ORF307 – Optimization

13. Duality

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Ed Forum

- I was still a little confused about the notation for the bases, and was wondering if we could cover this in more detail (i.e. B = {1, 2}, A_B, etc.)?
- A question I have is about the software, is python the best platform to do optimization problems on? What do large companies use?

Complexity

Complexity of a single simplex iteration

1. Compute the reduced costs $\bar{c}$
    - Solve $A_B^T p = c_B$
    - $\bar{c} = c - A^T p$
2. If $\bar{c} \geq 0$, x **optimal. break**
3. Choose j such that $\bar{c}_j < 0$
4. Compute search direction d with $d_j = 1$ and $A_B d_B = -A_j$
5. If $d_B \geq 0$, the problem is **unbounded** and the optimal value is $-\infty$. **break**
6. Compute step length $\theta^\star = \min_{\{i \in B \mid d_i < 0\}} \left(-\frac{x_i}{d_i}\right)$
7. Define y such that $y = x + \theta^\star d$
8. Get new basis $\bar{B}$ (i exits and j enters)

Bottleneck
Two linear systems

Linear system solutions

Very similar linear systems

$$A_B^T p = c_B$$
$$A_B d_B = -A_j$$

$\longrightarrow$

LU factorization

$(2/3)n^3$ flops

$$A_B = PLU$$

$\longrightarrow$

Easy linear systems

$4n^2$ flops

$$U^T L^T P^T p = c_B$$
$$PLU d_B = -A_j$$

Factorization is expensive

Do we need to recompute it at every iteration?

Basis update

Index update

- j enters (x_j becomes $\theta^\star$)
- $i = B(\ell)$ exists (x_i becomes 0)

$\longrightarrow$

Basis matrix change

$$A_{\bar{B}} = A_B + (A_i - A_j)e_\ell^T$$

Example

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$B = \{4, 1, 6\} \quad \rightarrow \quad \bar{B} = \{4, 1, 2\}$$

- 2 enters
- $6 = B(3)$ exists

$$A_{\bar{B}} = \underset{A_B}{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}} + \underset{A_2 e_3^T}{\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}} - \underset{A_6 e_3^T}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

Smarter linear system solution

Basis matrix change

$$A_{\bar{B}} = A_B + \overbrace{(A_i - A_j)}^{v} e_\ell^T \longrightarrow$$

Matrix inversion lemma
(from homework 2)

$$(A_B + v e_\ell^T)^{-1} = \left(I - \frac{1}{1 + e_\ell^T A_B^{-1} v} A_B^{-1} v e_\ell^T \right) A_B^{-1}$$

Solve $A_{\bar{B}} d_{\bar{B}} = -A_j$

1. Solve $A_B z^1 = e_\ell$ ($2n^2$ flops)
2. Solve $A_B z^2 = -A_j$ ($2n^2$ flops)
3. Solve $d_{\bar{B}} = z^2 - \frac{v^T z^2}{1 + v^T z^1} z^1$

Remarks

- Same complexity for $A_B^T p = c_B$ ($4n^2$ flops)
- k-th next iteration ($4kn^2$ flops, derive as exercise...)
- Once in a while (e.g., $k = 100$), better to refactor A_B

Complexity of a single simplex iteration

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8. Get new basis $\bar{B}$ (i exits and j enters)

Bottleneck
Two linear systems

$\longrightarrow$

Matrix inversion lemma trick
$\approx n^2$ per iteration
(very cheap)

How many iterations do we need?

Complexity of the simplex method

Example of worst-case behavior

Innocent-looking problem

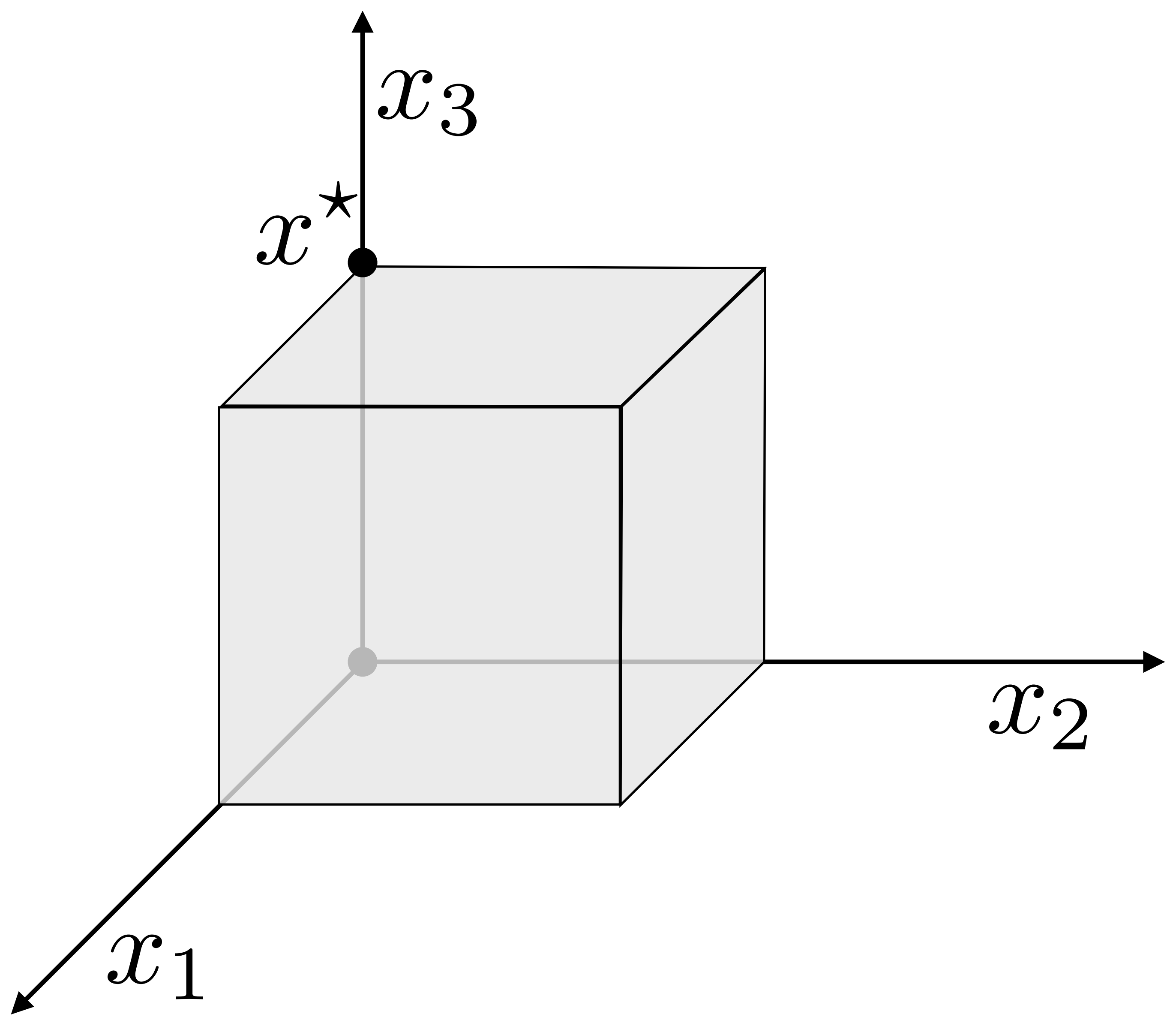
minimize $\quad -x_n$

subject to $\quad 0 \leq x \leq 1$

2^n **vertices**

$2^n/2$ vertices: cost = 1

$2^n/2$ vertices: cost = 0



Perturb unit cube

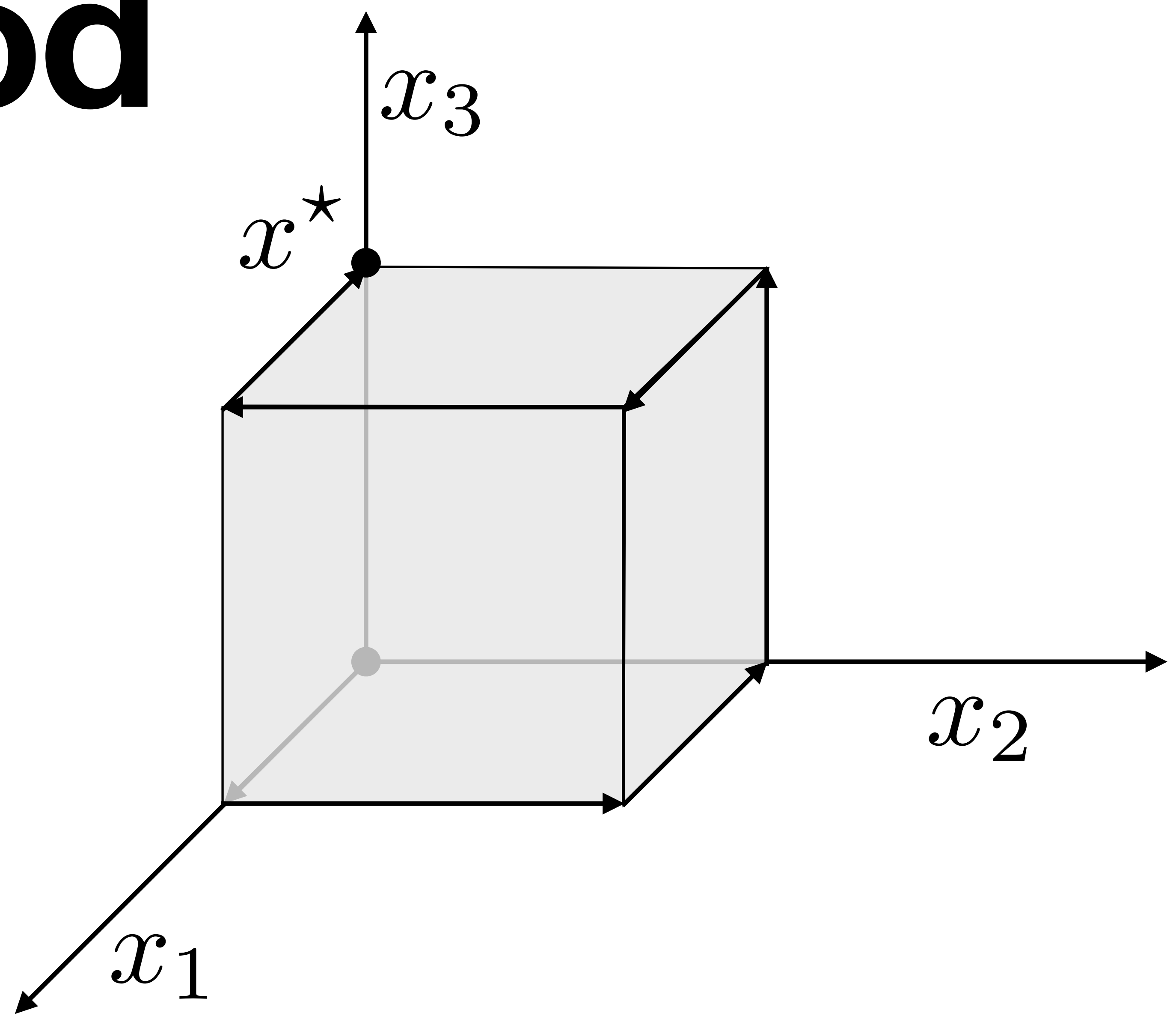
minimize $\quad -x_n$

subject to $\quad \epsilon \leq x_1 \leq 1$

$$\epsilon x_{i-1} \leq x_i \leq 1 - \epsilon x_{i-1}, \quad i = 2, \dots, n$$

Complexity of the simplex method

Example of worst-case behavior



$$
\begin{array}{ll}
\text{minimize} & -x_n \\
\text{subject to} & \epsilon \leq x_1 \leq 1 \\
& \epsilon x_{i-1} \leq x_i \leq 1 - \epsilon x_{i-1}, \quad i = 2, \dots, n
\end{array}
$$

Theorem

- The vertices can be ordered so that each one is adjacent to and has a **lower cost than the previous one**
- There exists a pivoting rule under which the simplex method terminates after $2^n - 1$ **iterations**

Remark

- A **different pivot rule** would have converged in one iteration.
- We have a bad example for every pivot rule.

Complexity of the simplex method

We do **not know any polynomial version of the simplex method**, no matter which pivoting rule we pick. → Still **open research question!**

Worst-case

There are problem instances where the simplex method will run an **exponential number of iterations** in terms of the dimensions, e.g. 2^n

Good news: average-case
Practical performance is very good. On average, it stops in n iterations.

Average simplex complexity

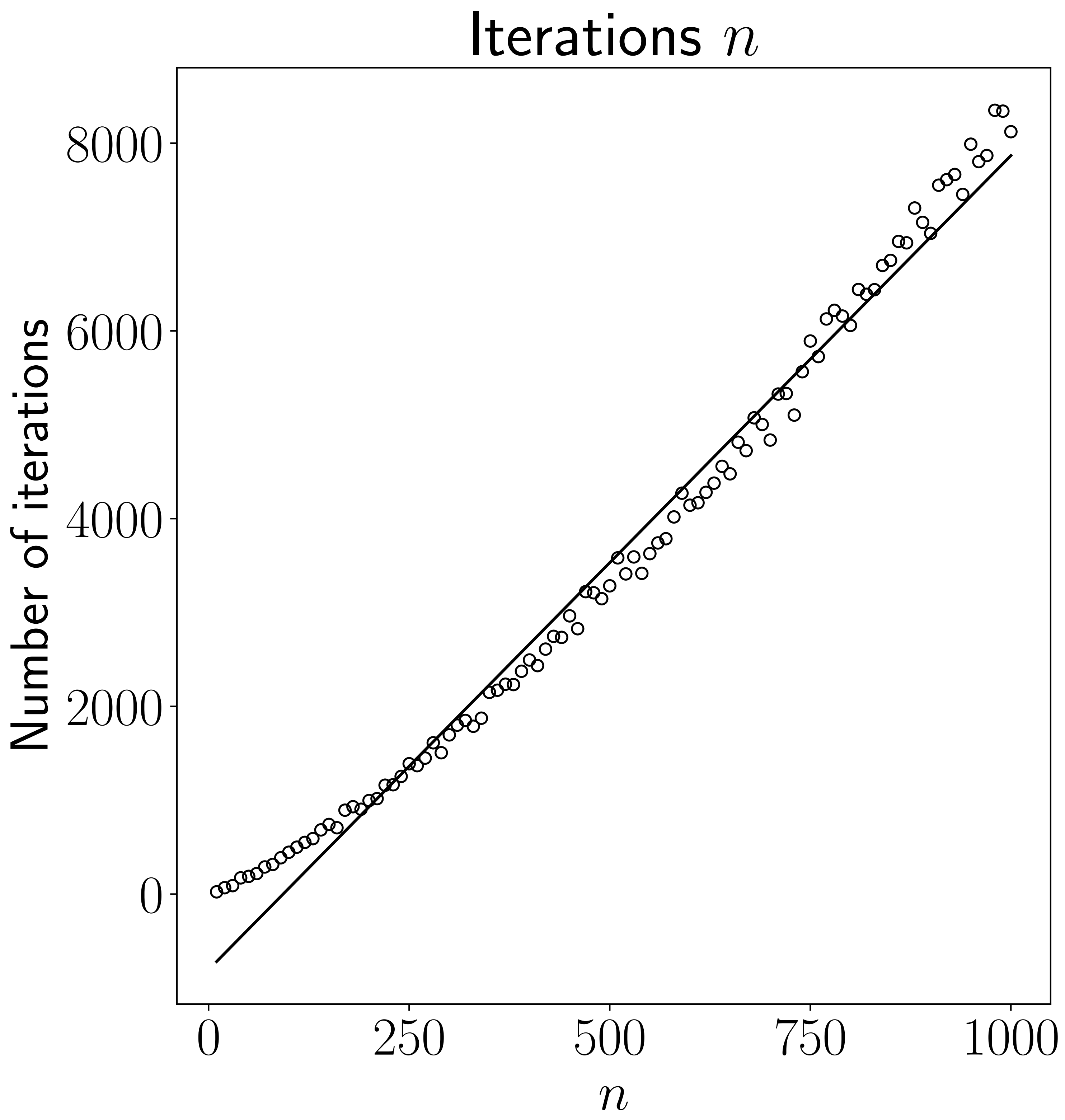
Random LPs

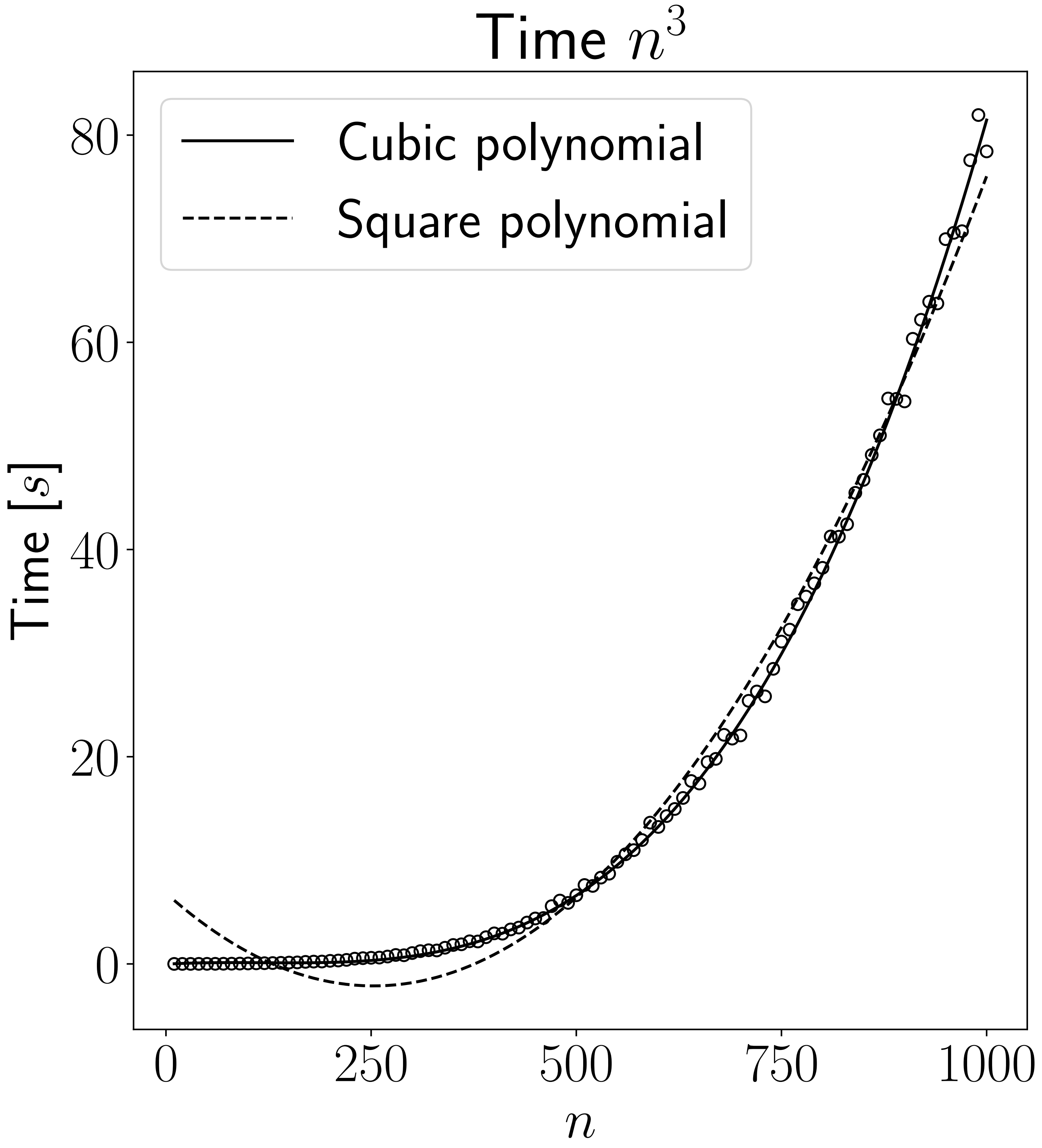
minimize $\quad c^T x$

subject to $\quad Ax \le b$

n variables

$3n$ constraints





Recap

Linear optimization formulations

Standard form LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Inequality form LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

Today’s agenda

Duality

- Obtaining lower bounds
- The dual problem
- Weak and strong duality

Obtaining lower bounds

Obtaining lower bounds

A simple example

$$\begin{array}{ll} \text{minimize} & x_1 + 3x_2 \\ \text{subject to} & x_1 + 3x_2 \geq 2 \end{array}$$

What is a **lower bound** on the optimal cost?

A lower bound is 2 because $x_1 + 3x_2 \geq 2$

Obtaining lower bounds

Another example

$$\begin{array}{ll} \text{minimize} & x_1 + 3x_2 \\ \text{subject to} & x_1 + x_2 \geq 2 \\ & x_2 \geq 1 \end{array}$$

What is a **lower bound** on the optimal cost?

Let's sum the constraints

$$\begin{aligned} & 1 \cdot (x_1 + x_2 \geq 2) \\ + & \, 2 \cdot (x_2 \geq 1) \\ = & \, x_1 + 3x_2 \geq 4 \end{aligned}$$

A lower bound is 4

Obtaining lower bounds

A more interesting example

$$\begin{array}{ll} \text{minimize} & x_1 + 3x_2 \\ \text{subject to} & x_1 + x_2 \geq 2 \\ & x_2 \geq 1 \\ & x_1 - x_2 \geq 3 \end{array}$$

How can we obtain a lower bound?

Add constraints

$$\begin{aligned} & y_1 \cdot (x_1 + x_2 \geq 2) \\ + & y_2 \cdot (x_2 \geq 1) \\ + & y_3 \cdot (x_1 - x_2 \geq 3) \\ \Rightarrow & (y_1 + y_3)x_1 + (y_1 + y_2 - y_3)x_2 \geq 2y_1 + y_2 + 3y_3 \end{aligned}$$

Match cost coefficients

$$\begin{aligned} y_1 + y_3 &= 1 \\ y_1 + y_2 - y_3 &= 3 \\ y_1, y_2, y_3 &\geq 0 \end{aligned}$$

Bound

Many options

$y = (1, 2, 0) \Rightarrow$ Bound 4

$y = (0, 4, 1) \Rightarrow$ Bound 7

How can we get the **best one**?

Obtaining lower bounds

A more interesting example — Best lower bound

We can obtain the **best lower bound** by solving the following problem

$$
\begin{array}{ll}
\text{maximize} & 2y_1 + y_2 + 3y_3 \\
\text{subject to} & y_1 + y_3 = 1 \\
& y_1 + y_2 - y_3 = 3 \\
& y_1, y_2, y_3 \geq 0
\end{array}
$$

This linear optimization problem is called the **dual problem**

The dual problem

Lagrange multipliers

Consider the LP in standard form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Relax the constraint

$$g(y) = \begin{array}{ll} \underset{x}{\text{minimize}} & c^T x + y^T (Ax - b) \\ \text{subject to} & x \geq 0 \end{array}$$

Lower bound

$$g(y) \leq c^T x^\star + y^T (Ax^\star - b) = c^T x^\star$$

Best lower bound

$$\underset{y}{\text{maximize}} \quad g(y)$$

The dual

Dual function

$$g(y) = \underset{x \geq 0}{\text{minimize}} \left(c^T x + y^T (Ax - b)\right)$$

$$= -b^T y + \underset{x \geq 0}{\text{minimize}} \left(c + A^T y\right)^T x$$

$$g(y) = \begin{cases} -b^T y & \text{if } c + A^T y \geq 0 \\ -\infty & \text{otherwise} \end{cases}$$

Dual problem (find the best bound)

$$\underset{y}{\text{maximize}} \quad g(y) \quad = \quad \begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c \geq 0 \end{array}$$

Primal and dual problems

Primal problem

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax = b \\
& x \geq 0
\end{array}
$$

Primal variable $x \in \mathbf{R}^n$

Dual problem

$$
\begin{array}{ll}
\text{maximize} & -b^T y \\
\text{subject to} & A^T y + c \geq 0
\end{array}
$$

Dual variable $y \in \mathbf{R}^m$

The dual problem carries **useful information** for the primal problem

Duality is useful also to **solve** optimization problems

Dual of inequality form LP

What if you find an LP with inequalities?

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

1. We could first transform it to standard form
2. We can compute the dual function (same procedure as before)

Relax the constraint

$$g(y) = \underset{x}{\text{minimize}} \quad c^T x + y^T(Ax - b)$$

Lower bound

$$g(y) \leq c^T x^\star + y^T(Ax^\star - b) \leq c^T x^\star$$

we must have $y \geq 0$

Dual of LP with inequalities

Derivation

Dual function

$$g(y) = \underset{x}{\text{minimize}} \left(c^T x + y^T (Ax - b)\right)$$

$$= -b^T y + \underset{x}{\text{minimize}} \left(c + A^T y\right)^T x$$

$$g(y) = \begin{cases} -b^T y & \text{if } c + A^T y = 0 \quad \text{(and } y \geq 0\text{)} \\ -\infty & \text{otherwise} \end{cases}$$

Dual problem (find the best bound)

$$\underset{y}{\text{maximize}} \quad g(y) \quad = \quad \begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c = 0 \\ & y \geq 0 \end{array}$$

General forms

Standard form LP

Primal

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Dual

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c \geq 0 \end{array}$$

Inequality form LP

Primal

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

Dual

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c = 0 \\ & y \geq 0 \end{array}$$

LP with inequalities and equalities

Primal

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \\ & Cx = d \end{array}$$

Dual

$$\begin{array}{ll} \text{maximize} & -b^T y - d^T z \\ \text{subject to} & A^T y + C^T z + c = 0 \\ & y \geq 0 \end{array}$$

Example from before

minimize	$x_1 + 3x_2$
subject to	$x_1 + x_2 \geq 2$
	$x_2 \geq 1$
	$x_1 - x_2 \geq 3$

$\longrightarrow$

Inequality form LP

minimize	$c^T x$
subject to	$Ax \leq b$

$$c = (1, 3)$$

$$A = \begin{bmatrix} -1 & -1 \\ 0 & -1 \\ -1 & 1 \end{bmatrix}$$

$$b = (-2, -1, -3)$$

Dual

maximize	$-b^T y$
subject to	$A^T y + c = 0$
	$y \geq 0$

$\longrightarrow$

maximize	$2y_1 + y_2 + 3y_3$
subject to	$-y_1 - y_3 = -1$
	$-y_1 - y_2 + y_3 = -3$
	$y_1, y_2, y_3 \geq 0$

To memorize

Ways to get the dual

- Derive dual function directly
- Transform the problem in inequality form LP and dualize

Sanity-checks and signs convention

- Consider constraints as $Ax - b \leq 0$ or $Ax - b = 0$ (not ≥ 0)
- Each dual variable is associated to a primal constraint
- y free for primal equalities and $y \geq 0$ for primal inequalities

Dual of the dual

Theorem

If we transform the primal into its dual and then transform the dual to its dual, we obtain a problem equivalent to the original problem. In other words, the **dual of the dual is the primal**.

Exercise

Derive dual and dualize again

Primal

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \\ & Cx = d \end{array}$$

Dual

$$\begin{array}{ll} \text{maximize} & -b^T y - d^T z \\ \text{subject to} & A^T y + C^T z + c = 0 \\ & y \geq 0 \end{array}$$

Theorem

If we **transform a linear optimization problem to another form** (inequality form, standard form, inequality and equality form), **the dual of the two problems will be equivalent**.

Weak and strong duality

Optimal objective values

Primal

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

Dual

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c = 0 \\ & y \geq 0 \end{array}$$

$p^\star$ is the primal optimal value

$d^\star$ is the dual optimal value

Primal infeasible: $p^\star = +\infty$
Primal unbounded: $p^\star = -\infty$

Dual infeasible: $d^\star = -\infty$
Dual unbounded: $d^\star = +\infty$

Weak duality

Theorem

If x, y satisfy:

- x is a feasible solution to the primal problem
- y is a feasible solution to the dual problem

$$\longrightarrow \quad -b^T y \leq c^T x$$

Proof

We know that $Ax \leq b$, $A^T y + c = 0$ and $y \geq 0$. Therefore,

$$0 \leq y^T(b - Ax) = b^T y - y^T Ax = c^T x + b^T y \qquad \blacksquare$$

Remark

- Any dual feasible y gives a **lower bound** on the primal optimal value
- Any primal feasible x gives an **upper bound** on the dual optimal value
- $c^T x + b^T y$ is the **duality gap**

Weak duality

Corollaries

Unboundedness vs feasibility

- Primal unbounded ($p^\star = -\infty$) $\Rightarrow$ dual infeasible ($d^\star = -\infty$)
- Dual unbounded ($d^\star = +\infty$) $\Rightarrow$ primal infeasible ($p^\star = +\infty$)

Optimality condition

If x, y satisfy:

- x is a feasible solution to the primal problem
- y is a feasible solution to the dual problem
- The duality gap is zero, *i.e.*, $c^T x + b^T y = 0$

Then x and y are **optimal solutions** to the primal and dual problem respectively

Strong duality

Theorem
If a linear optimization problem has an optimal solution, so does its dual, and the optimal value of primal and dual are equal

$$d^\star = p^\star$$

Strong duality

Constructive proof

Given a primal optimal solution $x^\star$ we will construct a dual optimal solution $y^\star$

Apply simplex to problem in **standard form**

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \longrightarrow$$

- optimal basis B
- optimal solution $x^\star$ with $A_B x_B^\star = b$
- reduced costs $\bar{c} = c - A^T A_B^{-T} c_B \geq 0$

Define $y^\star$ such that $y^\star = -A_B^{-T} c_B$. Therefore, $A^T y^\star + c \geq 0$ ($y^\star$ dual feasible).

$$-b^T y^\star = -b^T(-A_B^{-T} c_B) = c_B^T(A_B^{-1} b) = c_B^T x_B^\star = c^T x^\star$$

By weak duality theorem corollary, $y^\star$ is an optimal solution of the dual.
Therefore, $d^\star = p^\star$. ∎

Exception to strong duality

Primal

$$\begin{array}{ll} \text{minimize} & x \\ \text{subject to} & 0 \cdot x \leq -1 \end{array}$$

Dual

$$\begin{array}{ll} \text{maximize} & y \\ \text{subject to} & 0 \cdot y + 1 = 0 \\ & y \geq 0 \end{array}$$

Optimal value is $p^\star = +\infty$

Optimal value is $d^\star = -\infty$

Both **primal** and **dual infeasible**

Relationship between primal and dual

	$p^\star = +\infty$	$p^\star$ finite	$p^\star = -\infty$
$d^\star = +\infty$	primal inf. dual unb.		
$d^\star$ finite		optimal values equal	
$d^\star = -\infty$	exception		primal unb. dual inf

- Upper-right excluded by **weak duality**
- $(1,1)$ and $(3,3)$ proven by **weak duality**
- $(3,1)$ and $(2,2)$ proven by **strong duality**

Example

Production problem

maximize	$x_1 + 2x_2$	⟵ Profits
subject to	$x_1 \leq 100$	
	$2x_2 \leq 200$	⟵ Resources
	$x_1 + x_2 \leq 150$	
	$x_1, x_2 \geq 0$	

$$c = (-1, -2)$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$b = (100, 200, 150, 0, 0)$$

Dualize

1. Transform in inequality form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

2. Derive dual

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c = 0 \\ & y \geq 0 \end{array}$$

Production problem

Dualized

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c = 0 \\ & y \geq 0 \end{array}$$

$$c = (-1, -2)$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$b = (100, 200, 150, 0, 0)$$

Fill-in data

$$\begin{array}{ll} \text{minimize} & 100y_1 + 200y_2 + 150y_3 \\ \text{subject to} & y_1 + y_3 \cancel{- y_4} = 1 \\ & 2y_2 + y_3 \cancel{- y_5} = 2 \\ & y_1, y_2, y_3, y_4, y_5 \geq 0 \end{array}$$

$$\longrightarrow$$

Eliminate variables

$$\begin{array}{ll} \text{minimize} & 100y_1 + 200y_2 + 150y_3 \\ \text{subject to} & y_1 + y_3 \geq 1 \\ & 2y_2 + y_3 \geq 2 \\ & y_1, y_2, y_3 \geq 0 \end{array}$$

Production problem

The dual

$$
\begin{aligned}
\text{minimize} \quad & 100y_1 + 200y_2 + 150y_3 \\
\text{subject to} \quad & y_1 + y_3 \geq 1 \\
& 2y_2 + y_3 \geq 2 \\
& y_1, y_2, y_3 \geq 0
\end{aligned}
$$

Interpretation

- **Sell all your resources** at a fair (minimum) price
- Selling must be **more convenient than producing**:
  - Product 1 (price 1, needs $1\times$ resource 1 and 3): $y_1 + y_3 \geq 1$
  - Product 2 (price 2, needs $2\times$ resource 2 and $1\times$ resource 3): $2y_2 + y_3 \geq 2$

Linear optimization duality

Today, we learned to:

- **Dualize** linear optimization problems
- **Prove** weak and strong duality conditions
- **Interpret** simple dual optimization problems

References

- Bertsimas and Tsitsiklis: Introduction to Linear Optimization
  - Chapter 4: Duality theory
- R. Vanderbei: Linear Programming — Foundations and Extensions
  - Chapter 5: Duality theory

Next lecture

More on duality:

- Game theory
- Complementary slackness
- Farkas lemma