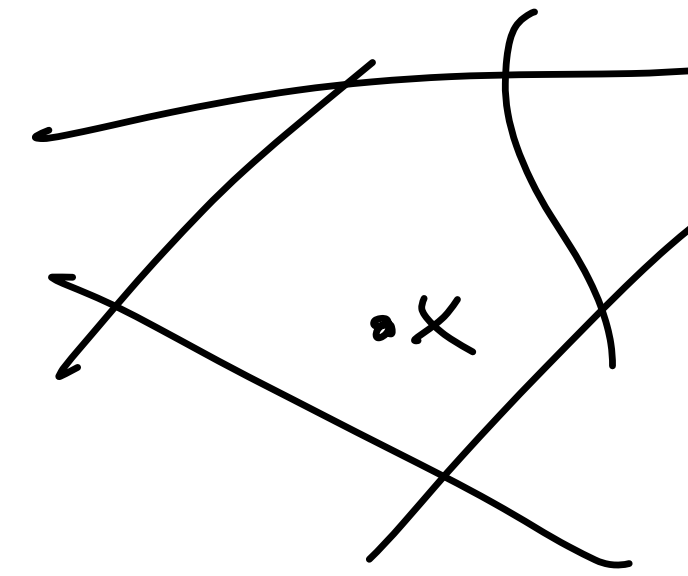


ORF307 – Optimization

15. Sensitivity analysis

Ed Forum

$$x \in P$$



$$y$$

$$a^T x \leq b$$

$$\underline{a^T x^* = b}$$

- I was wondering what the word "certificate" and "tight" means
- When is a game not zero-sum?
- How does the minmax theorem apply when there is no optimal strategy/Nash Equilibrium for both players?

$$x^T A y \quad \text{or} \quad y^T A x$$

Recap

Optimal objective values

Primal

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax \leq b \end{aligned}$$

p^* is the primal optimal value

Primal infeasible: $p^* = +\infty$

Primal unbounded: $p^* = -\infty$

Dual

$$\begin{aligned} &\text{maximize} && -b^T y \\ &\text{subject to} && A^T y + c = 0 \\ &&& y \geq 0 \end{aligned}$$

d^* is the dual optimal value

Dual infeasible: $d^* = -\infty$

Dual unbounded: $d^* = +\infty$

Weak duality

Theorem

If x, y satisfy:

- x is a feasible solution to the primal problem
 - y is a feasible solution to the dual problem
- $-b^T y \leq c^T x$

Weak duality

Theorem

If x, y satisfy:

- x is a feasible solution to the primal problem
 - y is a feasible solution to the dual problem
- $-b^T y \leq c^T x$

Proof

We know that $Ax \leq b$, $A^T y + c = 0$ and $y \geq 0$. Therefore,

$$0 \leq y^T (b - Ax) = b^T y - y^T Ax = c^T x + b^T y \quad \blacksquare$$

Weak duality

Theorem

If x, y satisfy:

- x is a feasible solution to the primal problem
 - y is a feasible solution to the dual problem
- $\longrightarrow -b^T y \leq c^T x$

Proof

We know that $Ax \leq b$, $A^T y + c = 0$ and $y \geq 0$. Therefore,

$$0 \leq y^T (b - Ax) = b^T y - y^T Ax = c^T x + b^T y \quad \blacksquare$$

Remark

- Any dual feasible y gives a **lower bound** on the primal optimal value
- Any primal feasible x gives an **upper bound** on the dual optimal value
- $c^T x + b^T y$ is the **duality gap**

Strong duality

Theorem

If a linear optimization problem has an optimal solution, so does its dual, and the optimal value of primal and dual are equal

$$d^* = p^*$$

Minmax theorem

Theorem

$$\max_{y \in P_n} \min_{x \in P_m} x^T A y = \min_{x \in P_m} \max_{y \in P_n} x^T A y$$

Minmax theorem

Theorem

$$\max_{y \in P_n} \min_{x \in P_m} x^T A y = \min_{x \in P_m} \max_{y \in P_n} x^T A y$$

Proof

The optimal x^* is the solution of

$$\begin{aligned} &\text{minimize} && t \\ &\text{subject to} && A^T x \leq t \mathbf{1} \\ &&& \mathbf{1}^T x = 1 \\ &&& x \geq 0 \end{aligned}$$

Minmax theorem

Theorem

$$\max_{y \in P_n} \min_{x \in P_m} x^T A y = \min_{x \in P_m} \max_{y \in P_n} x^T A y$$

Proof

The optimal x^* is the solution of

$$\begin{aligned} &\text{minimize} && t \\ &\text{subject to} && A^T x \leq t \mathbf{1} \\ & && \mathbf{1}^T x = 1 \\ & && x \geq 0 \end{aligned}$$

The optimal y^* is the solution of

$$\begin{aligned} &\text{maximize} && w \\ &\text{subject to} && A y \geq w \mathbf{1} \\ & && \mathbf{1}^T y = 1 \\ & && y \geq 0 \end{aligned}$$

Minmax theorem

Theorem

$$\max_{y \in P_n} \min_{x \in P_m} x^T A y = \min_{x \in P_m} \max_{y \in P_n} x^T A y$$

Proof

The optimal x^* is the solution of

$$\begin{aligned} &\text{minimize} && t \\ &\text{subject to} && A^T x \leq t \mathbf{1} \\ & && \mathbf{1}^T x = 1 \\ & && x \geq 0 \end{aligned}$$

The optimal y^* is the solution of

$$\begin{aligned} &\text{maximize} && w \\ &\text{subject to} && A y \geq w \mathbf{1} \\ & && \mathbf{1}^T y = 1 \\ & && y \geq 0 \end{aligned}$$

The two LPs are **duals** and by **strong duality** the equality follows. ■

Nash equilibrium

Theorem

$$\max_{y \in P_n} \min_{x \in P_m} x^T A y = \min_{x \in P_m} \max_{y \in P_n} x^T A y$$

Consequence

The pair of mixed strategies (x^*, y^*) attains the **Nash equilibrium** of the two-person matrix game, i.e.,

$$x^T A y^* \geq x^{*T} A y^* \geq x^{*T} A y, \quad \forall x \in P_m, \forall y \in P_n$$

Lagrangian and duality

Primal

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax \leq b \end{aligned}$$

Dual

$$\begin{aligned} &\text{maximize} && -b^T y \\ &\text{subject to} && A^T y + c = 0 \\ &&& y \geq 0 \end{aligned}$$

Lagrangian and duality

Primal

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax \leq b \end{aligned}$$

Dual

$$\begin{aligned} &\text{maximize} && -b^T y \\ &\text{subject to} && A^T y + c = 0 \\ &&& y \geq 0 \end{aligned}$$

Dual function

$$\begin{aligned} g(y) &= \underset{x}{\text{minimize}} (c^T x + y^T (Ax - b)) \\ &= -b^T y + \underset{x}{\text{minimize}} (c + A^T y)^T x \\ &= \begin{cases} -b^T y & \text{if } c + A^T y = 0 \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

Lagrangian and duality

Primal

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax \leq b \end{aligned}$$

Dual

$$\begin{aligned} &\text{maximize} && -b^T y \\ &\text{subject to} && A^T y + c = 0 \\ &&& y \geq 0 \end{aligned}$$

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Lagrangian

$$L(x, y) = c^T x + y^T (Ax - b)$$

Lagrangian and duality

Primal

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax \leq b \end{aligned}$$

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Lagrangian

$$L(x, y) = c^T x + y^T (Ax - b)$$

$$\nabla_x L(x, y) = c + A^T y = 0$$

Karush-Kuhn-Tucker conditions

Optimality conditions for linear optimization

Primal

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax \leq b \end{aligned}$$

Dual

$$\begin{aligned} &\text{maximize} && -b^T y \\ &\text{subject to} && A^T y + c = 0 \\ &&& y \geq 0 \end{aligned}$$

Primal feasibility

$$Ax \leq b$$

Dual feasibility

$$\nabla_x L(x, y) = A^T y + c = 0 \quad \text{and} \quad y \geq 0$$

Complementary slackness

$$y_i (Ax - b)_i = 0, \quad i = 1, \dots, m$$

Karush-Kuhn-Tucker conditions

Solving linear optimization problems

Primal

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax \leq b \quad (y) \\ &&& Cx = d \quad (z) \end{aligned}$$

Dual

$$\begin{aligned} &\text{maximize} && -b^T y \\ &\text{subject to} && A^T y + \cancel{c} + C^T z + c = 0 \\ &&& y \geq 0 \end{aligned}$$

We can solve our optimization problem by solving a system of equations

$$\begin{aligned} \nabla_x L(x, y) &= A^T y + \cancel{c} + C^T z + c = 0 \\ b - Ax &\geq 0 \\ y &\geq 0 \\ y^T (b - Ax) &= 0 \end{aligned}$$

$Cx - d = 0$

Today's lecture

Sensitivity analysis and ~~game theory~~

- Primal and dual simplex
- Adding variables and constraints
- Global sensitivity
- Local sensitivity

Primal and dual simplex

Optimality conditions

Primal problem

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Dual problem

$$\begin{aligned} \text{maximize} \quad & -b^T y \\ \text{subject to} \quad & A^T y + c \geq 0 \end{aligned}$$

Optimality conditions

Primal problem

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax = b \\ &&& x \geq 0 \end{aligned}$$

Dual problem

$$\begin{aligned} &\text{maximize} && -b^T y \\ &\text{subject to} && A^T y + c \geq 0 \end{aligned}$$

x and y are **primal** and **dual** optimal if and only if

- x is **primal feasible**: $Ax = b$ and $x \geq 0$
- y is **dual feasible**: $A^T y + c \geq 0$
- The **duality gap** is zero: $c^T x + b^T y = 0$

Primal and dual basic feasible solutions

Primal problem

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax = b \\ &&& x \geq 0 \end{aligned}$$

Dual problem

$$\begin{aligned} &\text{maximize} && -b^T y \\ &\text{subject to} && A^T y + c \geq 0 \end{aligned}$$

Given a **basis** ~~matrix~~ B

$$\text{Primal feasible: } Ax = b, x \geq 0 \quad \Rightarrow \quad x_B = A_B^{-1} b \geq 0$$

Primal and dual basic feasible solutions

Primal problem

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax = b \\ &&& x \geq 0 \end{aligned}$$

Dual problem

$$\begin{aligned} &\text{maximize} && -b^T y \\ &\text{subject to} && A^T y + c \geq 0 \end{aligned}$$

Given a **basis** matrix B

Primal feasible: $Ax = b, x \geq 0 \Rightarrow x_B = A_B^{-1}b \geq 0$

Dual feasible: $A^T y + c \geq 0$. Set $y = -A_B^{-T}c_B$. Dual feasible if $\bar{c} = c + A^T y \geq 0$

Primal and dual basic feasible solutions

Primal problem

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax = b \\ &&& x \geq 0 \end{aligned}$$

Dual problem

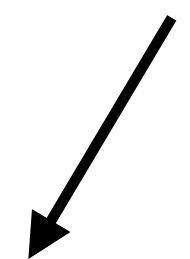
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Given a **basis** matrix B

Primal feasible: $Ax = b, x \geq 0 \Rightarrow x_B = A_B^{-1}b \geq 0$

Dual feasible: $A^T y + c \geq 0$. Set $y = -A_B^{-T}c_B$. Dual feasible if $\bar{c} = c + A^T y \geq 0$

Reduced costs



Primal and dual basic feasible solutions

Primal problem

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax = b \\ &&& x \geq 0 \end{aligned}$$

Dual problem

$$\begin{aligned} &\text{maximize} && -b^T y \\ &\text{subject to} && A^T y + c \geq 0 \end{aligned}$$

Given a **basis** matrix B

Primal feasible: $Ax = b, x \geq 0 \Rightarrow \underline{x_B} = A_B^{-1} b \geq 0$

Dual feasible: $A^T y + c \geq 0$. Set $y = -A_B^{-T} c_B$. Dual feasible if $\bar{c} = c + A^T y \geq 0$

Zero duality gap: $c^T x + b^T y = c_B^T x_B - \underbrace{b^T A_B^{-T} c_B}_{\text{TRANSPOSE}} = c_B^T x_B - c_B^T \underbrace{A_B^{-1} b}_{\text{Reduced costs}} = 0$

Reduced costs

Primal and dual basic feasible solutions

Primal problem

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax = b \\ &&& x \geq 0 \end{aligned}$$

Dual problem

$$\begin{aligned} &\text{maximize} && -b^T y \\ &\text{subject to} && A^T y + c \geq 0 \end{aligned}$$

Given a **basis** matrix B

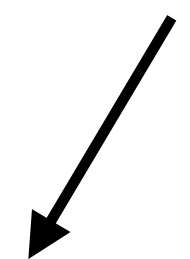
Primal feasible: $Ax = b, x \geq 0 \Rightarrow x_B = A_B^{-1}b \geq 0$

Dual feasible: $A^T y + c \geq 0$. Set $y = -A_B^{-T}c_B$. Dual feasible if $\bar{c} = c + A^T y \geq 0$

Zero duality gap: $c^T x + b^T y = c_B^T x_B - b^T A_B^{-T}c_B = c_B x_B - c_B^T A_B^{-1}b = 0$

(by construction)

Reduced costs



The primal (dual) simplex method

Primal problem

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax = b \\ &&& x \geq 0 \end{aligned}$$

Dual problem

$$\begin{aligned} &\text{maximize} && -b^T y \\ &\text{subject to} && A^T y + c \geq 0 \end{aligned}$$

The primal (dual) simplex method

Primal problem

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax = b \\ &&& x \geq 0 \end{aligned}$$

Dual problem

$$\begin{aligned} &\text{maximize} && -b^T y \\ &\text{subject to} && A^T y + c \geq 0 \end{aligned}$$

Primal simplex

- Primal feasibility
- Zero duality gap



Dual feasibility

The primal (dual) simplex method

Primal problem

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax = b \\ &&& x \geq 0 \end{aligned}$$

Primal simplex

- Primal feasibility
- Zero duality gap



Dual feasibility $\bar{c} \geq 0$

Dual problem

$$\begin{aligned} &\text{maximize} && -b^T y \\ &\text{subject to} && A^T y + c \geq 0 \end{aligned}$$

Dual simplex (solve dual instead)

- Dual feasibility
- Zero duality gap



Primal feasibility $x \geq 0$

**Adding new constraints and
variables**

Adding new variables

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Solution x^*, y^*

Adding new variables

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & c^T x + c_{n+1} x_{n+1} \\ \text{subject to} & Ax + A_{n+1} x_{n+1} = b \\ & x, x_{n+1} \geq 0 \end{array}$$

Solution x^*, y^*

Adding new variables

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & c^T x + c_{n+1} x_{n+1} \\ \text{subject to} & Ax + A_{n+1} x_{n+1} = b \\ & x, x_{n+1} \geq 0 \end{array}$$

Solution x^*, y^*

Is the solution $(x^*, 0), y^*$ **optimal** for the new problem?

Adding new variables

Optimality conditions

minimize $c^T x + c_{n+1} x_{n+1}$

subject to $Ax + A_{n+1} x_{n+1} = b \longrightarrow$ Solution $(x^*, 0)$ is still **primal feasible**

$x, x_{n+1} \geq 0$

Adding new variables

Optimality conditions

minimize $c^T x + c_{n+1} x_{n+1}$

subject to $Ax + A_{n+1} x_{n+1} = b \longrightarrow$ Solution $(x^*, 0)$ is still **primal feasible**

$$x, x_{n+1} \geq 0$$

Is y^* still **dual feasible**?

$$A_{n+1}^T y^* + c_{n+1} \geq 0$$

Adding new variables

Optimality conditions

$$\begin{array}{l} \max \quad -b^T y \\ \text{s.t.} \quad A^T y + c \geq 0 \\ \quad \quad [A_{n+1}^T y + c_{n+1} \geq 0 \end{array}$$

$$\text{minimize} \quad c^T x + c_{n+1} x_{n+1}$$

$$\text{subject to} \quad Ax + A_{n+1} x_{n+1} = b \longrightarrow \text{Solution } (x^*, 0) \text{ is still } \mathbf{primal\ feasible}$$

$$x, x_{n+1} \geq 0$$

Is y^* still **dual feasible**?

$$A_{n+1}^T y^* + c_{n+1} \geq 0$$

Yes

$(x^*, 0)$ still **optimal** for new problem

Otherwise

Primal simplex

Adding new variables

Example

$$\text{minimize} \quad -60x_1 - 30x_2 - 20x_3$$

$$\text{subject to} \quad 8x_1 + 6x_2 + x_3 \leq 48$$

$$4x_1 + 2x_2 + 1.5x_3 \leq 20$$

$$2x_1 + 1.5x_2 + 0.5x_3 \leq 8$$

$$x \geq 0$$

Adding new variables

Example

$$\begin{array}{ll} \text{minimize} & -60x_1 - 30x_2 - 20x_3 \\ \text{subject to} & 8x_1 + 6x_2 + x_3 \leq 48 \\ & 4x_1 + 2x_2 + 1.5x_3 \leq 20 \\ & 2x_1 + 1.5x_2 + 0.5x_3 \leq 8 \\ & x \geq 0 \end{array} \quad \text{-profit}$$

Adding new variables

Example

minimize

$$-60x_1 - 30x_2 - 20x_3$$

-profit

subject to

$$8x_1 + 6x_2 + x_3 \leq 48$$

material

$$4x_1 + 2x_2 + 1.5x_3 \leq 20$$

$$2x_1 + 1.5x_2 + 0.5x_3 \leq 8$$

$$x \geq 0$$

Adding new variables

Example

minimize

$$-60x_1 - 30x_2 - 20x_3$$

-profit

subject to

$$8x_1 + 6x_2 + x_3 \leq 48$$

material

$$4x_1 + 2x_2 + 1.5x_3 \leq 20$$

production

$$2x_1 + 1.5x_2 + 0.5x_3 \leq 8$$

$$x \geq 0$$

Adding new variables

Example

minimize

$$-60x_1 - 30x_2 - 20x_3$$

subject to

$$8x_1 + 6x_2 + x_3 \leq 48$$

$$4x_1 + 2x_2 + 1.5x_3 \leq 20$$

$$2x_1 + 1.5x_2 + 0.5x_3 \leq 8$$

$$x \geq 0$$

-profit

material

production

quality control

Adding new variables

Example

$$\begin{array}{ll} \text{minimize} & -60x_1 - 30x_2 - 20x_3 \\ \text{subject to} & 8x_1 + 6x_2 + x_3 \leq 48 \\ & 4x_1 + 2x_2 + 1.5x_3 \leq 20 \\ & 2x_1 + 1.5x_2 + 0.5x_3 \leq 8 \\ & x \geq 0 \end{array}$$

-profit
material
production
quality control

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

$$c = (-60, -30, -20, 0, 0, 0)$$

$$A = \begin{bmatrix} 8 & 6 & 1 & 1 & 0 & 0 \\ 4 & 2 & 1.5 & 0 & 1 & 0 \\ 2 & 1.5 & 0.5 & 0 & 0 & 1 \end{bmatrix}$$

$$b = (48, 20, 8)$$

Adding new variables

Example

$$\begin{array}{ll} \text{minimize} & -60x_1 - 30x_2 - 20x_3 \\ \text{subject to} & 8x_1 + 6x_2 + x_3 \leq 48 \\ & 4x_1 + 2x_2 + 1.5x_3 \leq 20 \\ & 2x_1 + 1.5x_2 + 0.5x_3 \leq 8 \\ & x \geq 0 \end{array}$$

-profit
material
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quality control

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$$b = (48, 20, 8)$$

$$x^* = (2, 0, 8, 24, 0, 0), \quad y^* = (0, 10, 10), \quad c^T x^* = -280, \quad \text{basis } \{1, 3, 4\}$$

Adding new variables

Example: add new product?

$$\begin{aligned} \text{minimize} \quad & c^T x + c_{n+1} x_{n+1} \\ \text{subject to} \quad & Ax + A_{n+1} x_{n+1} = b \\ & x, x_{n+1} \geq 0 \end{aligned}$$

$$c = (-60, -30, -20, 0, 0, 0, -15)$$

$$A = \begin{bmatrix} 8 & 6 & 1 & 1 & 0 & 0 & 1 \\ 4 & 2 & 1.5 & 0 & 1 & 0 & 1 \\ 2 & 1.5 & 0.5 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$b = (48, 20, 8)$$

Adding new variables

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$$\begin{aligned} \text{minimize} \quad & c^T x + c_{n+1} x_{n+1} \\ \text{subject to} \quad & Ax + A_{n+1} x_{n+1} = b \\ & x, x_{n+1} \geq 0 \end{aligned}$$

$$c = (-60, -30, -20, 0, 0, 0, -15)$$

$$A = \begin{bmatrix} 8 & 6 & 1 & 1 & 0 & 0 & 1 \\ 4 & 2 & 1.5 & 0 & 1 & 0 & 1 \\ 2 & 1.5 & 0.5 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$b = (48, 20, 8)$$

Previous solution

$$x^* = (2, 0, 8, 24, 0, 0), \quad y^* = (0, 10, 10), \quad c^T x^* = -280, \quad \text{basis } \{1, 3, 4\}$$

Adding new variables

Example: add new product?

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$$c = (-60, -30, -20, 0, 0, 0, -15)$$

$$A = \begin{bmatrix} 8 & 6 & 1 & 1 & 0 & 0 & 1 \\ 4 & 2 & 1.5 & 0 & 1 & 0 & 1 \\ 2 & 1.5 & 0.5 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$b = (48, 20, 8)$$

Previous solution

$$x^* = (2, 0, 8, 24, 0, 0), \quad y^* = (0, 10, 10), \quad c^T x^* = -280, \quad \text{basis } \{1, 3, 4\}$$

$(x^*, 0)$ is still optimal

$$A_{n+1}^T y^* + c_{n+1} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 10 \\ 10 \end{bmatrix} - 15 = 5 \geq 0$$

Adding new variables

Example: add new product?

$$\begin{aligned} \text{minimize} \quad & c^T x + c_{n+1} x_{n+1} \\ \text{subject to} \quad & Ax + A_{n+1} x_{n+1} = b \\ & x, x_{n+1} \geq 0 \end{aligned}$$

$$c = (-60, -30, -20, 0, 0, 0, \overbrace{-15}^{-25})$$

$$A = \begin{bmatrix} 8 & 6 & 1 & 1 & 0 & 0 & 1 \\ 4 & 2 & 1.5 & 0 & 1 & 0 & 1 \\ 2 & 1.5 & 0.5 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$b = (48, 20, 8)$$

Previous solution

$$x^* = (2, 0, 8, 24, 0, 0), \quad y^* = (0, 10, 10), \quad c^T x^* = -280, \quad \text{basis } \{1, 3, 4\}$$

$(x^*, 0)$ is still optimal

$$A_{n+1}^T y^* + c_{n+1} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 10 \\ 10 \end{bmatrix} - 15 = 5 \geq 0$$

-25 = -5 < 0

Shall we add a new product?

Adding new constraints

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Solution x^*, y^*

Adding new constraints

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Solution x^*, y^*



$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & a_{m+1}^T x = b_{m+1} \\ & x \geq 0 \end{array}$$

Adding new constraints

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & a_{m+1}^T x = b_{m+1} \\ & x \geq 0 \end{array}$$

Solution x^*, y^*

Dual

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + a_{m+1} y_{m+1} + c \geq 0 \end{array}$$

Adding new constraints

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & a_{m+1}^T x = b_{m+1} \\ & x \geq 0 \end{array}$$

Solution x^*, y^*

Dual

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + a_{m+1} y_{m+1} + c \geq 0 \end{array}$$

Is the solution $x^*, (y^*, 0)$ **optimal** for the new problem?

Adding new constraints

Optimality conditions

maximize $-b^T y$
subject to $A^T y + \cancel{a_{m+1}} y_{m+1} + c \geq 0 \longrightarrow$ Solution $(y^*, 0)$ is still **dual feasible**

Adding new constraints

Optimality conditions

maximize $-b^T y$
subject to $A^T y + a_{m+1} y_{m+1} + c \geq 0 \longrightarrow$ Solution $(y^*, 0)$ is still **dual feasible**

Is x^* still **primal feasible**?

$$Ax = b$$

$$a_{m+1}^T x = b_{m+1}$$

$$x \geq 0$$

Adding new constraints

Optimality conditions

maximize $-b^T y$
subject to $A^T y + a_{m+1} y_{m+1} + c \geq 0 \longrightarrow$ Solution $(y^*, 0)$ is still **dual feasible**

Is x^* still **primal feasible**?

$$Ax = b$$

$$a_{m+1}^T x = b_{m+1}$$

$$x \geq 0$$

Yes

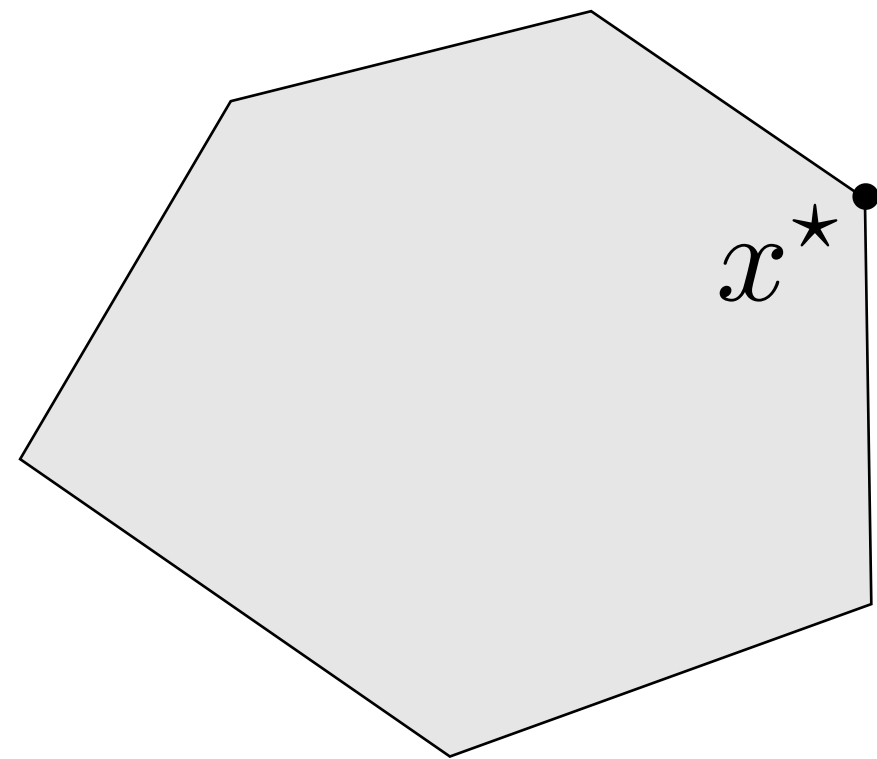
x^* still **optimal** for new problem

Otherwise

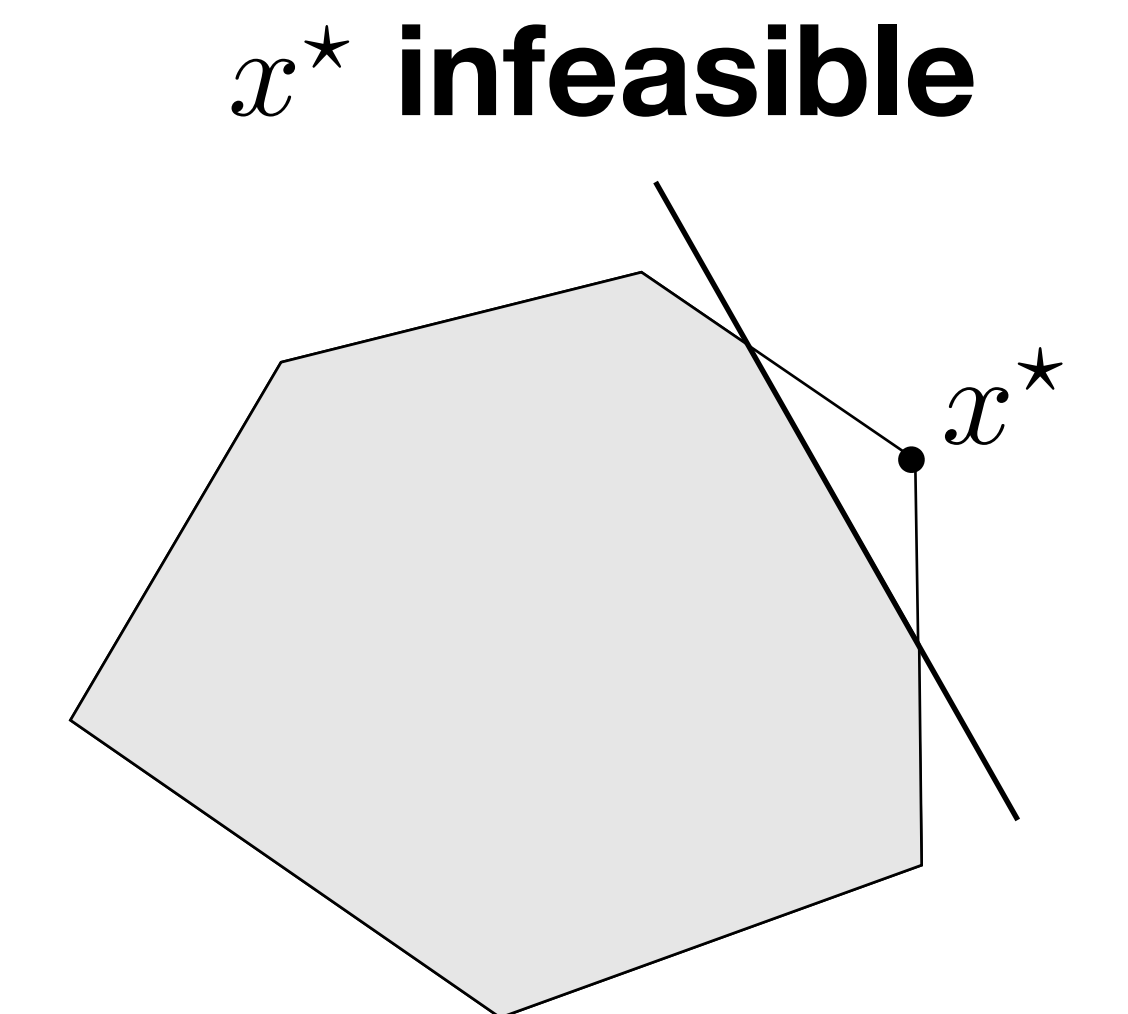
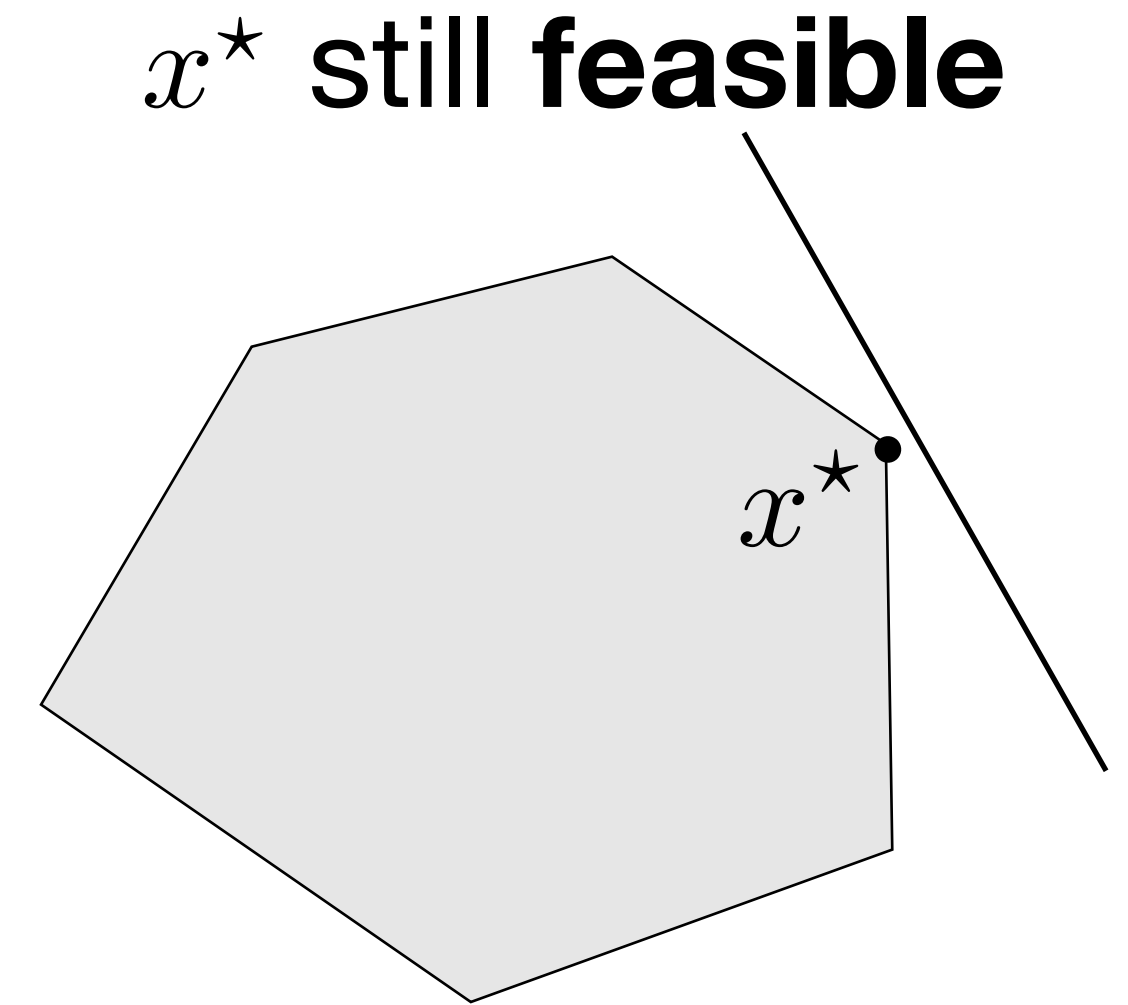
Dual simplex

Adding new constraints

Example



Add new constraint



Global sensitivity analysis

Changes in problem data

Goal: extract information from x^*, y^* about their sensitivity with respect to changes in problem data

Modified LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b + u \\ & x \geq 0 \end{array}$$

Optimal cost $p^*(u)$

Global sensitivity

Dual of modified LP

$$\begin{aligned} &\text{maximize} && -(b + u)^T y \\ &\text{subject to} && A^T y + c \geq 0 \end{aligned}$$

Global sensitivity

Dual of modified LP

$$\begin{aligned} &\text{maximize} && -(b + u)^T y \\ &\text{subject to} && A^T y + c \geq 0 \end{aligned}$$

Global lower bound

Given y^* a dual optimal solution for $u = 0$, then

$$\begin{aligned} p^*(u) &\geq -(b + u)^T y^* && \text{(from weak duality and} \\ &= \underbrace{p^*(0)}_{-b^T y^*} - u^T y^* && \text{dual feasibility)} \end{aligned}$$

Global sensitivity

Dual of modified LP

$$\begin{aligned} &\text{maximize} && -(b + u)^T y \\ &\text{subject to} && A^T y + c \geq 0 \end{aligned}$$

Global lower bound

Given y^* a dual optimal solution for $u = 0$, then

$$\begin{aligned} p^*(u) &\geq -(b + u)^T y^* && \text{(from weak duality and} \\ &= p^*(0) - u^T y^* && \text{dual feasibility)} \end{aligned}$$

It holds for any u

Global sensitivity

Example

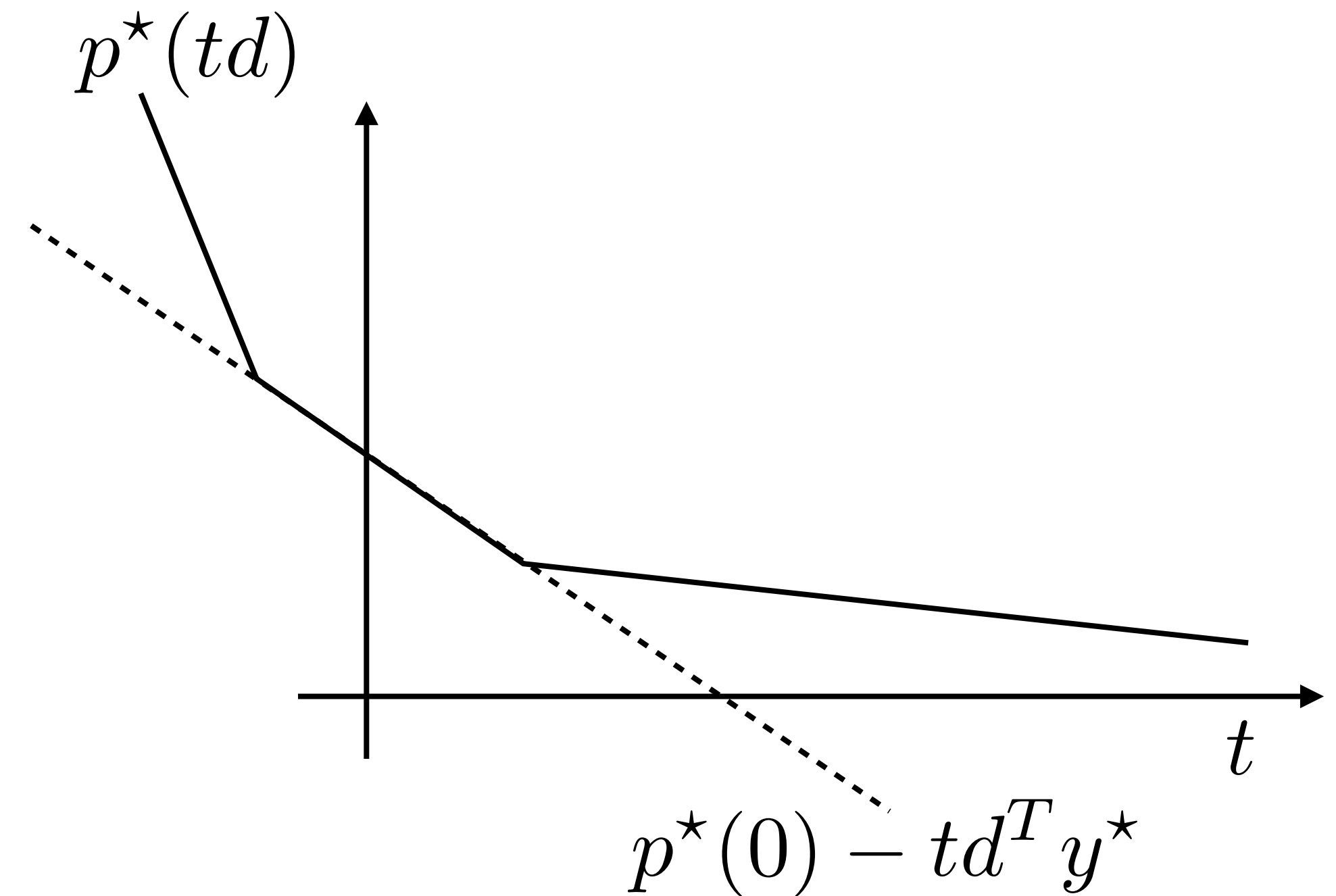
Take $u = td$ with $d \in \mathbf{R}^m$ fixed

$$\text{minimize } c^T x$$

$$\text{subject to } Ax = b + td$$

$$x \geq 0$$

$p^*(td)$ is the optimal value as a function of t



Global sensitivity

Example

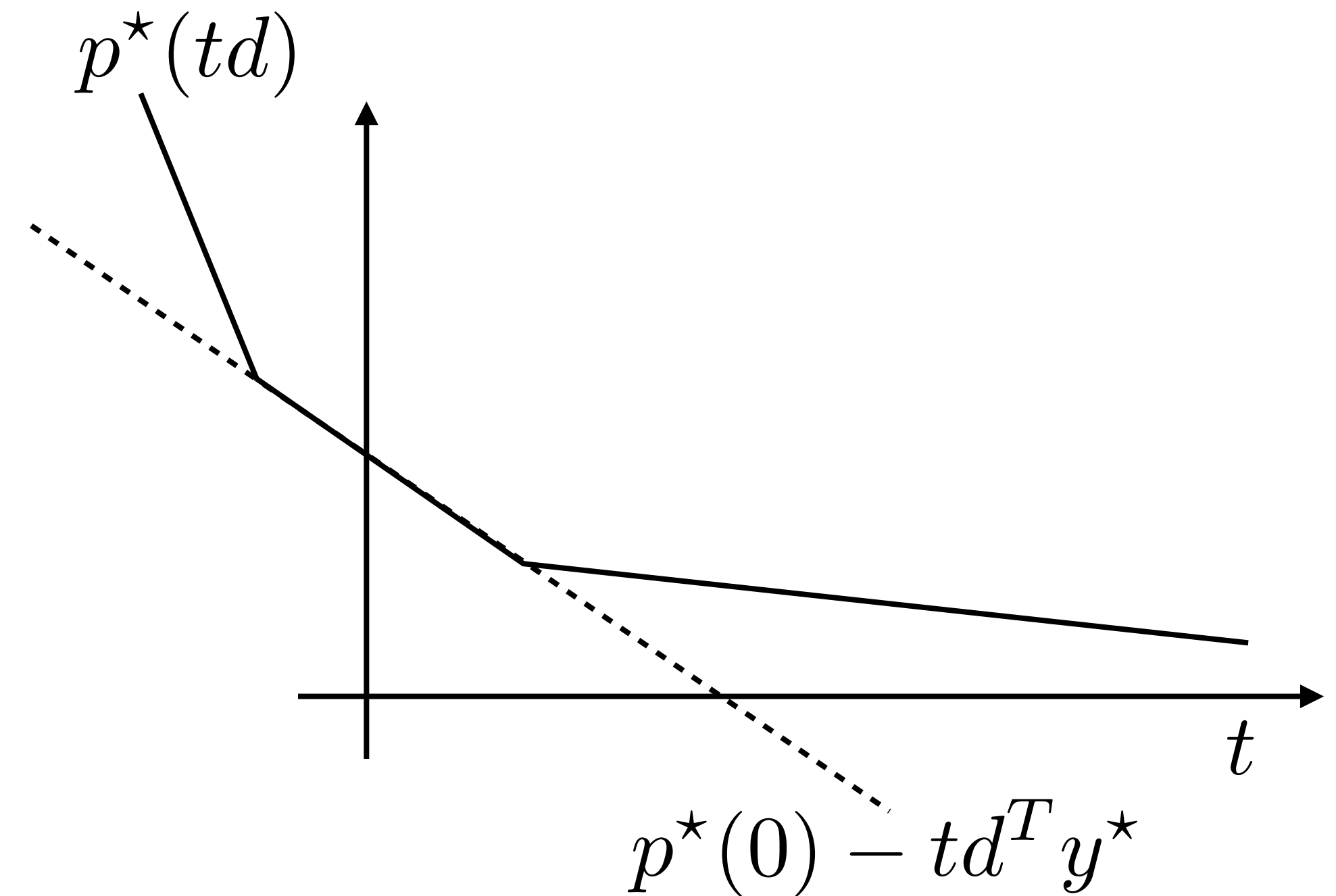
Take $u = td$ with $d \in \mathbf{R}^m$ fixed

$$\text{minimize } c^T x$$

$$\text{subject to } Ax = b + td$$

$$x \geq 0$$

$p^*(td)$ is the optimal value as a function of t



Sensitivity information (assuming $d^T y^* \geq 0$)

- $t < 0$ the optimal value increases
- $t > 0$ the optimal value decreases (not so much if t is small)

Optimal value function

$$p^*(u) = \min\{c^T x \mid Ax = b + u, x \geq 0\}$$

Assumption: $p^*(0)$ is finite

Properties

- $p^*(u) > -\infty$ everywhere (from global lower bound)
- $p^*(u)$ is piecewise-linear on its domain

Optimal value function is piecewise linear

Proof

$$p^*(u) = \min\{c^T x \mid Ax = b + u, x \geq 0\}$$

Optimal value function is piecewise linear

Proof

$$p^*(u) = \min\{c^T x \mid Ax = b + u, x \geq 0\}$$

Dual feasible set

$$D = \{y \mid A^T y + c \geq 0\}$$

Assumption: $p^*(0)$ is finite

Optimal value function is piecewise linear

Proof

Dual feasible set

$$p^*(u) = \min\{c^T x \mid Ax = b + u, x \geq 0\}$$

$$D = \{y \mid A^T y + c \geq 0\}$$

Assumption: $p^*(0)$ is finite

If $p^*(u)$ finite

$$\underline{p^*(u)} = \max_{y \in D} -(b + u)^T y = \max_{k=1, \dots, r} \overline{Q_k^T u + T_k}$$

y_1, \dots, y_r are the extreme points of D

Local sensitivity analysis

Local sensitivity

u in neighborhood of the origin

Original LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \longrightarrow$$

Optimal solution

$$\begin{array}{ll} \text{Primal} & x_i^* = 0, \quad i \notin B \\ & x_B^* = A_B^{-1} b \\ \text{Dual} & y^* = -A_B^{-T} c_B \end{array}$$

Local sensitivity

u in neighborhood of the origin

Original LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \longrightarrow$$

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Modified LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b + u \\ & x \geq 0 \end{array}$$

Modified dual

$$\begin{array}{ll} \text{maximize} & -(b + u)^T y \\ \text{subject to} & A^T y + c \geq 0 \end{array}$$

**Optimal basis
does not change**

Local sensitivity

u in neighborhood of the origin

Original LP

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax = b \\ &&& x \geq 0 \end{aligned}$$



Optimal solution

$$\begin{aligned} &\text{Primal} && x_i^* = 0, \quad i \notin B \\ &&& x_B^* = A_B^{-1} b \\ &\text{Dual} && y^* = -A_B^{-T} c_B \end{aligned}$$

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Modified dual

$$\begin{aligned} &\text{maximize} && -(b + u)^T y \\ &\text{subject to} && A^T y + c \geq 0 \end{aligned}$$

**Optimal basis
does not change**

Modified optimal solution

$$\begin{aligned} x_B^*(u) &= A_B^{-1} (b + u) = x_B^* + A_B^{-1} u \\ y^*(u) &= y^* \end{aligned}$$

Derivative of the optimal value function

Modified optimal solution

$$x_B^*(u) = A_B^{-1}(b + u) = x_B^* + A_B^{-1}u$$

$$y^*(u) = y^*$$

Derivative of the optimal value function

Modified optimal solution

$$x_B^*(u) = A_B^{-1}(b + u) = \underbrace{x_B^* + A_B^{-1}u}_{\text{blue arrow pointing to } c^T x^*}$$
$$y^*(u) = y^*$$

$$c^T x^* = c_B^T x_B^*$$

Optimal value function

$$p^*(u) = c^T x^*(u) \quad \text{blue arrow pointing to } x^*(u)$$
$$= c^T x^* + c_B^T A_B^{-1}u$$
$$= p^*(0) - y^{*T}u \quad \text{(affine for small } u)$$

Derivative of the optimal value function

Modified optimal solution

$$x_B^*(u) = A_B^{-1}(b + u) = x_B^* + A_B^{-1}u$$

$$y^*(u) = y^*$$

Optimal value function

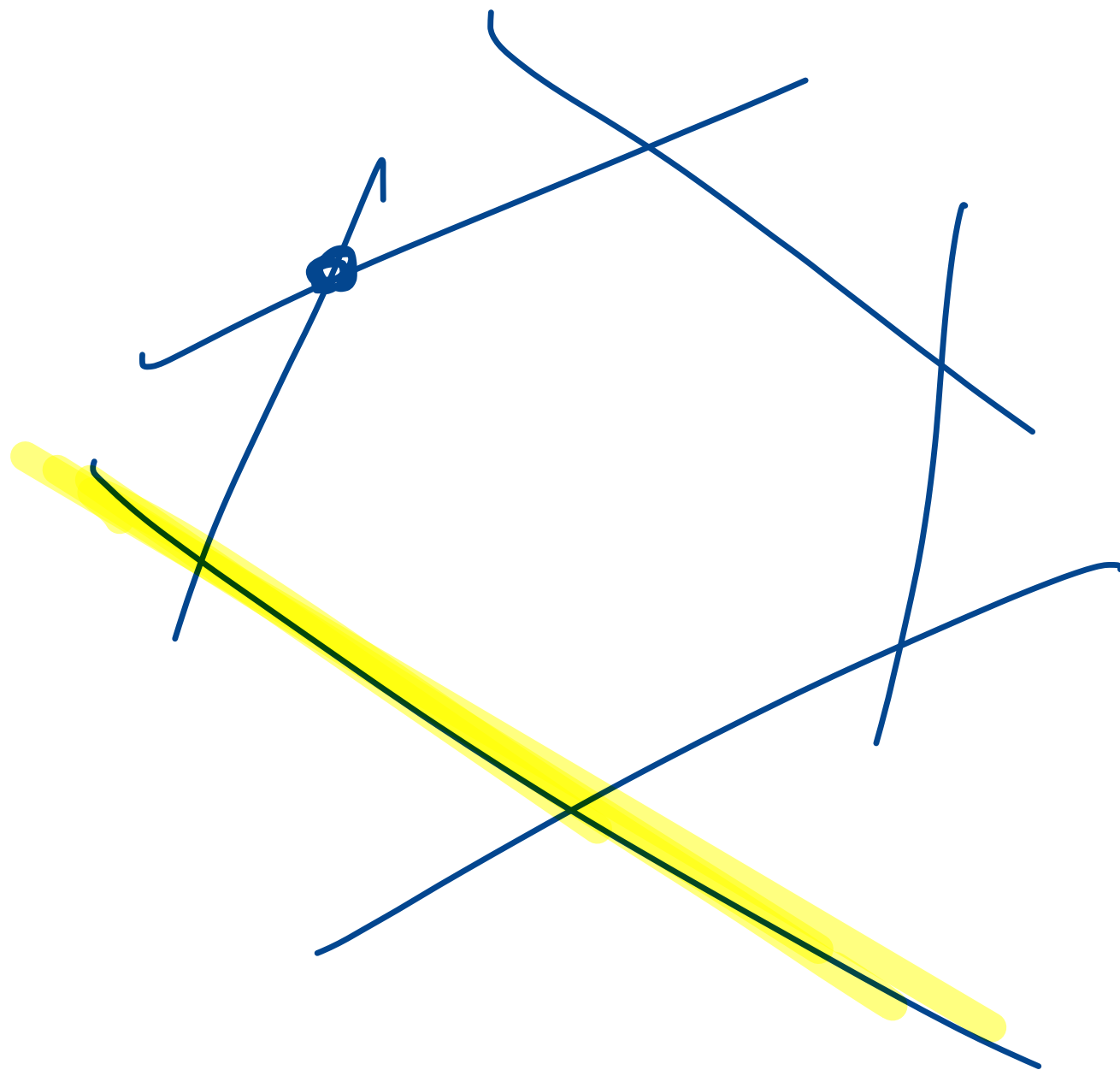
$$p^*(u) = c^T x^*(u)$$

$$= c^T x^* + c_B^T A_B^{-1}u$$

$$= p^*(0) - y^{*T}u \quad (\text{affine for small } u)$$

Local derivative

$$\nabla p^*(u) = -y^* \quad (y^* \text{ are the shadow prices})$$



Sensitivity example

$$\begin{array}{ll} \text{minimize} & -60x_1 - 30x_2 - 20x_3 \\ \text{subject to} & 8x_1 + 6x_2 + x_3 \leq 48 \\ & 4x_1 + 2x_2 + 1.5x_3 \leq 20 \\ & 2x_1 + 1.5x_2 + 0.5x_3 \leq 8 \\ & x \geq 0 \end{array}$$

Sensitivity example

minimize $-60x_1 - 30x_2 - 20x_3$ -profit

subject to $8x_1 + 6x_2 + x_3 \leq 48$

$4x_1 + 2x_2 + 1.5x_3 \leq 20$

$2x_1 + 1.5x_2 + 0.5x_3 \leq 8$

$x \geq 0$

Sensitivity example

minimize

$$-60x_1 - 30x_2 - 20x_3$$

-profit

subject to

$$8x_1 + 6x_2 + x_3 \leq 48$$

material

$$4x_1 + 2x_2 + 1.5x_3 \leq 20$$

$$2x_1 + 1.5x_2 + 0.5x_3 \leq 8$$

$$x \geq 0$$

Sensitivity example

minimize

$$-60x_1 - 30x_2 - 20x_3$$

subject to

$$8x_1 + 6x_2 + x_3 \leq 48$$

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$$2x_1 + 1.5x_2 + 0.5x_3 \leq 8$$

$$x \geq 0$$

-profit

material

production

Sensitivity example

minimize

$$-60x_1 - 30x_2 - 20x_3$$

subject to

$$8x_1 + 6x_2 + x_3 \leq 48$$

$$4x_1 + 2x_2 + 1.5x_3 \leq 20$$

$$2x_1 + 1.5x_2 + 0.5x_3 \leq 8$$

$$x \geq 0$$

-profit

material

production

quality control

Sensitivity example

minimize

$$-60x_1 - 30x_2 - 20x_3$$

-profit

subject to

$$8x_1 + 6x_2 + x_3 \leq 48$$

material

$$4x_1 + 2x_2 + 1.5x_3 \leq 20$$

production

$$2x_1 + 1.5x_2 + 0.5x_3 \leq 8$$

quality control

$$x \geq 0$$

$$x^* = (2, 0, 8, 24, 0, 0), \quad y^* = (0, 10, 10), \quad c^T x^* = -280, \quad \text{basis } \{1, 3, 4\}$$

Sensitivity example

minimize

$$-60x_1 - 30x_2 - 20x_3$$

-profit

subject to

$$8x_1 + 6x_2 + x_3 \leq 48$$

material

$$4x_1 + 2x_2 + 1.5x_3 \leq 20$$

production

$$2x_1 + 1.5x_2 + 0.5x_3 \leq 8$$

quality control

$$x \geq 0$$

$$x^* = (2, 0, 8, 24, 0, 0), \quad y^* = (0, 10, 10), \quad c^T x^* = -280, \quad \text{basis } \{1, 3, 4\}$$

What does $y_3^* = 10$ mean?

Sensitivity example

minimize	$-60x_1 - 30x_2 - 20x_3$	-profit
subject to	$8x_1 + 6x_2 + x_3 \leq 48$	material
	$4x_1 + 2x_2 + 1.5x_3 \leq 20$	production
	$2x_1 + 1.5x_2 + 0.5x_3 \leq 8$	quality control
	$x \geq 0$	

$$x^* = (2, 0, 8, 24, 0, 0), \quad y^* = (0, 10, 10), \quad c^T x^* = -280, \quad \text{basis } \{1, 3, 4\}$$

What does $y_3^* = 10$ mean?

Let's increase the quality control budget by 1, i.e., $u = (0, 0, 1)$

$$p^*(u) = p^*(0) - y^{*T} u = -280 - 10 = -290$$

Sensitivity analysis

Today, we learned to:

- **Reuse** primal and dual solutions when variables or constraints are added
- **Analyze** value function as problem parameters change
- **Compute** local sensitivity to parameter perturbations

References

- D. Bertsimas and J. Tsitsiklis: Introduction to Linear Optimization
 - Chapter 5: Sensitivity analysis
- R. Vanderbei: “Linear Programming”
 - Chapter 7: Sensitivity and parametric analysis

Next lecture

- Network optimization