Few simplex steps

Interior-point methods implementation

Today, we learned to:

- **Apply** Mehrotra predictor-corrector algorithm
- **Exploit** linear algebra to speedup computations
- **Analyze** empirical complexity
- **Compare** interior-point and simplex methods

References

- D. Bertsimas and J. Tsitsiklis: Introduction to Linear Optimization
  - Chapter 9.4 — 9.6: Interior point methods
- R. Vanderbei: Linear Programming
  - Chapter 17: The Central Path
  - Chapter 15: A Path-Following Method

Next lecture

- Overview for linear optimization