ORF307 – Optimization

17. Interior-point methods

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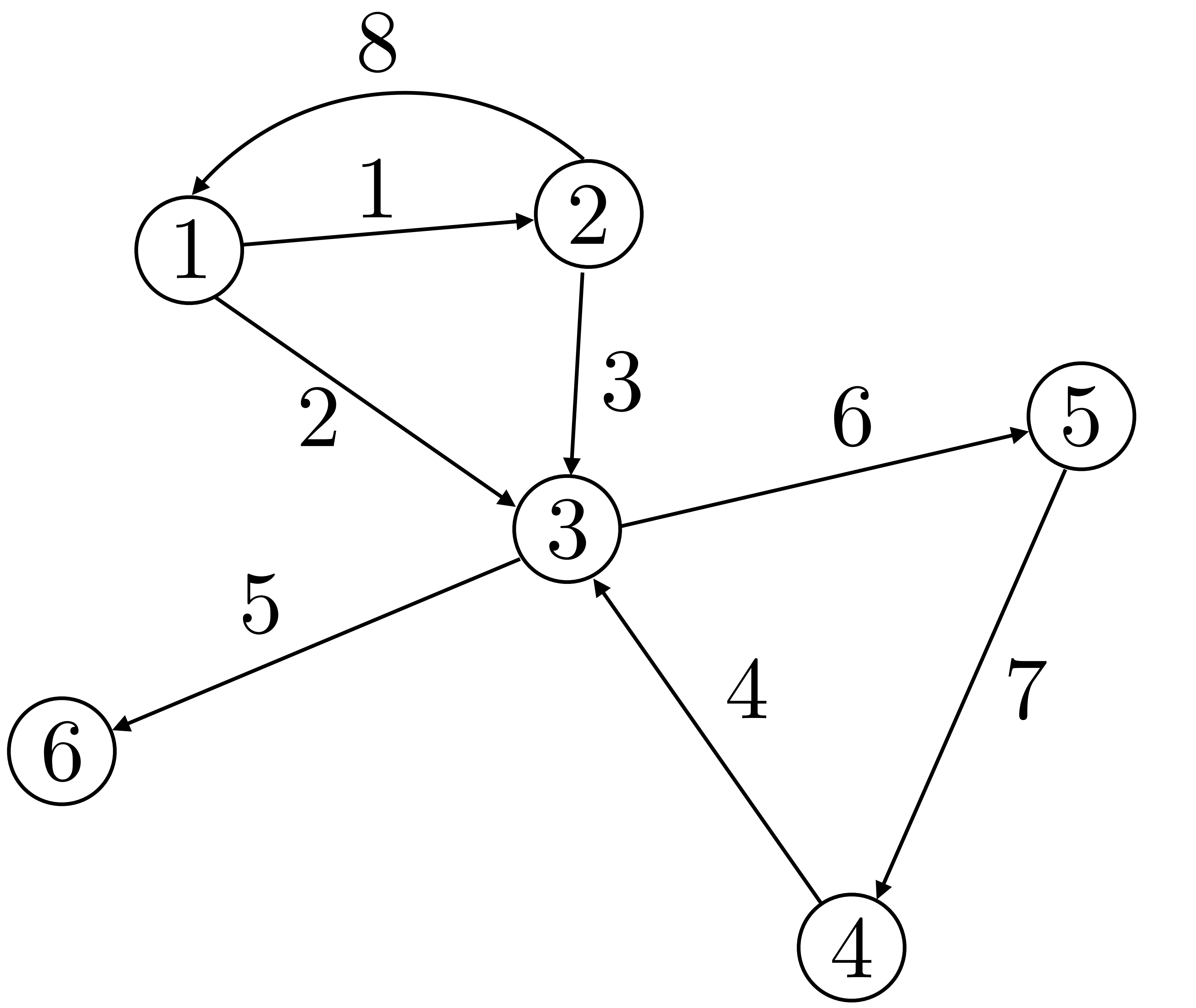
Ed Forum

- I was confused about the meaning of matrix A
- How does solving a min cost / maxflow problem in this way compare to running the Ford Fulkerson algorithm? Are they similar in terms of complexity, or is one better suited than the other in certain cases?

Recap

Arc-node incidence matrix



$m \times n$ matrix A with entries

$$A_{ij} = \begin{cases} 1 & \text{if arc } j \text{ starts at node } i \\ -1 & \text{if arc } j \text{ ends at node } i \\ 0 & \text{otherwise} \end{cases}$$

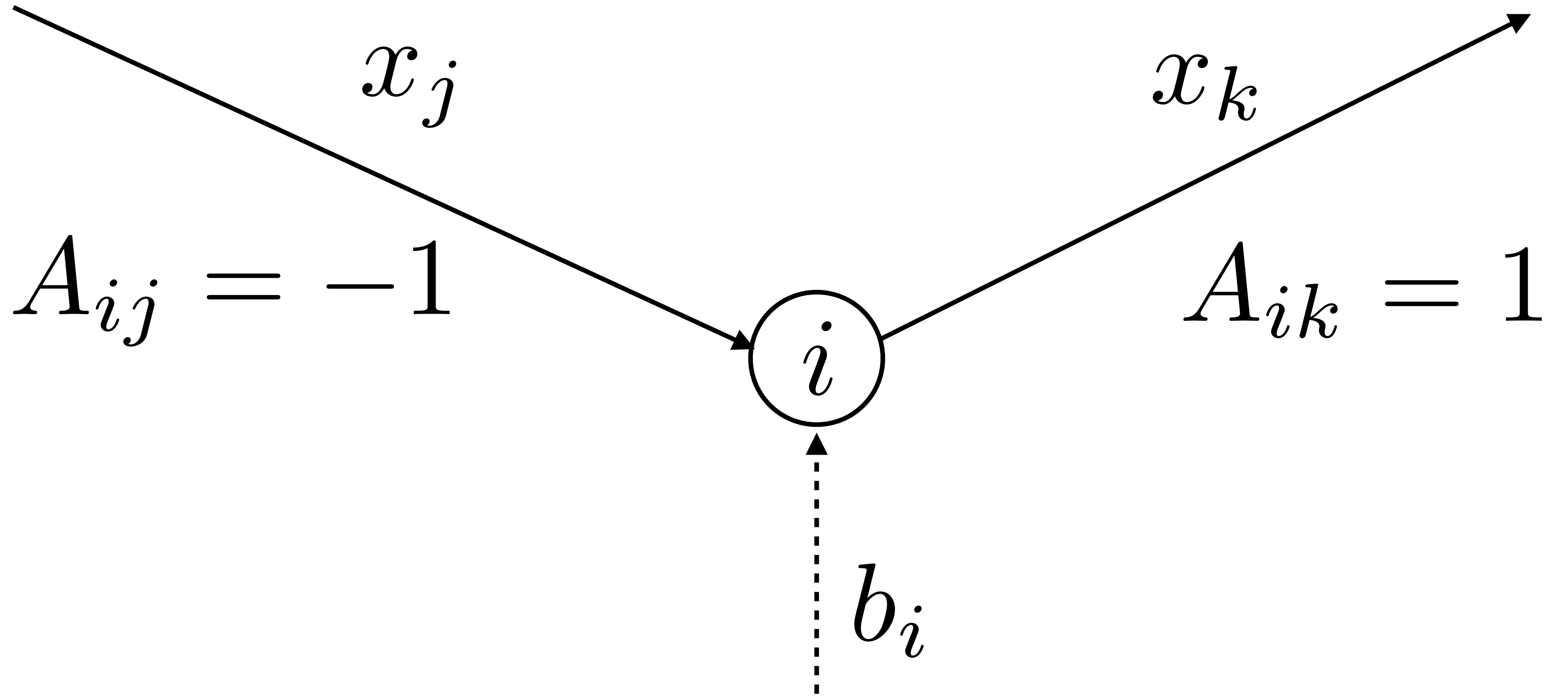
Note Each column has one -1 and one 1

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}$$

External supply

supply vector $b \in \mathbf{R}^m$

- b_i is the external supply at node i (if $b_i < 0$, it represents demand)
- We must have $\mathbf{1}^T b = 0$ (total supply = total demand)



Balance equations

$$\sum_{j=1}^{n} A_{ij} x_j = (Ax)_i = b_i, \quad \text{for all } i \qquad \longrightarrow \qquad Ax = b$$

Total leaving flow

Supply

Minimum cost network flow problem

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & 0 \leq x \leq u \end{array}$$

- c_i is unit cost of flow through arc i
- Flow x_i must be nonnegative
- u_i is the maximum flow capacity of arc i
- Many network optimization problems are just special cases

Integrality theorem

Given a polyhedron $P = \{x \in \mathbf{R}^n \mid Ax = b, \quad x \geq 0\}$

where

- A is totally unimodular
- b is an integer vector

$\longrightarrow$ all the extreme points of P are integer vectors.

Proof

- All extreme points are basic feasible solutions with $x_B = A_B^{-1}b$ and $x_i = 0, \ i \neq B$
- A_B^{-1} has integer components because of total unimodularity of A
- b has also integer components
- Therefore, also x is integral

$\blacksquare$

Implications for network and combinatorial optimization

Minimum cost network flow

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & 0 \leq x \leq u \end{array}$$

$\longrightarrow$ If b and u are integral solutions $x^\star$ are integral

Integer linear programs

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & 0 \leq x \leq u \\ & x \in \mathbf{Z}^n \end{array}$$

Very difficult in general
(more on this in a few weeks)

$\downarrow$

If A totally unimodular and b, u integral, we can relax integrality and solve a fast LP instead

Today's lecture

Interior point methods

- History
- Newton's method
- Central path
- Primal-dual path-following algorithm
- Logarithmic barrier functions

History

A brief history of linear optimization

1940s :

- Foundations and applications in economics and logistics (Kantorovich, Koopmans)
- **1947** : Development of the **simplex method** by Dantzig

1950s—70s:

- Applications expand to engineering, OR, computer science…
- **1975** : Nobel prize in economics for Kantorovich and Koopmans

1980s:

- Development of polynomial time algorithms for LPs
- **1984** : Development of the **interior point method** by Karmarkar

—Today:

- Continued algorithm development. Expansion to very large problems.

Ellipsoid method

Khachian (1979)

Answer to major question

Is worst-case LP complexity polynomial? **Yes!**

Drawbacks

Very inefficient. Much slower than simplex!

Benefits

Motivated new research directions

Shazam! A Shortcut for Computers

A garment manufacturer has three kinds of dresses — A, B and C. On hand he has 17 bolts of one cloth and 25 of another, as well as 200 buttons and 75 belts. He has three cutters, 10 sewers and one finisher. Dress A, on which he makes a profit of $1.25 a unit, requires one combination of material, accessories and work; the B dress, with a $1.50 profit, takes a different combination, and the $2.25 dress C has yet a third set of requirements. How should he schedule his production to make the most money?

That is an easy example of a kind of eminently practical problem that becomes computationally difficult because of the number of variable factors and constraints that must be handled to get a best solution. And, as the number of variables and restraints grows — as, for instance, in a model of the national economy or in the scheduling of production at any oil refinery — the difficulty mushrooms.

Even the most powerful computers might have to run for hours to tell a plant manager how to handle a small change in, say, the amount of crude oil being delivered to his tanks. And adding one new restriction can substantially increase the number of possible answers and thus the time required to check them for an optimum solution.

Last week, intrigued mathematicians were trying to sort out the meaning of what looked like a stunning theoretical breakthrough in the handling of these "linear programming" problems — and some were wondering why it had taken so long for the breakthrough to become generally known.

In January, the Soviet journal Doklady published an abstract of the new solution put forward by a Russian mathematician, L. G. Khachian, about whom no further biographical data has been made public. The abstract was generally overlooked until two mathematicians, working at Stanford University, analyzed the theory and refined its application. Reports of their work and Mr. (or Miss) Khachian's began appearing in American journals four weeks ago, opening up the floodgates of mathematical curiosity.

Ronald L. Graham, a leading computer expert at the Bell Laboratories in Murray Hill, N.J., said the significance of the new method is that it provides a fast way to test whether there is an optimum solution for any particular linear programming problem and, if there is, to assure that the solution can be computed within a reasonable length of time.

The older, "simplex" method involved having the computer "build" a flat-sided polyhedron in multidimensional space and then hop from vertex to vertex testing for a best answer. Mr. Khachian's solution has the computer design a multidimensional curved ellipsoid that surrounds the area of possible solutions and is then made smaller until it neatly encloses the optimum answer.

The practical effects of the breakthrough were not entirely clear last week, however. Although it seems to offer enormous advantages in areas ranging from industrial scheduling to weather forecasting, it has yet to be tested in the development of an actual major computer program. Dr. Graham said it might work for some kinds of linear programming problems and not for others, noting that, despite its theoretical limitations, simplex in fact works quite efficiently for the problems it has been asked to handle.

Nevertheless Laslo Lovász, a Hungarian mathematician who worked on the problem at Stanford, said he used the method to program his pocket calculator to solve a problem with six variables and six constraints, which it probably could not have handled with the simplex method. And George B. Dantzig, who devised the simplex method in 1947, said he felt "stupid that I didn't see" Mr. Khachian's method.

While some wondered about the delay between the publication of the abstract in Doklady and its reception now, others pointed out that "simplex" itself it did not get into wide use until several years after its theoretical formulation.

—JONATHAN FRIENDLY

The New York Times

Published: November 11, 1979

Interior-point methods

1980s-1990s: interior point methods

- Karmarkar's algorithm (1984)
- Competitive with simplex, often faster for larger problems
- Began huge effort in algorithm development for convex optimization

Breakthrough in Problem Solving

By JAMES GLEICK

A 28-year-old mathematician at A.T.&T. Bell Laboratories has made a startling theoretical breakthrough in the solving of systems of equations that often grow too vast and complex for the most powerful computers.

The discovery, which is to be formally published next month, is already circulating rapidly through the mathematical world. It has also set off a deluge of inquiries from brokerage houses, oil companies and airlines, industries with millions of dollars at stake in problems known as linear programming.

These problems are fiendishly complicated systems, often with thousands of variables. They arise in a variety of commercial and government applications, ranging from allocating time on a communications satellite to routing millions of telephone calls over long distances. Linear programming is particularly useful whenever a limited, expensive resource must be spread most efficiently among competing users. And investment companies use the approach in creating portfolios with the best mix of stocks and bonds.

Faster Solutions Seen

The Bell Labs mathematician, Dr. Narendra Karmarkar, has devised a radically new procedure that may speed the routine handling of such problems by businesses and Government agencies and also make it possible to tackle problems that are now far out of reach.

"This is a path-breaking result," said Dr. Ronald L. Graham, director of mathematical sciences for Bell Labs in Murray Hill, N.J. "Science has its moments of great progress, and this may well be one of them."

Because problems in linear programming can have billions or more possible answers, even high-speed computers cannot check every one. So computers must use a special procedure, an algorithm, to examine as few answers as possible before finding the best one — typically the one that minimizes cost or maximizes efficiency.

A procedure devised in 1947, the simplex method, is now used for such prob-

Continued on Page A19, Column 1

Newton’s method

Newton's root finding method

Goal: solve

$$h(x) = 0$$

Method

1. Make a guess x^k and a linear approximation

$$h(x) \approx h(x^k) + \frac{\partial h}{\partial x^k}(x^k)(x - x^k)$$

2. Iteratively set $h(x^k)$ to 0

$$h(x^k) + \frac{\partial h}{\partial x^k}(x^k)(x^{k+1} - x^k) = 0$$

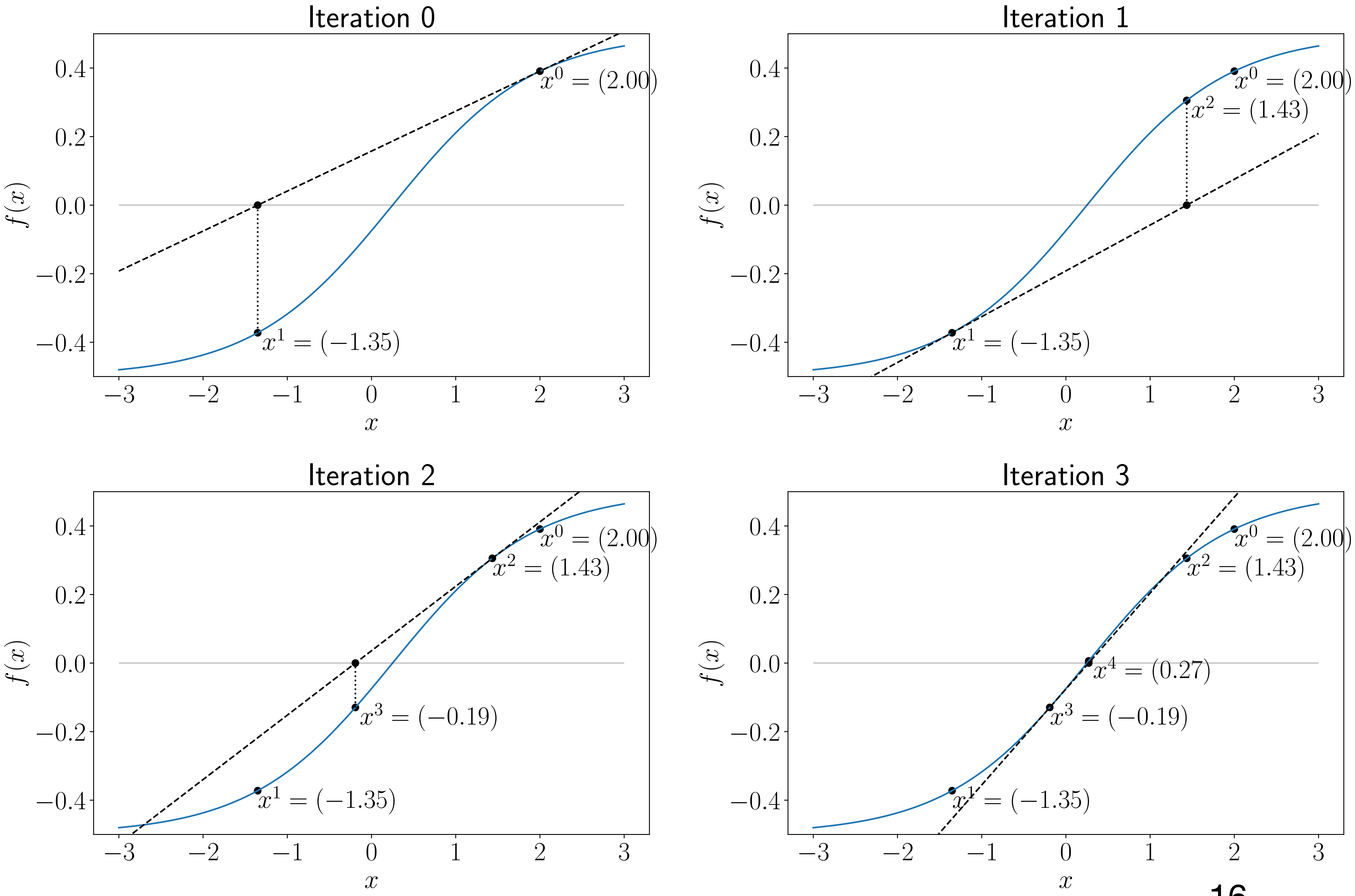
Newton's method example

$$f(x) = \frac{1}{1 + e^{-1.2x+0.3}} - 0.5$$

$$f(x) = 0$$

$$\downarrow$$

$$x^\star = 0.3$$



Newton's root finding method (multivariable)

Goal: solve

$$h(x) = 0$$

Method

1. Make a guess x^k and a linear approximation

$$h(x) \approx h(x^k) + Dh(x^k)(x - x^k)$$

2. Iteratively set $h(x^k)$ to 0

$$h(x^k) + Dh(x^k)(x^{k+1} - x^k) = 0$$

Derivative

$$Dh = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial h_m}{\partial x_1} & \cdots & \frac{\partial h_m}{\partial x_n} \end{bmatrix}$$

Newton method iterations

$$h(x^k) + Dh(x^k)\underbrace{(x^{k+1} - x^k)}_{\Delta x} = 0$$

Iterations

- Solve $Dh(x^k)\Delta x = -h(x^k)$
- $x^{k+1} \leftarrow x^k + \Delta x$

Remarks

- Iterations can be **expensive** (linear system solution)
- **Fast convergence** close to the solution $x^\star$

Linear optimization as a root finding problem

Optimality conditions

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax \leq b
\end{array}
\longrightarrow
$$

Primal

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax + s = b \\
& s \geq 0
\end{array}
$$

Dual

$$
\begin{array}{ll}
\text{maximize} & -b^T y \\
\text{subject to} & A^T y + c = 0 \\
& y \geq 0
\end{array}
$$

KKT conditions

$$
\begin{aligned}
Ax + s - b &= 0 \\
A^T y + c &= 0 \\
s_i y_i &= 0, \quad i = 1, \dots, m \\
s, y &\geq 0
\end{aligned}
$$

Linear optimization as a root finding problem

$$
\begin{aligned}
Ax + s - b &= 0 \\
A^T y + c &= 0 \\
s_i y_i &= 0, \quad i = 1, \dots, m \\
s, y &\geq 0
\end{aligned}
$$

Diagonalize complementary slackness

$$S = \mathbf{diag}(s) = \begin{bmatrix} s_1 & & & \\ & s_2 & & \\ & & \ddots & \\ & & & s_m \end{bmatrix}$$

$$Y = \mathbf{diag}(y) = \begin{bmatrix} y_1 & & & \\ & y_2 & & \\ & & \ddots & \\ & & & y_m \end{bmatrix}$$

$$SY\mathbf{1} = \mathbf{diag}(s)\mathbf{diag}(y)\mathbf{1} = \begin{bmatrix} s_1 y_1 & & & \\ & s_2 y_2 & & \\ & & \ddots & \\ & & & s_m y_m \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} s_1 y_1 \\ s_2 y_2 \\ \vdots \\ s_m y_m \end{bmatrix}$$

$$\downarrow$$

$$s_i y_i = 0, \quad i = 1, \dots, m \quad \iff \quad SY\mathbf{1} = 0$$

Main idea

Optimality conditions

$$h(y, x, s) = \begin{bmatrix} Ax + s - b \\ A^T y + c \\ SY\mathbf{1} \end{bmatrix} = \begin{bmatrix} r_p \\ r_d \\ SY\mathbf{1} \end{bmatrix} = 0$$

$$s, y \geq 0$$

$$S = \mathbf{diag}(s)$$
$$Y = \mathbf{diag}(y)$$

- Apply variants of Newton's method to solve $h(x, s, y) = 0$
- Enforce $s, y > 0$ (strictly) at every iteration
- **Motivation** avoid getting stuck in "corners"

Newton's method for optimality conditions

Optimality conditions

$$h(y, x, s) = \begin{bmatrix} Ax + s - b \\ A^T y + c \\ SY\mathbf{1} \end{bmatrix} = \begin{bmatrix} r_p \\ r_d \\ SY\mathbf{1} \end{bmatrix} = 0$$

$$s, y \geq 0$$

Derivative

$$Dh(y, x, s) = \begin{bmatrix} 0 & A & I \\ A^T & 0 & 0 \\ S & 0 & Y \end{bmatrix}$$

Iterations

- Solve $Dh(y^k, x^k, s^k)\Delta(y^k, x^k, s^k) = -h(y^k, x^k, s^k)$
- $\begin{bmatrix} y^{k+1} \\ x^{k+1} \\ s^{k+1} \end{bmatrix} \leftarrow \begin{bmatrix} y^k \\ x^k \\ s^k \end{bmatrix} + \Delta(y^k, x^k, s^k)$

Caution!

It might make (s, y) negative!

Central path

Line search to stay feasible

Root-finding equation

$$h(y, x, s) = \begin{bmatrix} Ax + s - b \\ A^T y + c \\ SY\mathbf{1} \end{bmatrix} = \begin{bmatrix} r_p \\ r_d \\ SY\mathbf{1} \end{bmatrix} = 0$$

Linear system

$$\underbrace{\begin{bmatrix} 0 & A & I \\ A^T & 0 & 0 \\ S & 0 & Y \end{bmatrix}}_{Dh} \begin{bmatrix} \Delta y \\ \Delta x \\ \Delta s \end{bmatrix} = \underbrace{\begin{bmatrix} -r_p \\ -r_d \\ -SY\mathbf{1} \end{bmatrix}}_{-h}$$

Residuals

$$r_p = Ax + s - b$$

$$r_d = A^T y + c$$

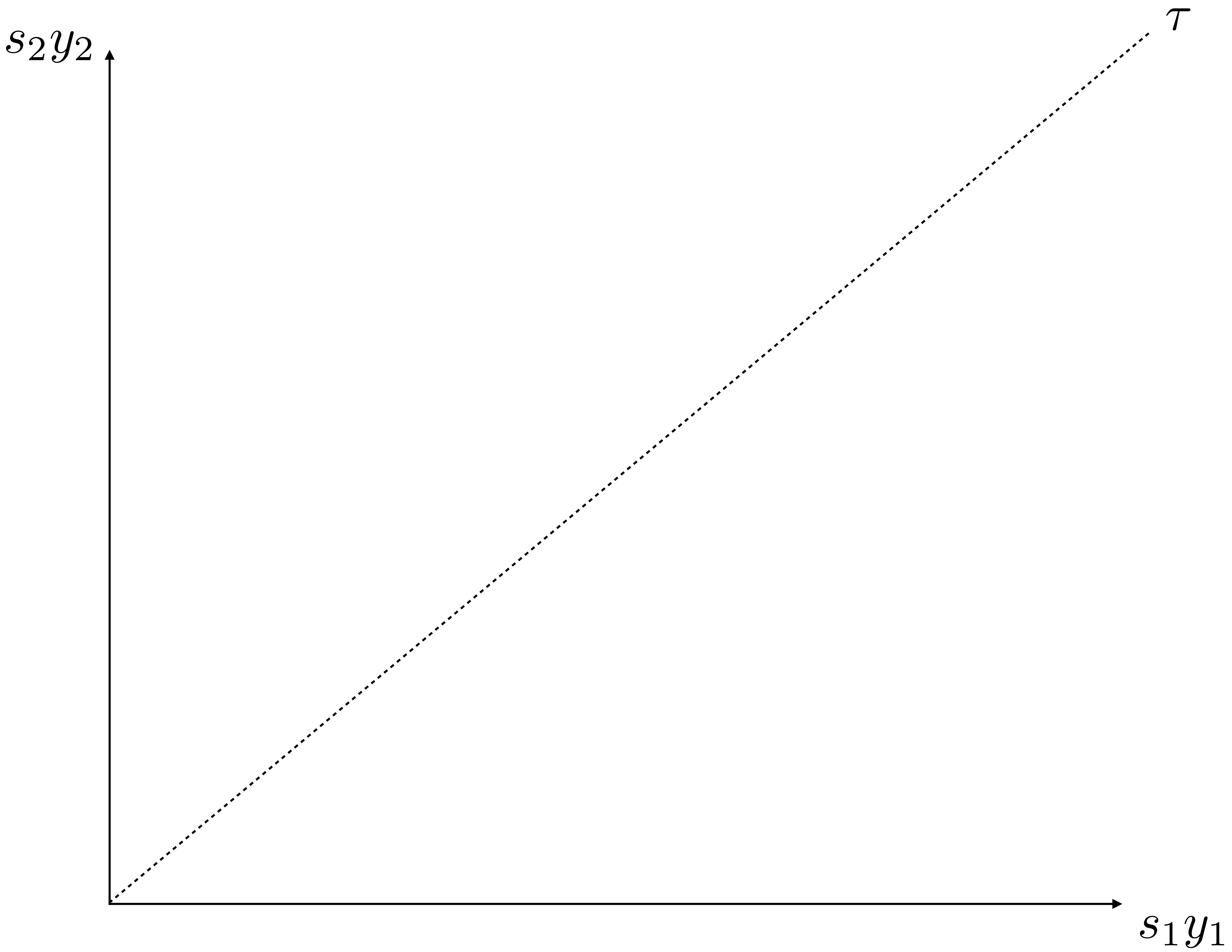
Issue

Pure **Newton's step** does not allow significant progress towards $h(y, x, s) = 0$ **and** $s, y \geq 0$.

Line search to enforce $s, y > 0$

$$(y, x, s) \leftarrow (y, x, s) + \alpha(\Delta y, \Delta x, \Delta s)$$

The central path



Smoothed optimality conditions

Optimality conditions

$$Ax + s - b = 0$$

$$A^T y + c = 0$$

$$s_i y_i = \boxed{\tau}$$

Same τ for every pair

$$s, y \geq 0$$

Same optimality conditions for a “smoothed” version of our problem

Duality gap

$$s^T y = (b - Ax)^T y = b^T x - x^T A^T y = b^T y + c^T x$$

Newton's method for smoothed optimality conditions

Smoothed optimality conditions

$$h_\tau(y, x, s) = \begin{bmatrix} Ax + s - b \\ A^T y + c \\ SY\mathbf{1} - \tau\mathbf{1} \end{bmatrix} = 0$$

$$s, y \geq 0$$

Linear system

$$\begin{bmatrix} 0 & A & I \\ A^T & 0 & 0 \\ S & 0 & Y \end{bmatrix} \begin{bmatrix} \Delta y \\ \Delta x \\ \Delta s \end{bmatrix} = \begin{bmatrix} -r_p \\ -r_d \\ -SY + \tau\mathbf{1} \end{bmatrix}$$

Line search to enforce $s, y > 0$

$$(y, x, s) \leftarrow (y, x, s) + \alpha(\Delta y, \Delta x, \Delta s)$$

The path parameter

Linear system

$$\begin{bmatrix} 0 & A & I \\ A^T & 0 & 0 \\ S & 0 & Y \end{bmatrix} \begin{bmatrix} \Delta y \\ \Delta x \\ \Delta s \end{bmatrix} = \begin{bmatrix} -r_p \\ -r_d \\ -SY\mathbf{1} + \sigma\mu\mathbf{1} \end{bmatrix}$$

Duality measure

$$\mu = \frac{s^T y}{m}$$ (average value of the pairs $s_i y_i$)

Centering parameter

$\sigma \in [0, 1]$

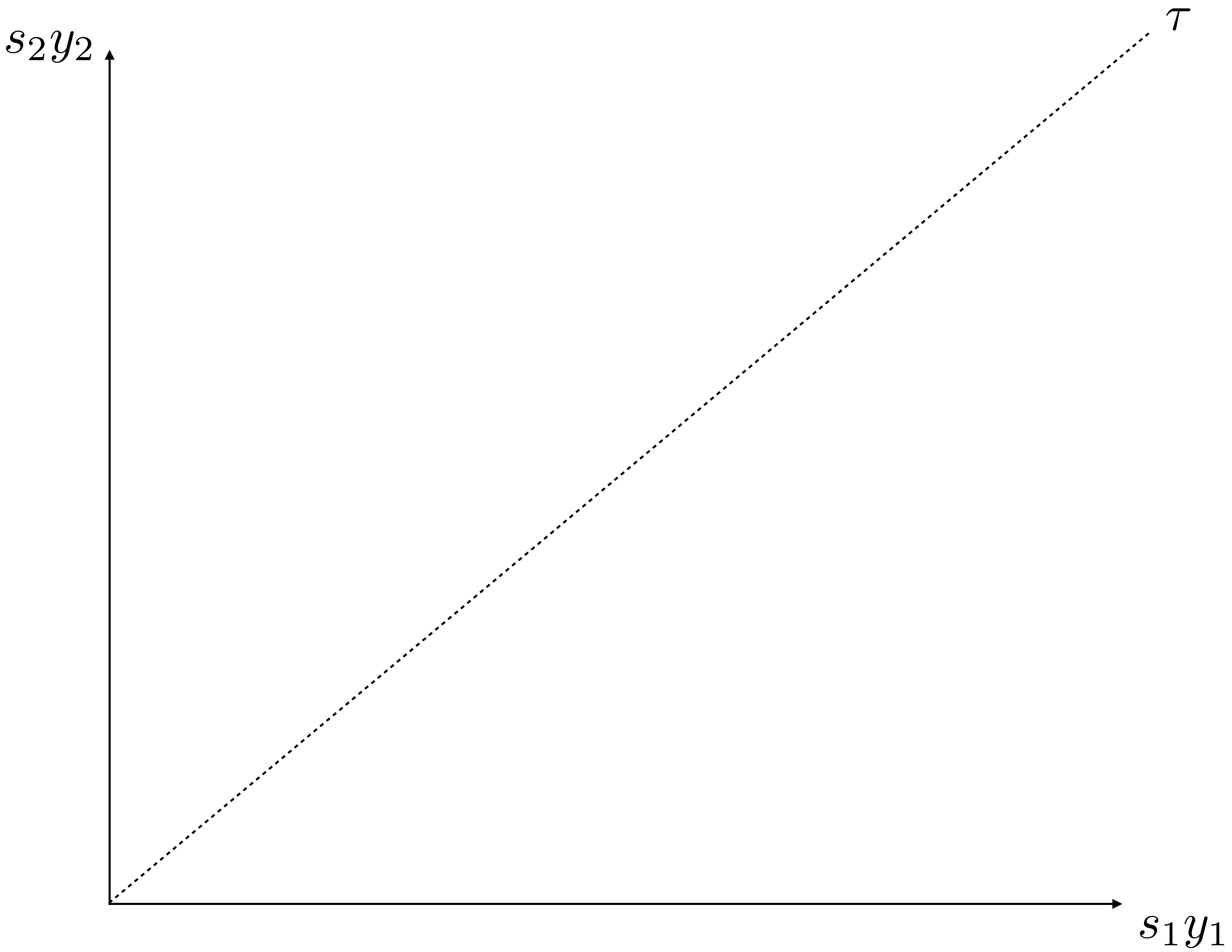
$\sigma = 0 \quad \Rightarrow$ Newton step

$\sigma = 1 \quad \Rightarrow$ Centering step towards $(y^\star(\mu), x^\star(\mu), s^\star(\mu))$

Line search to enforce $s, y > 0$

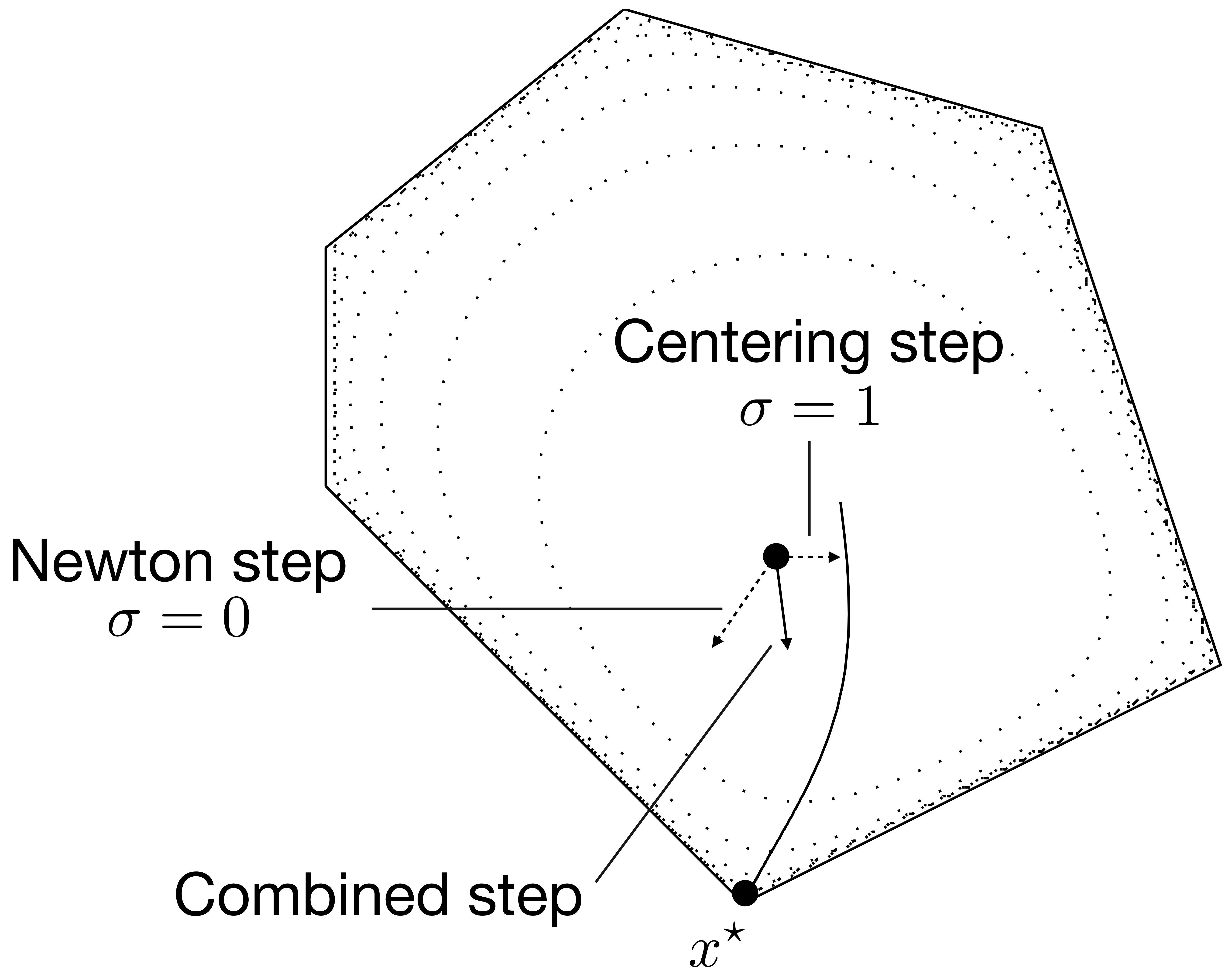
$$(y, x, s) \leftarrow (y, x, s) + \alpha(\Delta y, \Delta x, \Delta s)$$

The central path



Primal-dual path-following method

Path-following algorithm idea



Centering step
It brings towards the **central path** and is usually biased towards $s, y > 0$. **No progress** on duality measure μ

Newton step
It brings towards the **zero duality measure** μ. Quickly violates $s, y > 0$.

Combined step
Best of both worlds with longer steps

Primal-dual path-following algorithm

Initialization

1. Given (x_0, s_0, y_0) such that $s_0, y_0 > 0$

Iterations

1. Choose $\sigma \in [0, 1]$

2. Solve $\begin{bmatrix} 0 & A & I \\ A^T & 0 & 0 \\ S & 0 & Y \end{bmatrix} \begin{bmatrix} \Delta y \\ \Delta x \\ \Delta s \end{bmatrix} = \begin{bmatrix} -r_p \\ -r_d \\ -SY\mathbf{1} + \sigma\mu\mathbf{1} \end{bmatrix}$ where $\mu = s^T y/m$

3. Find maximum α such that $y + \alpha\Delta y > 0$ and $s + \alpha\Delta s > 0$

4. Update $(y, x, s) \leftarrow (y, x, s) + \alpha(\Delta y, \Delta x, \Delta s)$

Working towards optimality conditions

Optimality conditions satisfied **only at convergence**

Primal residual

$$r_p = Ax + s - b \to 0$$

Dual residual

$$r_d = A^T y + c \to 0$$

Complementary slackness

$$s^T y \to 0$$

Stopping criteria

$$\|r_p\| \leq \epsilon_{\text{pri}}$$

$$\|r_d\| \leq \epsilon_{\text{dua}}$$

$$s^T y \leq \epsilon_{\text{gap}}$$

Logarithmic barrier functions

Smoothed optimality conditions

Optimality conditions

$$\begin{aligned} Ax + s - b &= 0 \\ A^T y + c &= 0 \\ s_i y_i &= \tau \\ s, y &\geq 0 \end{aligned}$$

Same τ for every pair

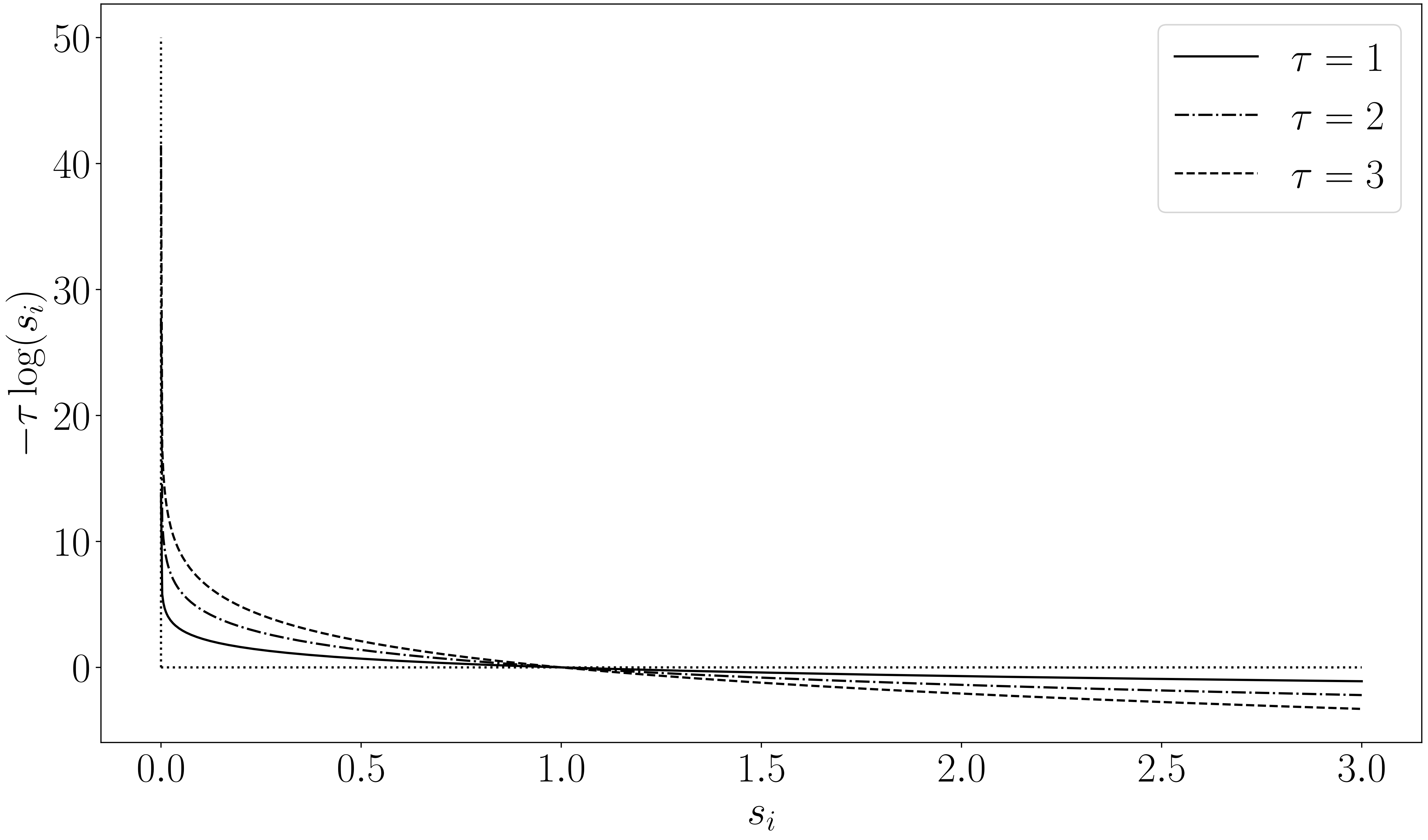
Same optimality conditions for a “smoothed” version of our problem

Do solutions actually exist?

What do they represent?

Logarithmic barrier

$\phi(s) = -\tau \sum_{i=1}^{m} \log(s_i)$ on domain $s_i > 0$



As $\tau \to 0$ it approximates

$$\mathcal{I}_{s_i \geq 0} = \begin{cases} 0 & \text{if } s_i \geq 0 \\ \infty & \text{otherwise} \end{cases}$$

Smoothed problem

minimize $c^T x$
subject to $Ax + s = b$
$s \geq 0$

$\longrightarrow$

minimize $c^T x + \phi(s) = c^T x - \tau \sum_{i=1}^m \log(s_i)$
subject to $Ax + s = b$

Lagrangian function

$$L(x, s, y) = c^T x - \tau \sum_{i=1}^{m} \log(s_i) + y^T (Ax + s - b)$$

$$\frac{\partial L}{\partial x} = A^T y + c = 0$$

$$\frac{\partial L}{\partial s_i} = -\tau \frac{1}{s_i} + y_i = 0 \quad \Longrightarrow s_i y_i = \tau$$

Central path

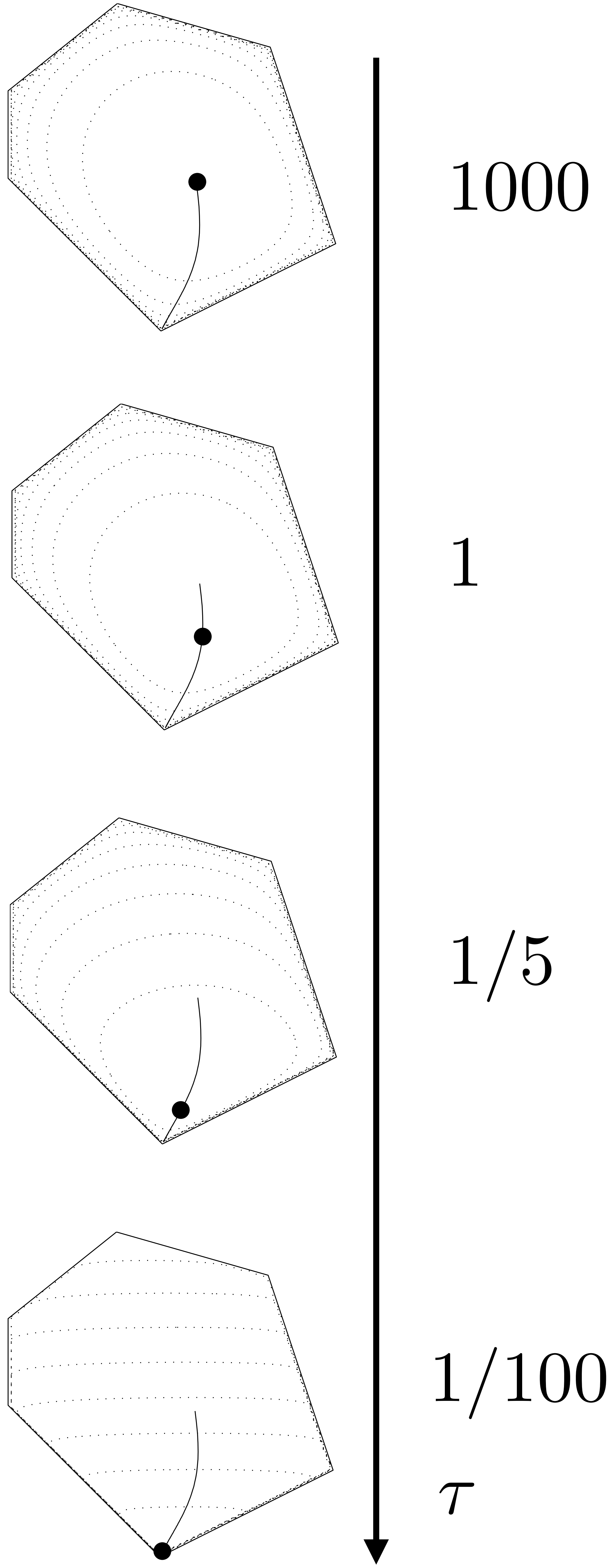
minimize $\quad c^T x - \tau \sum_{i=1}^{m} \log(s_i)$

subject to $\quad Ax + s = b$

Set of points $(x^\star(\tau), s^\star(\tau), y^\star(\tau))$
with $\tau > 0$ such that

$$
\begin{aligned}
Ax + s - b &= 0 \\
A^T y + c &= 0 \\
s_i y_i &= \tau \\
s, y &\geq 0
\end{aligned}
$$

Analytic Center
$\tau \to \infty$



Main idea

Follow central path as $\tau \to 0$

Interior-point methods for linear optimization

Today, we learned to:

- **Apply** Newton’s method to solve optimality conditions
- **Follow** the central path and the smoothed optimality conditions
- **Use logarithmic barrier functions** to interpret central path steps

References

- D. Bertsimas and J. Tsitsiklis: Introduction to Linear Optimization
  - Chapter 9.4 — 9.6: Interior point methods

- R. Vanderbei: Linear Programming
  - Chapter 17: The Central Path
  - Chapter 15: A Path-Following Method

Next lecture

- Practical interior-point method (Mehrotra predictor-corrector algorithm)
- Implementation details
- Interior-point vs simples