ORF307 – Optimization

16. Network optimization

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Ed Forum

- In local sensitivity analysis, does the optimal basis always remain the same?
- Could we also clarifying the meaning of shadow prices?
- Why we use the dual problem to calculate feasibility instead of just trying to solve the primal problem? Is it not possible to find out infeasibility by the simplex method? Or is it more efficient to use the dual? What if the dual is infeasible, and we try to use that to solve the primal problem?

Recap

Primal and dual basic feasible solutions

Primal problem

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Dual problem

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c \geq 0 \end{array}$$

Given a **basis** matrix A_B

Primal feasible: $Ax = b,\ x \geq 0 \quad \Rightarrow \quad x_B = A_B^{-1} b \geq 0$

Reduced costs

Dual feasible: $A^T y + c \geq 0$. Set $y = -A_B^{-T} c_B$. Dual feasible if $\bar{c} = c + A^T y \geq 0$

Zero duality gap: $c^T x + b^T y = c_B^T x_B - b^T A_B^{-T} c_B = c_B x_B - c_B^T A_B^{-1} b = 0$

(by construction)

The primal (dual) simplex method

Primal problem

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Dual problem

$$\begin{aligned} \text{maximize} \quad & -b^T y \\ \text{subject to} \quad & A^T y + c \geq 0 \end{aligned}$$

Primal simplex

- Primal feasibility
- Zero duality gap

$\downarrow$

Dual feasibility

Dual simplex
(solve dual instead)

- Dual feasibility
- Zero duality gap

$\downarrow$

Primal feasibility

Adding new variables

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & c^T x + c_{n+1} x_{n+1} \\ \text{subject to} & Ax + A_{n+1} x_{n+1} = b \\ & x, x_{n+1} \geq 0 \end{array}$$

Solution $x^\star, y^\star$

Is the solution $(x^\star, 0), y^\star$ **optimal** for the new problem?

Adding new variables

Optimality conditions

$$\begin{array}{ll}\text{minimize} & c^T x + c_{n+1}x_{n+1} \\ \text{subject to} & Ax + A_{n+1}x_{n+1} = b \\ & x, x_{n+1} \geq 0\end{array} \longrightarrow$$

Solution $(x^\star, 0)$ is still **primal feasible**

Is $y^\star$ still **dual feasible**?

$$A_{n+1}^T y^\star + c_{n+1} \geq 0$$

Yes

$(x^\star, 0)$ still **optimal** for new problem

Otherwise

Primal simplex

Optimal value function

$$p^\star(u) = \min\{c^T x \mid Ax = b + u,\ x \geq 0\}$$

Assumption: $p^\star(0)$ is finite

Properties

- $p^\star(u) > -\infty$ everywhere (from global lower bound)
- $p^\star(u)$ is piecewise-linear on its domain

Optimal value function is piecewise linear

Proof

Dual feasible set

$$p^\star(u) = \min\{c^T x \mid Ax = b + u,\ x \geq 0\}$$

$$D = \{y \mid A^T y + c \geq 0\}$$

Assumption: $p^\star(0)$ is finite

If $p^\star(u)$ finite

$$p^\star(u) = \max_{y \in D} -(b+u)^T y = \max_{k=1,\dots,r} -y_k^T u - b^T y_k$$

$y_1, \dots, y_r$ are the extreme points of D

Derivative of the optimal value function

Modified optimal solution

$$x_B^\star(u) = A_B^{-1}(b+u) = x_B^\star + A_B^{-1}u$$
$$y^\star(u) = y^\star$$

Optimal value function

$$\begin{aligned} p^\star(u) &= c^T x^\star(u) \\ &= c^T x^\star + c_B^T A_B^{-1} u \\ &= p^\star(0) - y^{\star T} u \end{aligned} \qquad \text{(affine for small } u\text{)}$$

Local derivative

$$\nabla p^\star(u) = -y^\star$$ ($y^\star$ are the **shadow prices**)

Today's lecture

Network optimization

- Network flows
- Minimum cost network flow problem
- Network flow solutions
- Examples: maximum flow, shortest path, assignment

Network flows

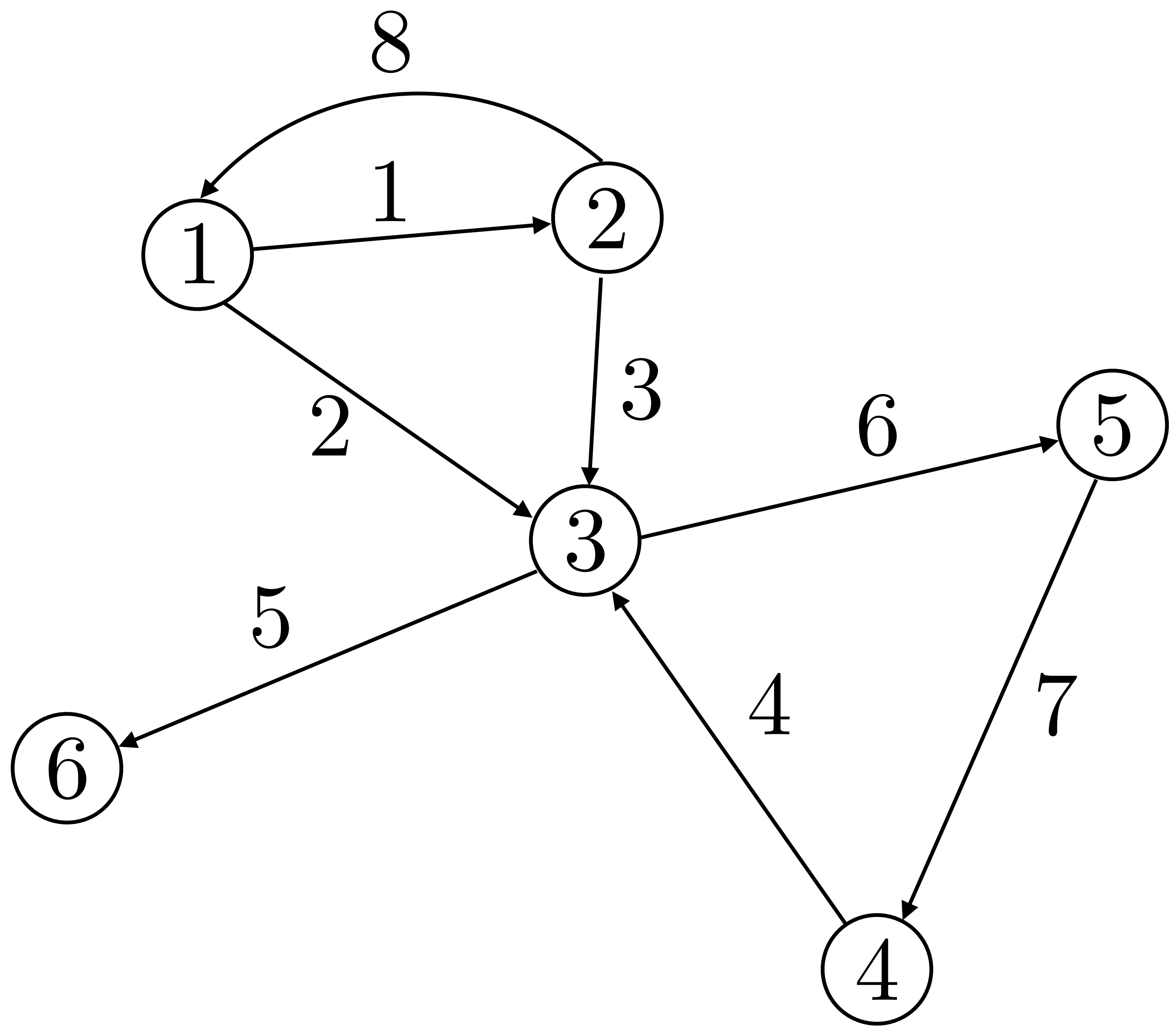
Networks

- Electrical and power networks
- Road networks
- Airline routes
- Printed circuit boards
- Social networks

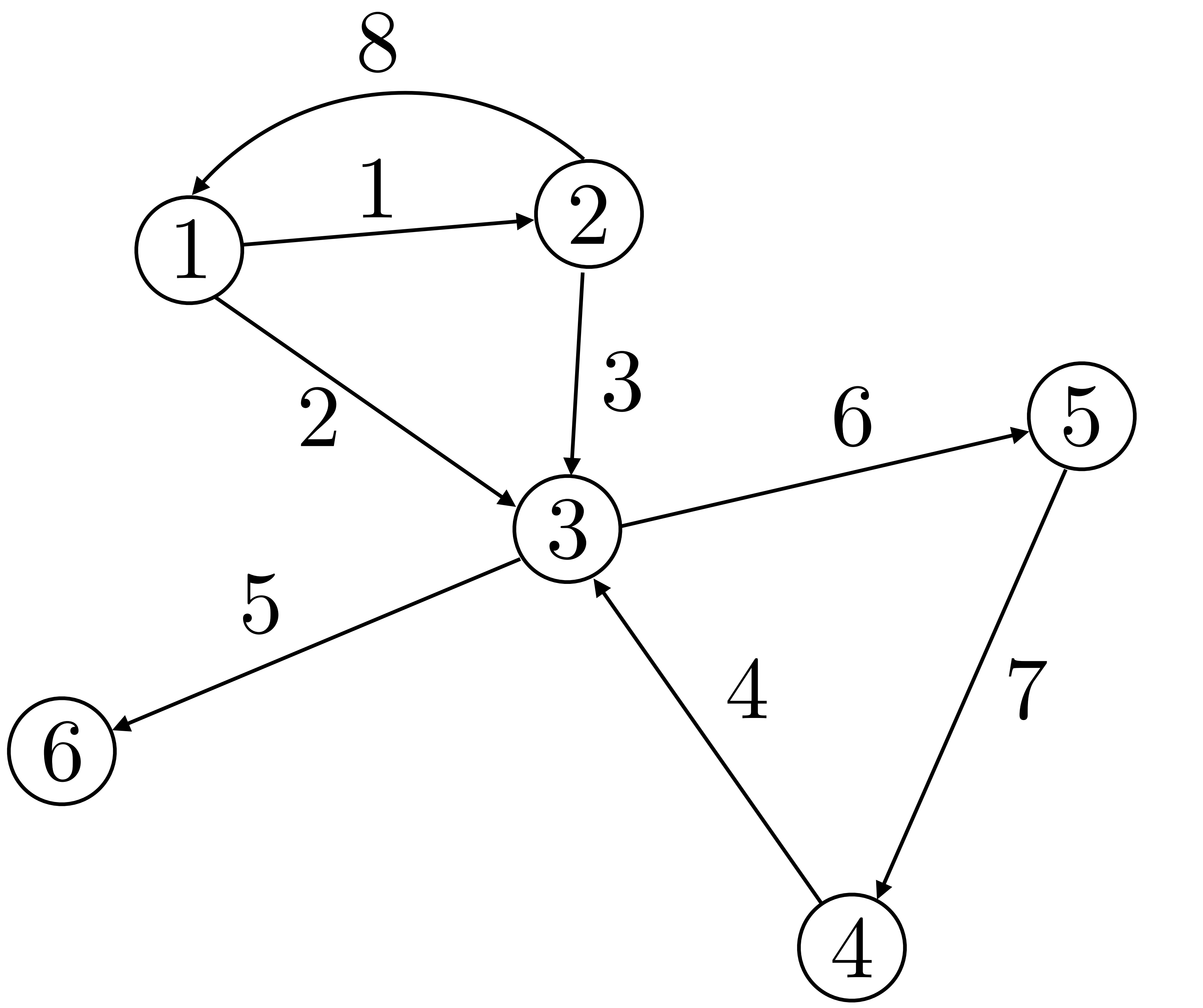
Network modelling

A **network** (or *directed graph*, or *digraph*)
is a set of m nodes and n directed arcs

- Arcs are ordered pairs of nodes (a, b) (leaves a, enters b)
- **Assumption** there is at most one arc from node a to node b
- There are no loops (arcs from a to a)



Arc-node incidence matrix



$m \times n$ matrix A with entries

$$A_{ij} = \begin{cases} 1 & \text{if arc } j \text{ starts at node } i \\ -1 & \text{if arc } j \text{ ends at node } i \\ 0 & \text{otherwise} \end{cases}$$

Note Each column has one -1 and one 1

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}$$

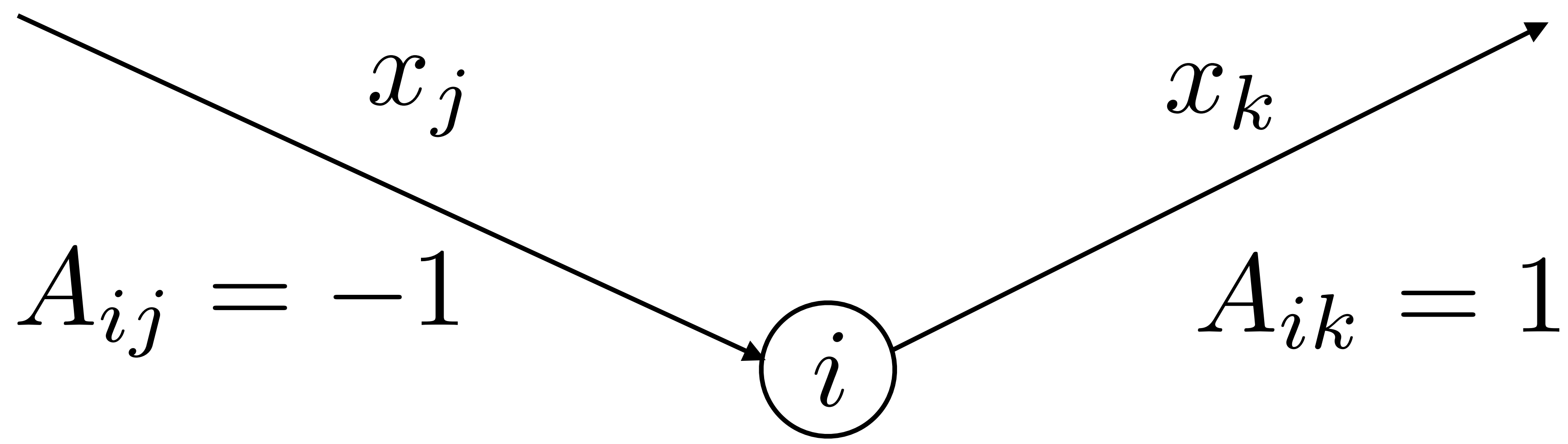
Network flow

flow vector $x \in \mathbf{R}^n$

x_j: flow (of material, traffic, information, electricity, etc) through arc j

total flow leaving node i

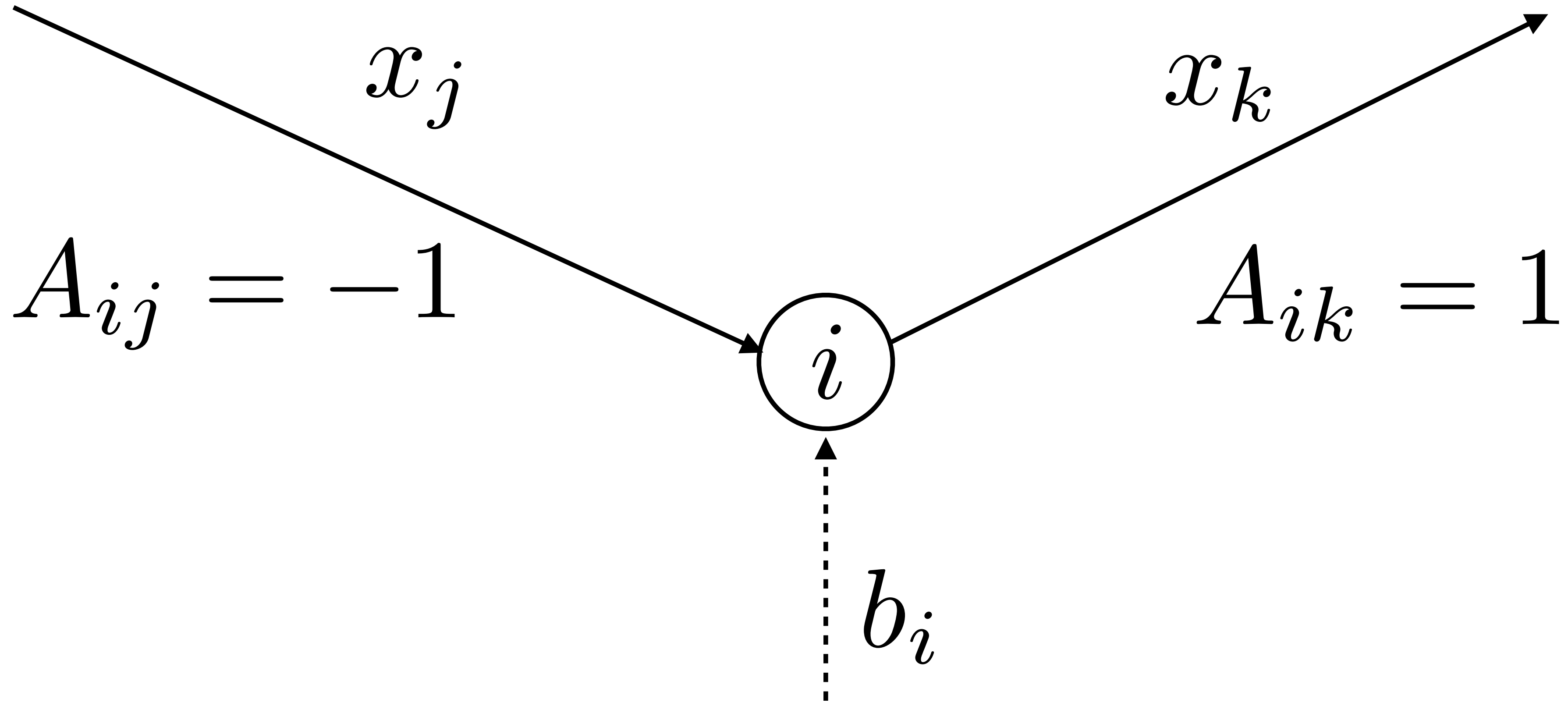
$$\sum_{j=1}^{n} A_{ij} x_j = (Ax)_i$$



External supply

supply vector $b \in \mathbf{R}^m$

- b_i is the external supply at node i (if $b_i < 0$, it represents demand)
- We must have $\mathbf{1}^T b = 0$ (total supply = total demand)



Balance equations

$$\sum_{j=1}^{n} A_{ij} x_j = \underbrace{(Ax)_i}_{\text{Total leaving flow}} = \underbrace{b_i}_{\text{Supply}}, \quad \text{for all } i \quad \longrightarrow \quad Ax = b$$

Minimum cost network flow problem

Minimum cost network flow problem

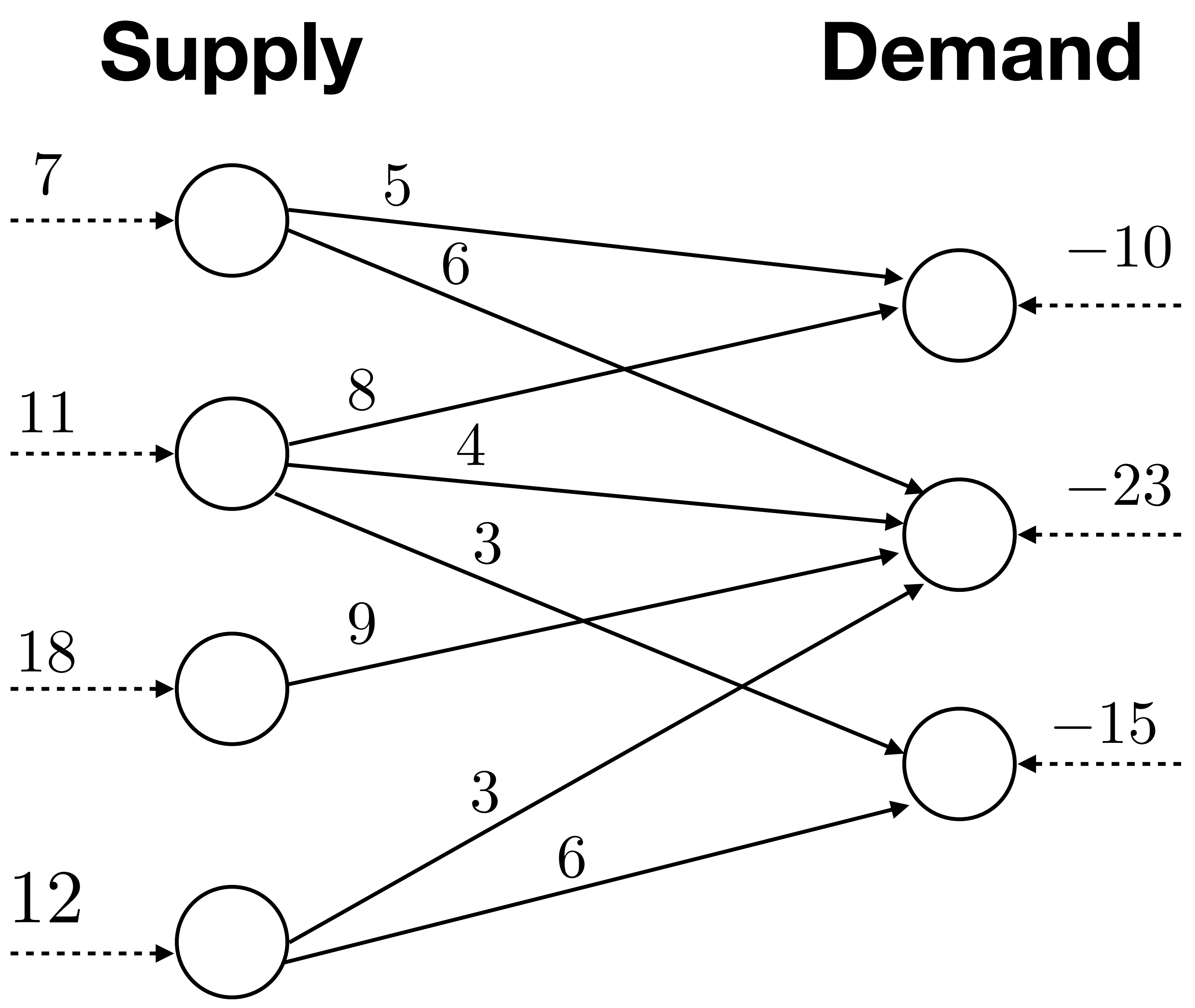
$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax = b \\
& 0 \leq x \leq u
\end{array}
$$

- c_i is unit cost of flow through arc i
- Flow x_i must be nonnegative
- u_i is the maximum flow capacity of arc i
- Many network optimization problems are just special cases

Example

Transportation

Goal ship $x \in \mathbf{R}^n$ to satisfy demand



(arc costs shown)
All capacities 20

$$c = (5, 6, 8, 4, 3, 9, 3, 6)$$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \end{bmatrix}$$

$$b = (7, 11, 18, 12, -10, -23, -15)$$

$$u = 20\,\mathbf{1}$$

Minimum cost network flow

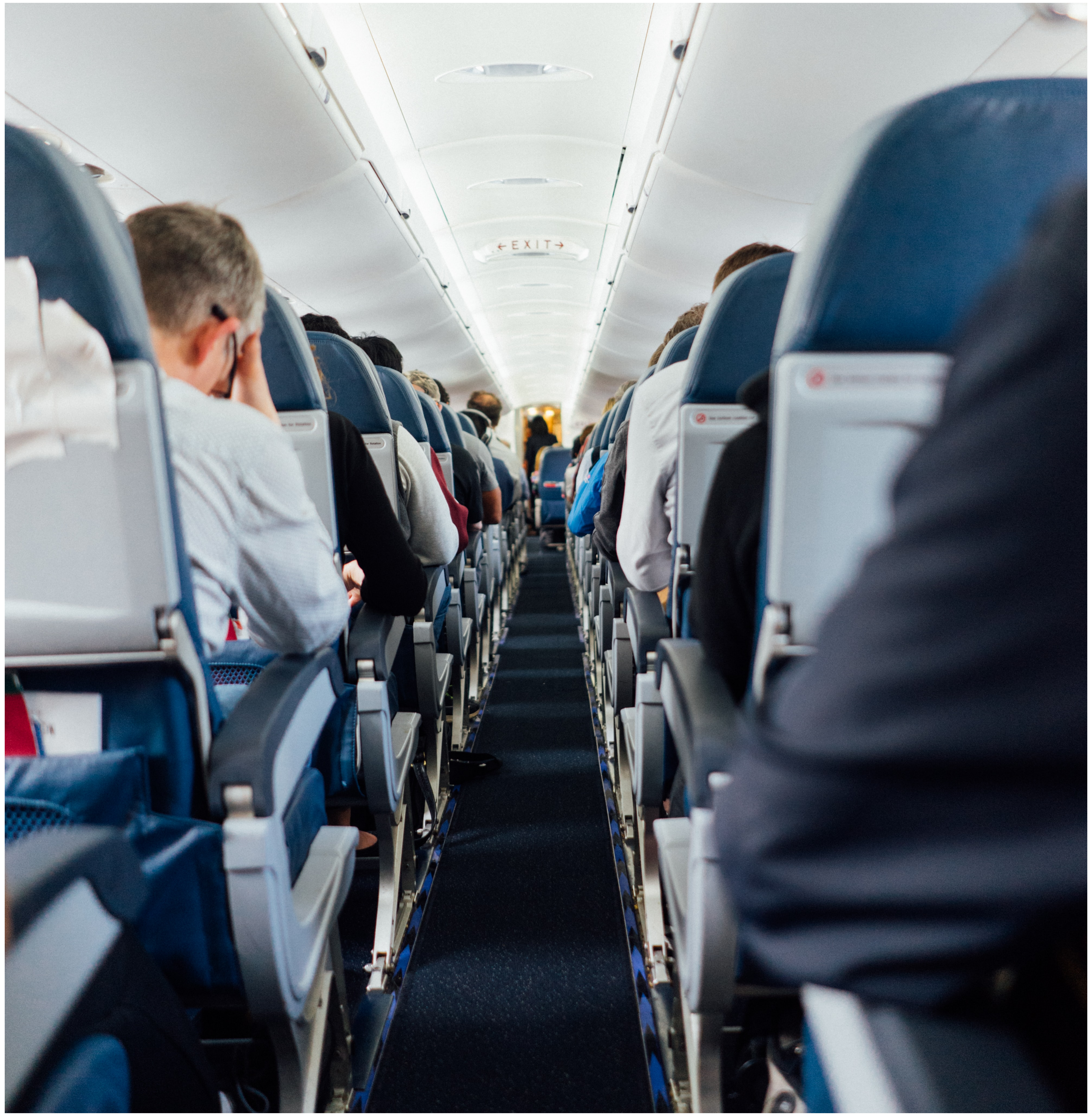
$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & 0 \le x \le u \end{array}$$

$$x^\star = (7, 0, 3, 0, 8, 18, 5, 7)$$

Example

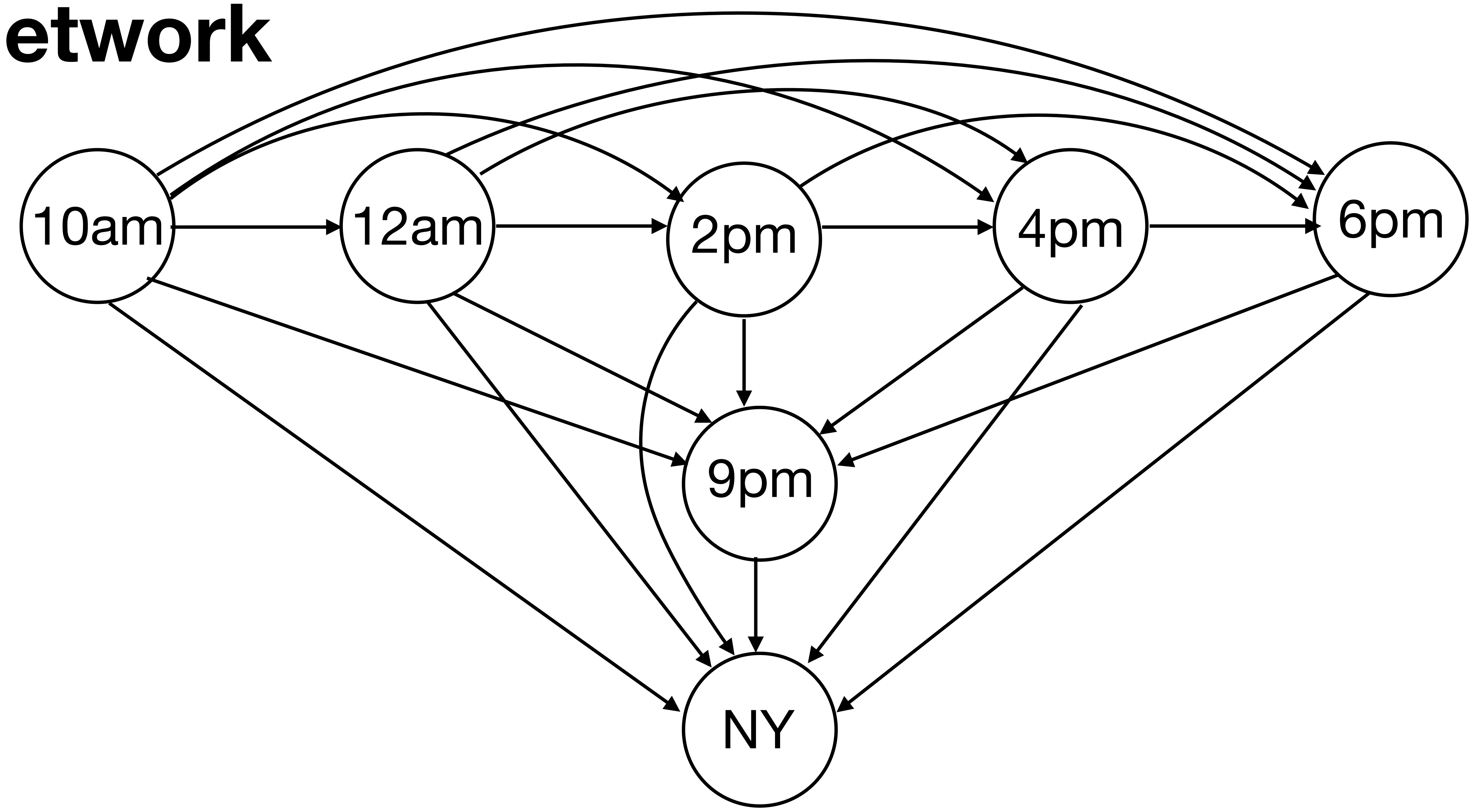
Airline passenger routing

- United Airlines has 5 flights per day from BOS to NY
(10am, 12pm, 2pm, 4pm, 6pm)
- Flight capacities
(100, 100, 100, 150, 150)
- Costs: $50/hour of delay
- Last option: 9pm flight with other company (additional cost $75)
- Today's reservations
(110, 118, 103, 161, 140)



Airline passenger routing

Network



Network flow formulation

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & 0 \leq x \leq u \end{array}$$

Decisions

x_j: passengers flowing on arc j

Costs

c_j: cost of moving passenger on arc j

- Between flights: $\$50$/hour
- To 9pm flight: $\$75$ additional
- To NY: $\$0$ (as scheduled)

Supplies

b_i reserved passengers for flight i

- 9pm flight: $b_i = 0$
- NY supply: - total reserved passeng.

Capacities

u_j maximum passengers over arc j

- Between flights: $u_j = \infty$
- To NY: $u_i =$ flight capacity

Network flow solutions

Remove arc capacities

Goal: create equivalent network without arc capacities

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & 0 \leq x \leq u \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & \tilde{c}^T \tilde{x} \\ \text{subject to} & \tilde{A}\tilde{x} = \tilde{b} \\ & \tilde{x} \geq 0 \end{array}$$

Standard form LP with arc-node incidence matrix

Remove arc capacities

Idea: slack variables

$$x_j \leq u_j \quad \Rightarrow \quad x_j + s_j = u_j,\ s_j \geq 0$$

$$\downarrow$$

Network structure lost
no longer one -1
and one 1 per column

$$\cdots + \boxed{x_j} \ldots = b_p$$
$$\cdots - x_j \ldots = b_q$$
$$\boxed{x_j + s_j} = u_j$$

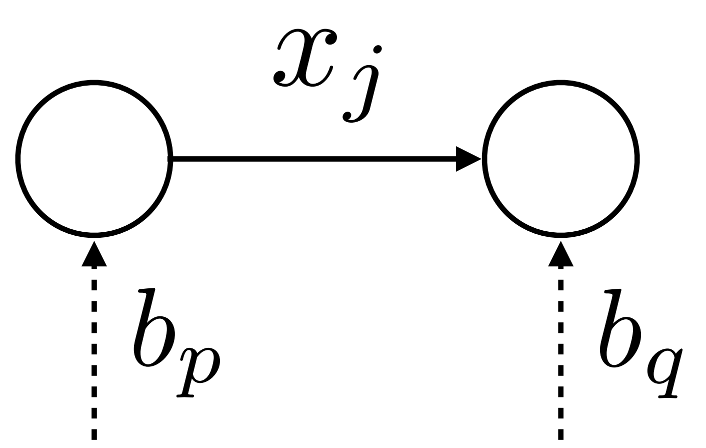
$$x_j = u_j - s_j$$

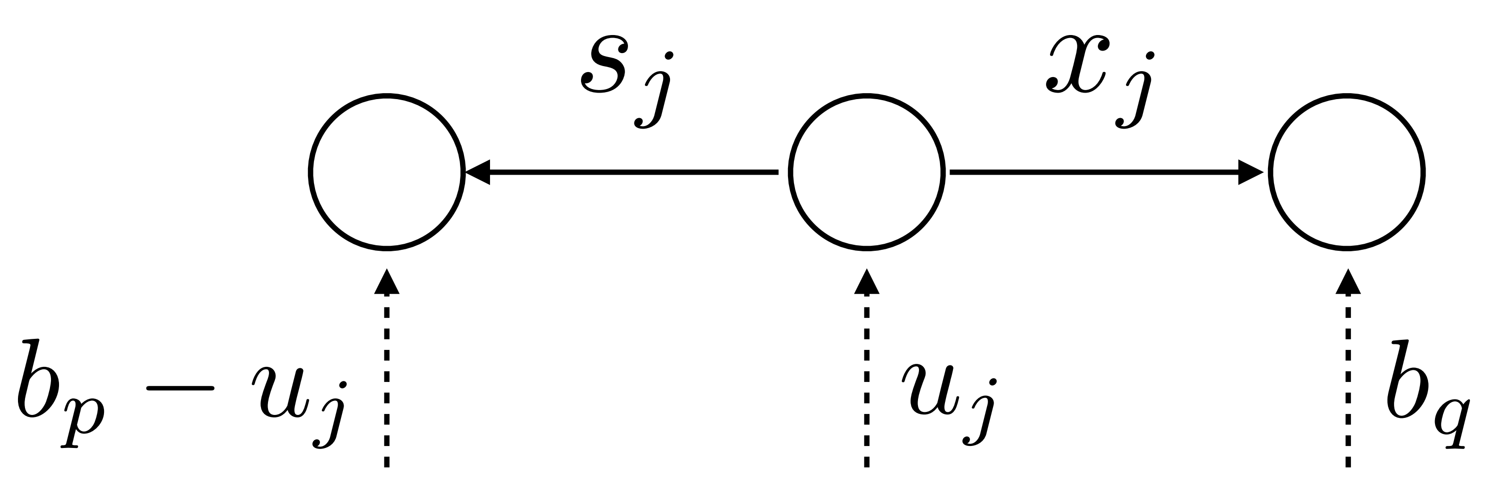
$$\downarrow$$

Network structure
recovered
(new node and new arc)

$$\cdots - s_j = b_p - u_j$$
$$\cdots - x_j \ldots = b_q$$
$$x_j + s_j = u_j$$

Nodes/arcs interpretation





Equivalent uncapacitated network flow

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax = b \\
& x \geq 0
\end{array}
$$

- A still an arc-node incidence matrix
- Can we say something about the extreme points?

Total unimodularity

A matrix is **totally unimodular** if all its minors are $-1, 0$ or 1
(minor is the determinant of a square submatrix of A)

example: a node-arc incidence matrix of a directed graph

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ -1 & -1 & 0 & 1 & 0 & -1 \end{bmatrix}$$

properties

- the entries of A_{ij} (i.e., its minors of order 1) are $-1, 0$, or 1
- The inverse of any nonsingular square submatrix of A has entries $+1, -1$, or 0

Integrality theorem

Given a polyhedron $P = \{x \in \mathbf{R}^n \mid Ax = b, \quad x \geq 0\}$

where

- A is totally unimodular
- b is an integer vector

$\longrightarrow$ all the extreme points of P are integer vectors.

Proof

- All extreme points are basic feasible solutions with $x_B = A_B^{-1}b$ and $x_i = 0, \ i \neq B$
- A_B^{-1} has integer components because of total unimodularity of A
- b has also integer components
- Therefore, also x is integral

$\blacksquare$

Implications for network and combinatorial optimization

Minimum cost network flow

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & 0 \leq x \leq u \end{array}$$

$\longrightarrow$ If b and u are integral solutions $x^\star$ are integral

Integer linear programs

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & 0 \leq x \leq u \\ & x \in \mathbf{Z}^n \end{array}$$

Very difficult in general
(more on this in a few weeks)

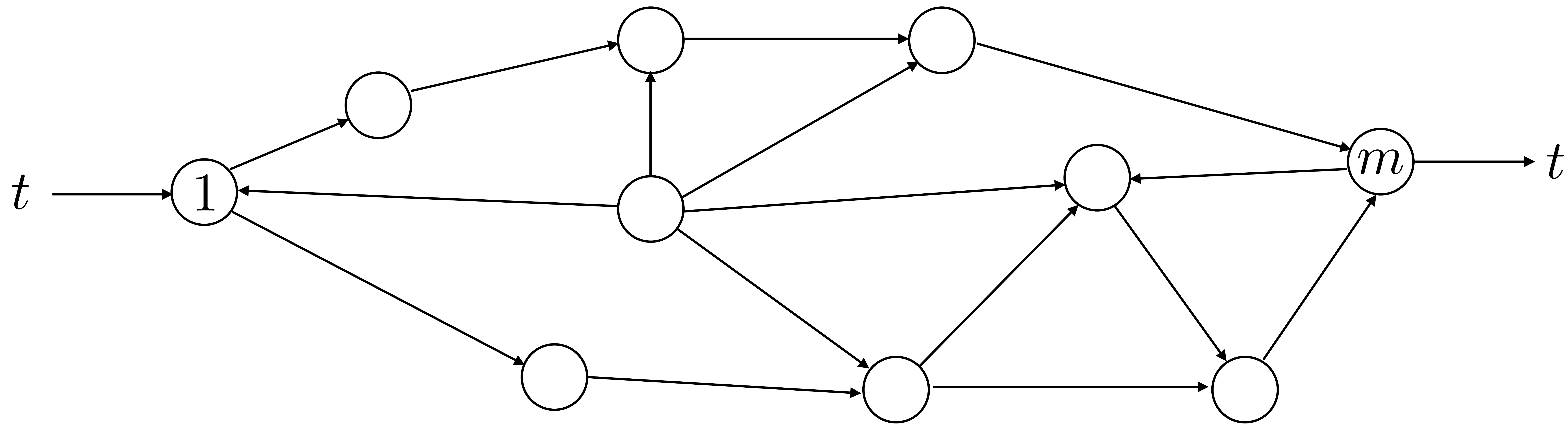
$\downarrow$

If A totally unimodular and b, u integral, we can relax integrality and solve a fast LP instead

Examples

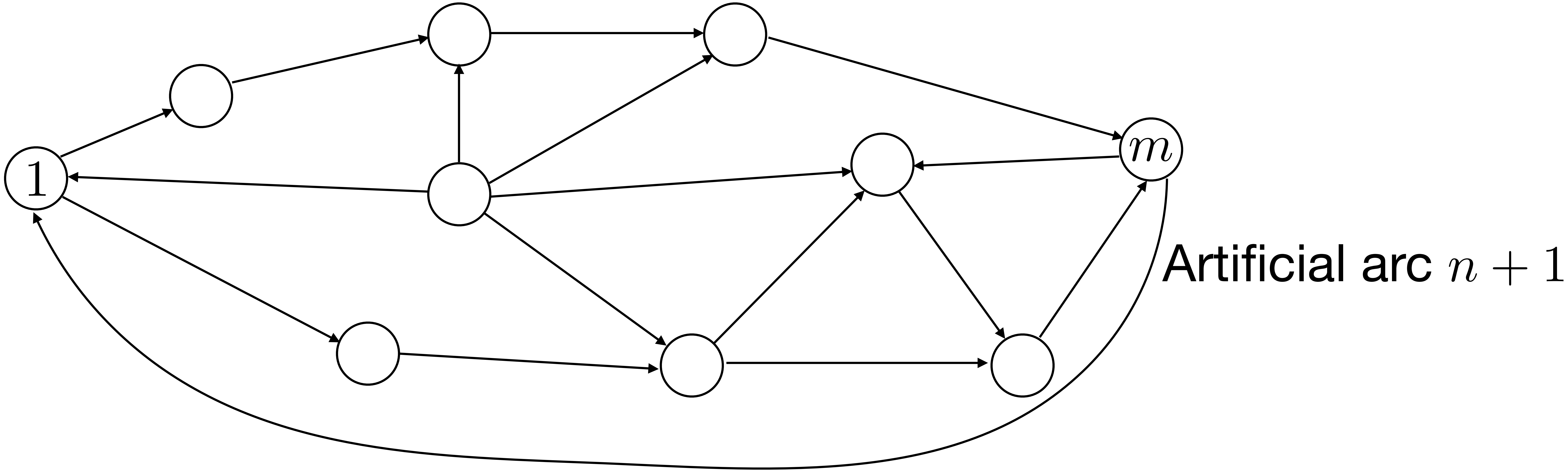
Maximum flow problem

Goal maximize flow from node 1 (source)
to node m (sink) through the network



$$
\begin{array}{ll}
\text{maximize} & t \\
\text{subject to} & Ax = te \\
& 0 \leq x \leq u
\end{array}
\qquad e = (1, 0, \ldots, 0, -1)
$$

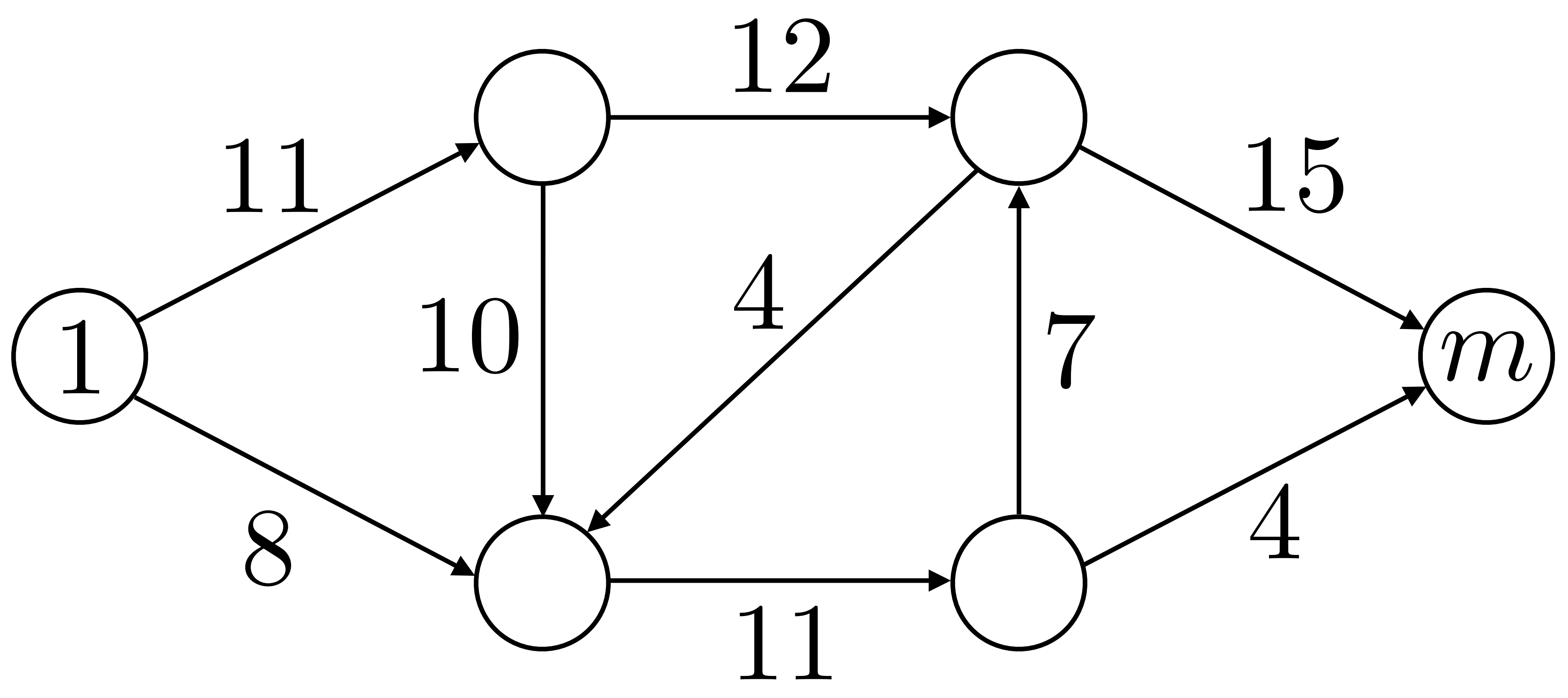
Maximum flow as minimum cost flow



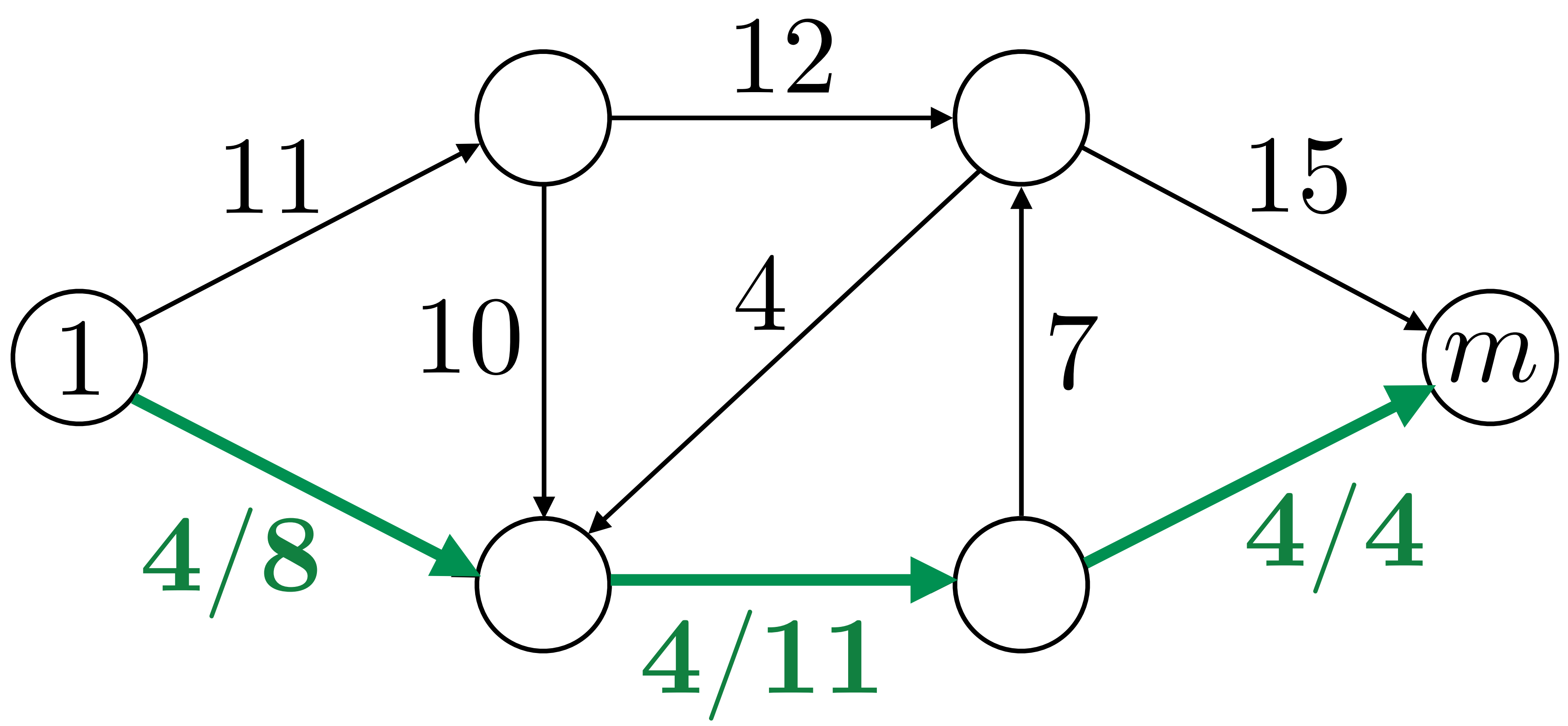
$$\begin{array}{ll} \text{minimize} & -t \\ \text{subject to} & \begin{bmatrix} A & -e \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} = 0 \\ & 0 \le \begin{bmatrix} x \\ t \end{bmatrix} \le \begin{bmatrix} u \\ \infty \end{bmatrix} \end{array}$$

Maximum flow example

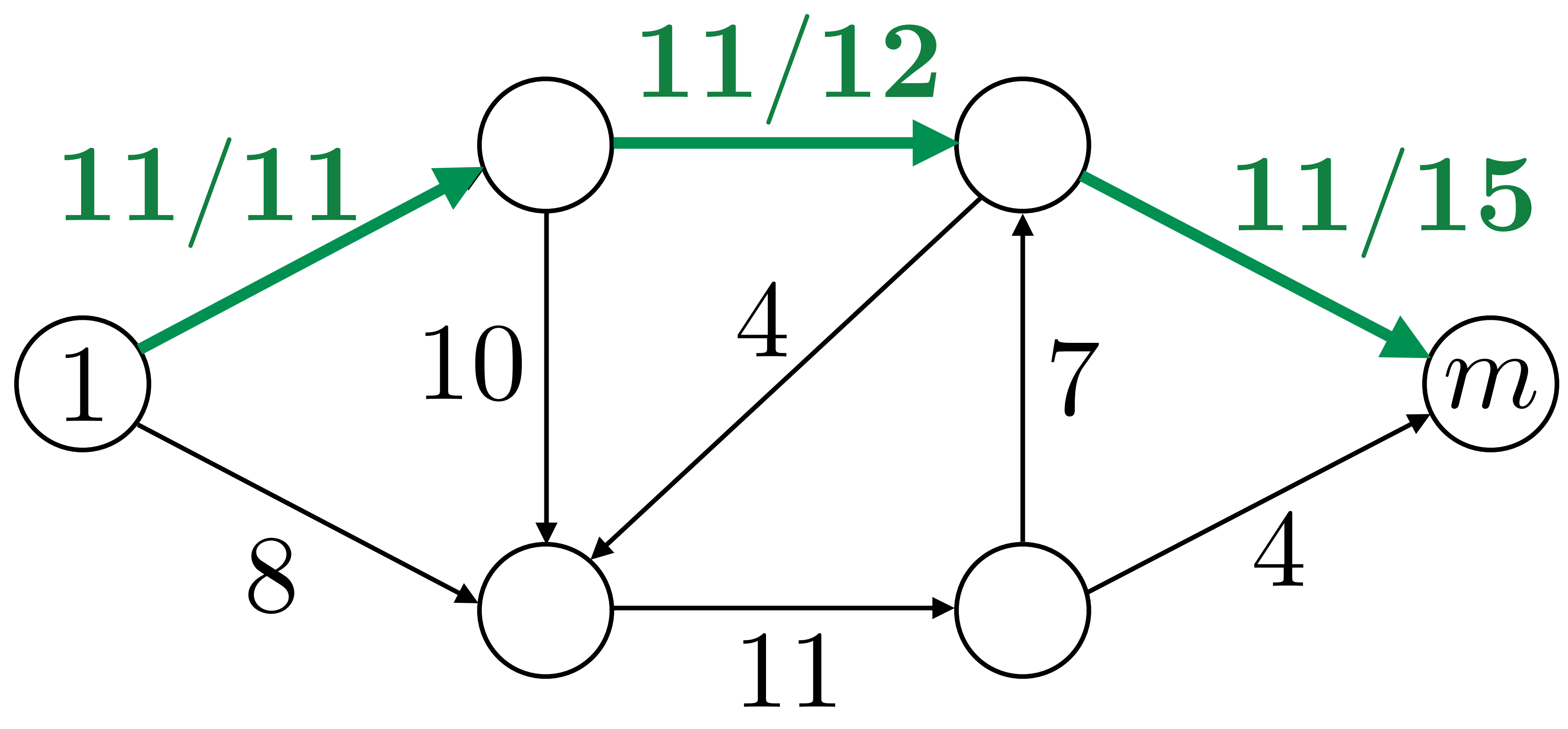
(arc capacities shown)



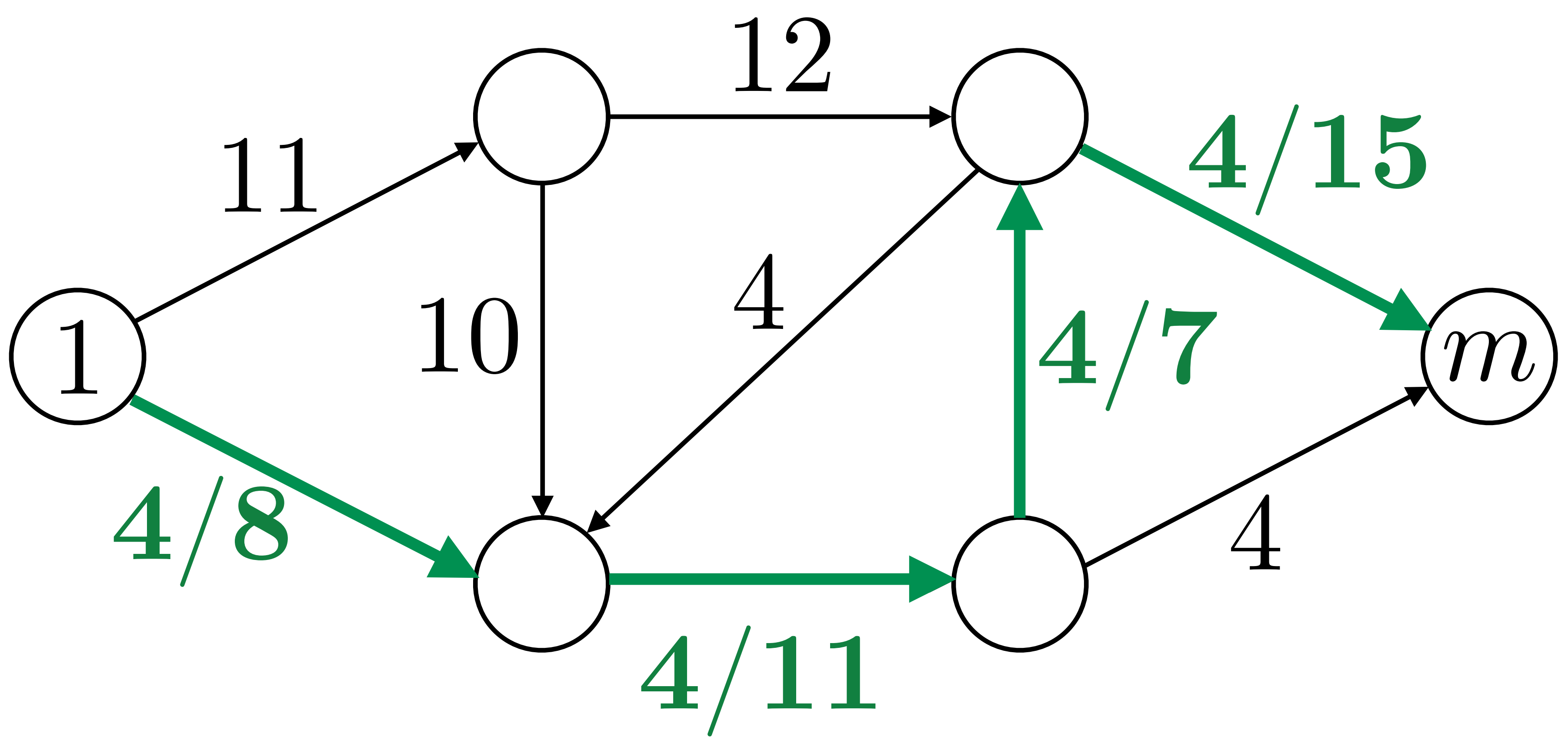
First flow



Second flow



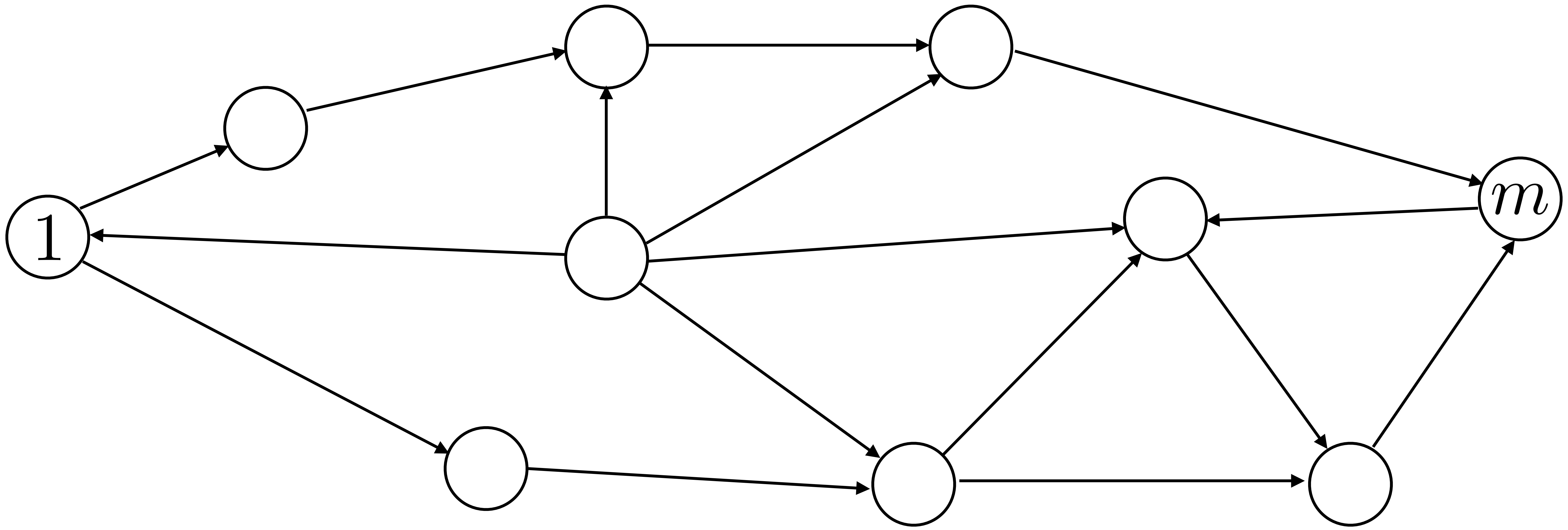
Third flow



Total flow: 19

Shortest path problem

Goal Find the shortest path between nodes 1 and m



paths can be represented as vectors $x \in \{0, 1\}^n$

Formulation

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = e \\ & x \in \{0, 1\}^n \end{array}$$

- c_j is the "length" of arc j
- $e = (1, 0, \ldots, 0, -1)$
- Variables are binary (include or not arc in path)

Shortest path as minimum cost flow

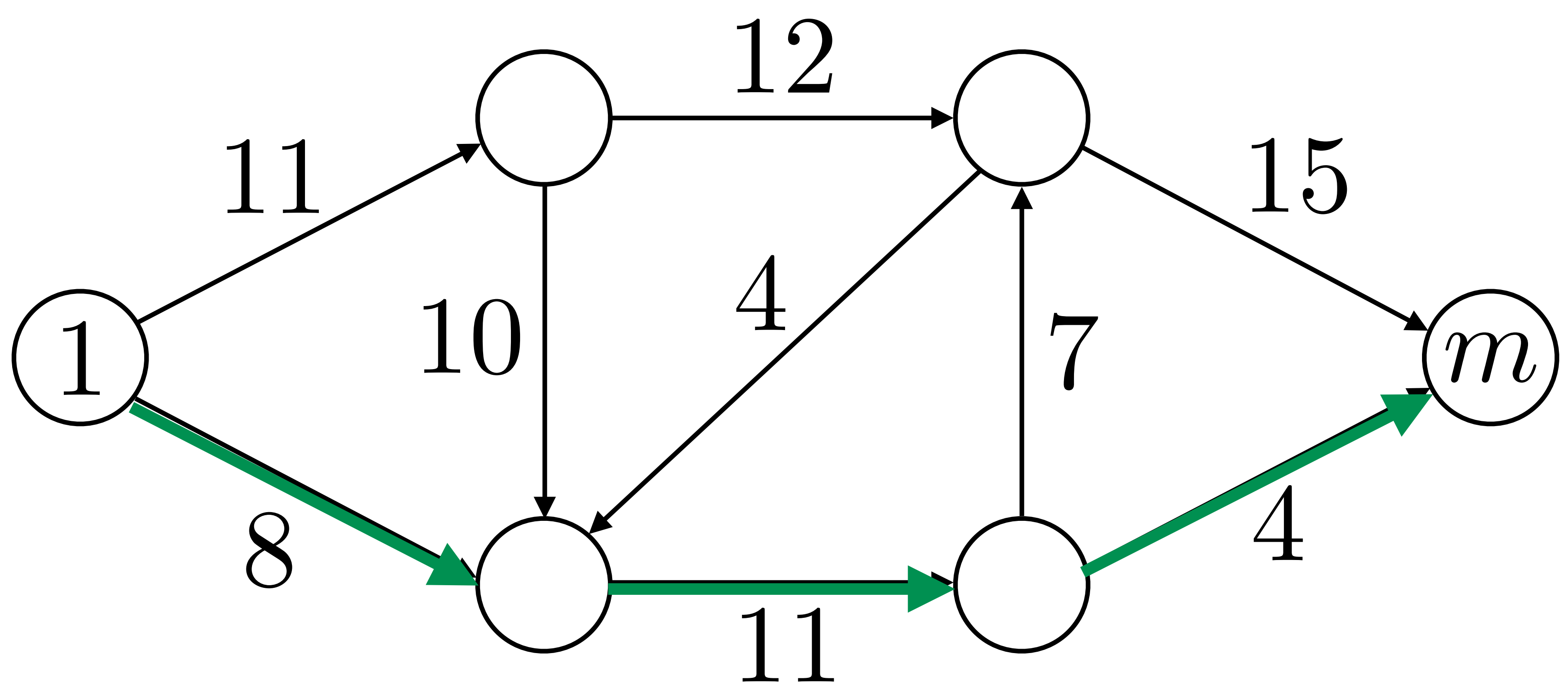
$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = e \\ & x \in \{0,1\}^n \end{array} \longrightarrow$$

Relaxation

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = e \\ & 0 \leq x \leq \mathbf{1} \end{array}$$

Extreme points satisfy $x_i \in \{0, 1\}$

Example (arc costs shown)



$$c = (11, 8, 10, 12, 4, 11, 7, 15, 4)$$
$$x^\star = (0, 1, 0, 0, 0, 1, 0, 0, 1)$$
$$c^T x^\star = 24$$

Assignment problem

Goal match N persons to N tasks

- Each person assigned to one task, each task to one person
- C_{ij} Cost of matching person i to task j

LP formulation

$$
\begin{array}{ll}
\text{minimize} & \displaystyle\sum_{i,j=1}^{N} C_{ij} X_{ij} \\
\text{subject to} & \displaystyle\sum_{i=1}^{N} X_{ij} = 1, \quad j = 1, \dots, N \\
& \displaystyle\sum_{j=1}^{N} X_{ij} = 1, \quad i = 1, \dots, N \\
& X_{ij} \in \{0, 1\}
\end{array}
$$

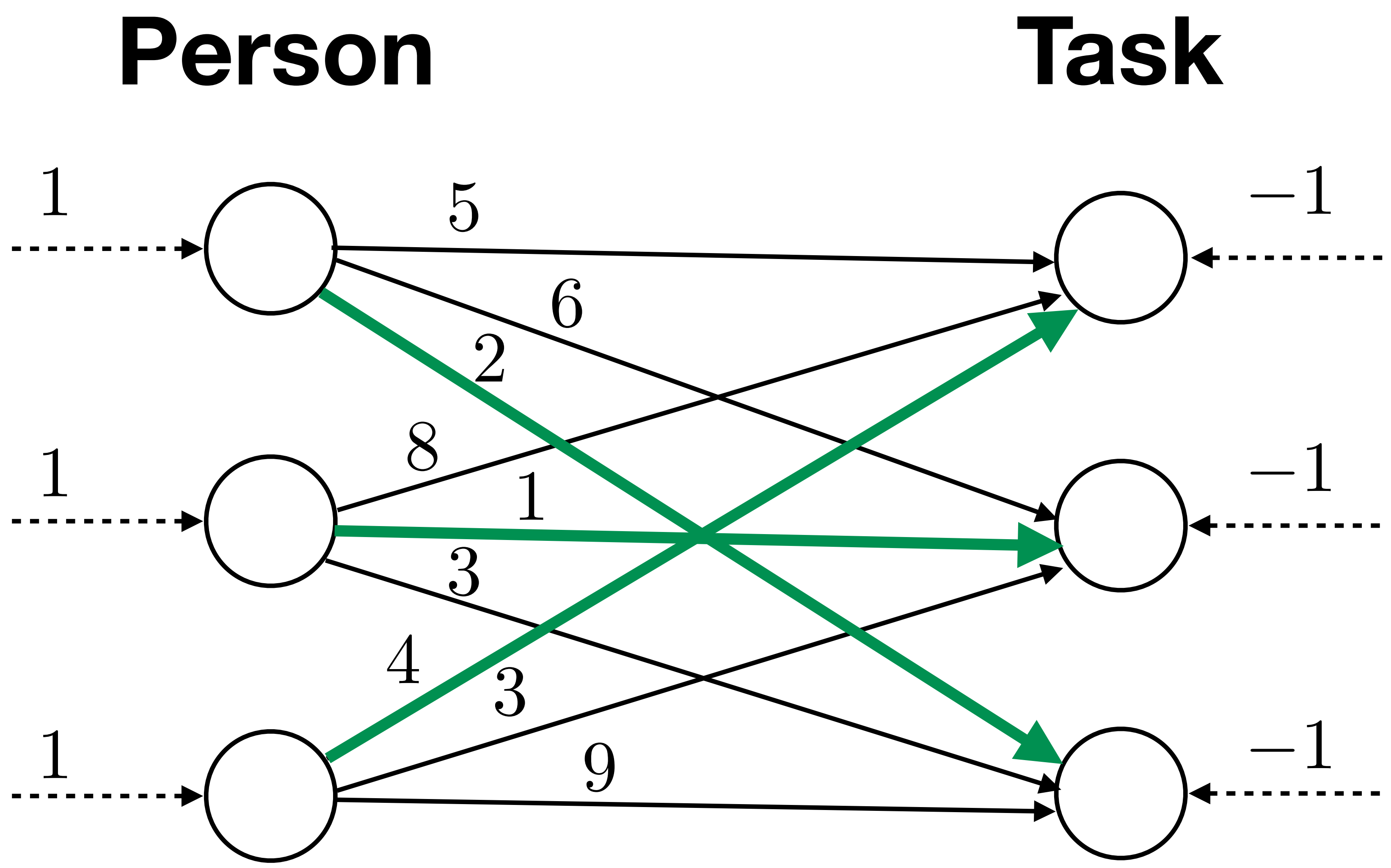
How do you define the network?

$$c = (5, 6, 2, 8, 1, 3, 4, 3, 9)$$

Task assignment as minimum cost network flow

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 \end{bmatrix}$$

$$b = (1, 1, 1, -1, -1, -1)$$



(arc costs shown)

Minimum cost network flow

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & \boxed{0 \leq x \leq \mathbf{1}} \end{array}$$

Extreme points satisfy $x_i \in \{0, 1\}$ $\longrightarrow$

Optimal solution

$$x^\star = (0, 0, 1, 0, 1, 0, 0, 0, 1)$$
$$c^T x^\star = 7$$

Network optimization

Today, we learned to:

- **Model** flows across networks
- **Formulate** minimum cost network flow problems
- **Analyze** network flow problem solutions (integrality theorem)
- **Formulate** maximum-flow, shortest path, and assignment problems as minimum cost network flows

References

- D. Bertsimas and J. Tsitsiklis: Introduction to Linear Optimization
  - Chapter 7: Network flow problems
- R. Vanderbei: Linear Programming
  - Chapter 14: Network Flow Problems
  - Chapter 15: Applications

Next lecture

- Interior point algorithms