

ORF307 – Optimization

12. The simplex method implementation

Ed Forum

- Final exam time window: **May 12 - May 17**. 24hours total take-home time.
- Midterm grades this week.
- Lecture questions:
 - I was hoping that in the next lecture we could review the two steps to computing the reduced cost vector on slide 22.
 - Towards the end of the lecture, we learned that in the case of finite convergence, the simplex method terminates after a finite number of iterations. How costly (in flops) is this algorithm, and why?

Recap

How does the cost change?

Cost improvement

$$c^T(x + \theta d) - c^T x = \theta c^T d$$

New cost

Old cost

We call \bar{c}_j the **reduced cost** of (introducing) variable x_j in the basis

$$\bar{c}_j = c^T d = \sum_{i=1}^n c_j d_j = c_j + c_B^T d_B = c_j - c_B^T A_B^{-1} A_j$$

- $\bar{c}_j > 0$: adding x_j will increase the objective (bad)
- $\bar{c}_j < 0$: adding x_j will decrease the objective (good)

Vector of reduced costs

Reduced costs

$$\bar{c}_j = c_j - c_B^T A_B^{-1} A_j$$

Isolate basis B -related components p
(they are the same across j)

$$\bar{c}_j = c_j - A_j^T (A_B^{-1})^T c_B = c_j - A_j^T p$$

Full vector in one shot?

$$\bar{c} = (\bar{c}_1, \dots, \bar{c}_n)$$

Obtain p by solving linear system

$$p = (A_B^{-1})^T c_B \quad \Rightarrow \quad A_B^T p = c_B$$

Note: $(M^{-1})^T = (M^T)^{-1}$
for any square invertible M

Computing reduced cost vector

1. Solve $A_B^T p = c_B$
2. $\bar{c} = c - A^T p$

Stepsize

What happens if some $\bar{c}_j < 0$?

We can decrease the cost by bringing x_j into the basis

How far can we go?

$$\theta^* = \max\{\theta \mid \theta \geq 0 \text{ and } x + \theta d \geq 0\} \quad d \text{ is the } j\text{-th basic direction}$$

Unbounded

If $d \geq 0$, then $\theta^* = \infty$. The LP is unbounded.

Bounded

If $d_i < 0$ for some i , then

$$\theta^* = \min_{\{i \mid d_i < 0\}} \left(-\frac{x_i}{d_i} \right) = \min_{\{i \in B \mid d_i < 0\}} \left(-\frac{x_i}{d_i} \right)$$

(Since $d_i \geq 0$, $i \notin B$)

Moving to a new basis

Next feasible solution

$$x + \theta^* d$$

Let $B(\ell) \in \{B(1), \dots, B(m)\}$ be the index such that $\theta^* = -\frac{x_{B(\ell)}}{d_{B(\ell)}}$. Then,

$$x_{B(\ell)} + \theta^* d_{B(\ell)} = 0$$

New solution

- $x_{B(\ell)}$ becomes 0 (exits)
- x_j becomes θ^* (enters)

New basis

$$A_{\bar{B}} = \left[A_{B(1)} \quad \dots \quad A_{B(\ell-1)} \quad A_j \quad A_{B(\ell+1)} \quad \dots \quad A_{B(m)} \right]$$

An iteration of the simplex method

Initialization

- a basic feasible solution x
- a basis matrix $A_B = \begin{bmatrix} A_{B(1)} & \dots & A_{B(m)} \end{bmatrix}$

Iteration steps

1. Compute the reduced costs \bar{c}
 - Solve $A_B^T p = c_B$
 - $\bar{c} = c - A^T p$
2. If $\bar{c} \geq 0$, x **optimal. break**
3. Choose j such that $\bar{c}_j < 0$
4. Compute search direction d with $d_j = 1$ and $A_B d_B = -A_j$
5. If $d_B \geq 0$, the problem is **unbounded** and the optimal value is $-\infty$. **break**
6. Compute step length $\theta^* = \min_{\{i \in B \mid d_i < 0\}} \left(-\frac{x_i}{d_i} \right)$
7. Define y such that $y = x + \theta^* d$
8. Get new basis \bar{B} (i exits and j enters)

Example

$$P = \{x \mid x_1 + x_2 + x_3 = 2, \quad x \geq 0\}$$

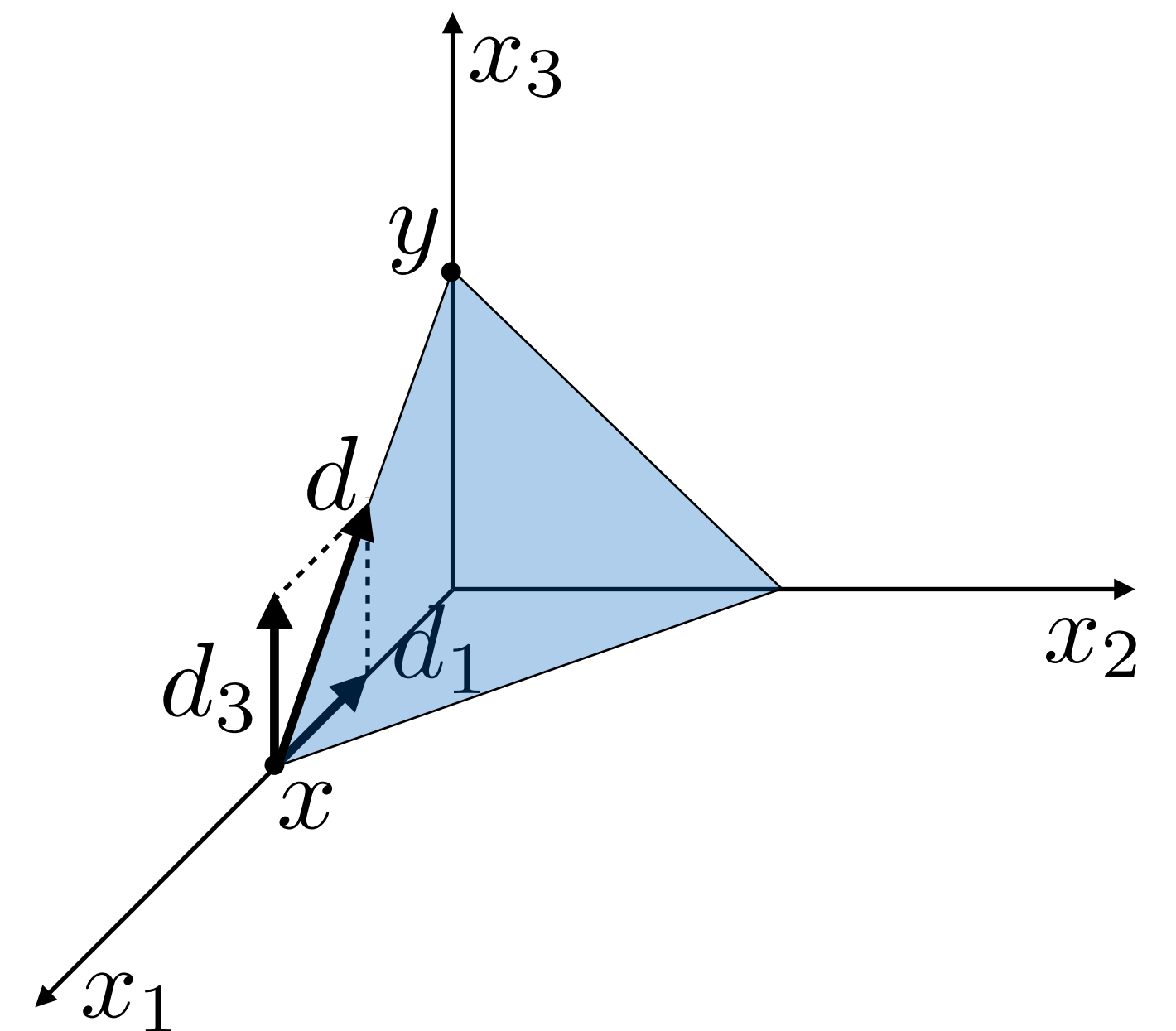
$$x = (2, 0, 0) \quad B = \{1\}$$

$$\text{Basic index } j = 3 \longrightarrow d = (-1, 0, 1) \\ d_j = 1$$

$$A_B d_B = -A_j \quad \Rightarrow \quad d_B = -1$$

$$\text{Stepsize } \theta^* = -\frac{x_1}{d_1} = 2$$

$$\text{New solution } y = x + \theta^* d = (0, 0, 2) \quad \bar{B} = \{3\}$$



Finite convergence

Assume that

- $P = \{x \mid Ax = b, x \geq 0\}$ not empty
- Every basic feasible solution **non degenerate**

Then

- The simplex method **terminates after a finite number of iterations**
- At termination we either have one of the following
 - an **optimal basis** B
 - a **direction** d such that $Ad = 0$, $d \geq 0$, $c^T d < 0$ and the optimal cost is $-\infty$

Finite convergence

Proof sketch

At each iteration the algorithm improves

- by a **positive** amount θ^*
- along the **direction** d such that $c^T d < 0$

Therefore

- The cost strictly decreases
- No basic feasible solution can be visited twice

Since there is a **finite number of basic feasible solutions**

The algorithm **must eventually terminate**



Today's lecture

The simplex method implementation

- Finding an initial basic feasible solution
- Degeneracy
- Full simplex example
- Efficiency

Find an initial point

Initial basic feasible solution

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

How do we get an initial **basic feasible solution** x and a **basis** B ?

Does it **exist**?

Finding an initial basic feasible solution

			Auxiliary problem	
minimize	$c^T x$		minimize	$\mathbf{1}^T y$ ← Minimize violations
subject to	$Ax = b$	→	subject to	$Ax + y = b$
	$x \geq 0$			$x \geq 0, y \geq 0$

Assumption $b \geq 0$ w.l.o.g. (if not multiply constraint by -1)

Trivial basic feasible solution: $x = 0, y = b$

Possible outcomes

- **Feasible problem** (cost = 0): $y^* = 0$ and x^* is a basic feasible solution
- **Infeasible problem** (cost > 0): $y^* > 0$ are the violations

Two-phase simplex method

Phase I

1. Construct **auxiliary problem** such that $b \geq 0$
2. Solve auxiliary problem using simplex method starting from $(x, y) = (0, b)$
3. If the optimal value is greater than 0, **problem infeasible. break.**

Phase II

1. Recover original problem (drop variables y and restore original cost)
2. Solve original problem starting from the solution x and its basis B .

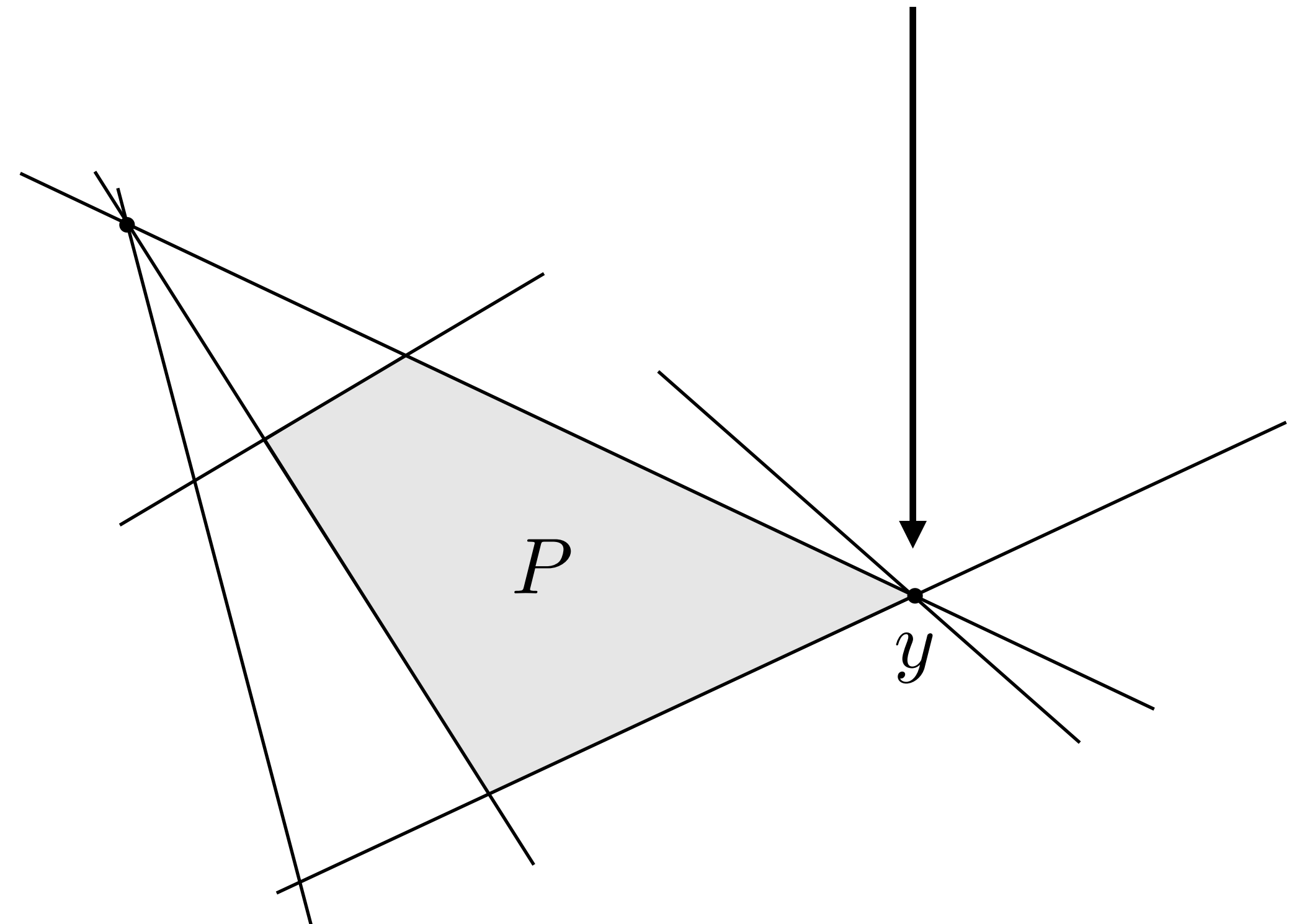
Degeneracy

Degenerate basic feasible solutions

Inequality form polyhedron

A solution y is degenerate if $|\mathcal{I}(y)| > n$

$$P = \{x \mid Ax \leq b\}$$



Degenerate basic feasible solutions

Standard form polyhedron

Given a basis matrix $A_B = \begin{bmatrix} A_{B(1)} & \dots & A_{B(m)} \end{bmatrix}$

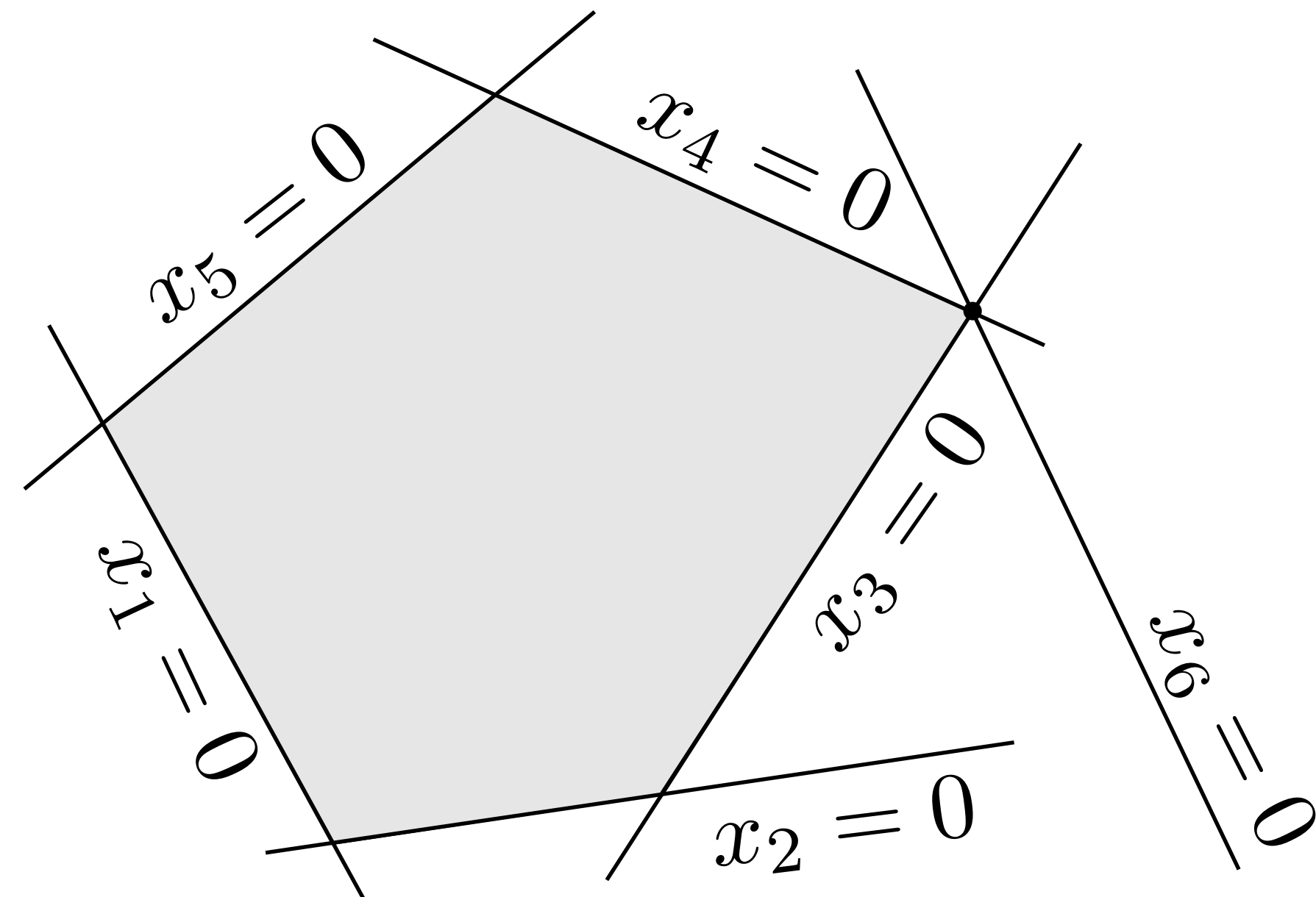
we have basic feasible solution x :

- $A_B x_B = b$
- $x_i = 0, \forall i \neq B(1), \dots, B(m)$



If some of the $x_B = 0$, then it is a **degenerate solution**

$$P = \{x \mid Ax = b, x \geq 0\}$$



Degenerate basic feasible solutions

Example

$$\begin{aligned}x_1 + x_2 + x_3 &= 1 \\ -x_1 + x_2 - x_3 &= 1 \\ x_1, x_2, x_3 &\geq 0\end{aligned}$$

Degenerate solutions

$$\text{Basis } B = \{1, 2\} \quad \longrightarrow \quad x = (0, 1, 0)$$

$$\text{Basis } B = \{2, 3\} \quad \longrightarrow \quad y = (0, 1, 0)$$

Cycling

Stepsize

6. Compute step length $\theta^* = \min_{\{i \in B \mid d_i < 0\}} \left(-\frac{x_i}{d_i} \right)$



If $i \in B$, $d_i < 0$ and $x_i = 0$ (degenerate)

$$\theta^* = 0$$

Therefore $y = x + \theta^* x = x$ and $B \neq \bar{B}$ **Same solution and cost**
Different basis

Finite termination no longer guaranteed!

How can we fix it?

Pivoting rules

Pivoting rules

Choose the index entering the basis

Simplex iterations

3. Choose j such that $\bar{c}_j < 0$ \longrightarrow **Which j ?**

Possible rules

- **Smallest subscript:** smallest j such that $\bar{c}_j < 0$
- **Most negative:** choose j with the most negative \bar{c}_j
- **Largest cost decrement:** choose j with the largest $\theta^* |\bar{c}_j|$

Pivoting rules

Choose index exiting the basis

Simplex iterations

6. Compute step length $\theta^* = \min_{\{i \in B \mid d_i < 0\}} \left(-\frac{x_i}{d_i} \right) \longrightarrow$

We can have more than one i for which $x_i = 0$
(**next solution is degenerate**)

Which i ?

Smallest index rule

Smallest i such that $\theta^* = -\frac{x_i}{d_i}$

Bland's rule to avoid cycles

Theorem

If we use the **smallest index rule** for choosing both the j entering the basis and the i leaving the basis, then **no cycling will occur**.

Proof idea [Vanderbei, Ch 3, Sec 4][Bertsimas and Tsitsiklis, Sec 3.4]

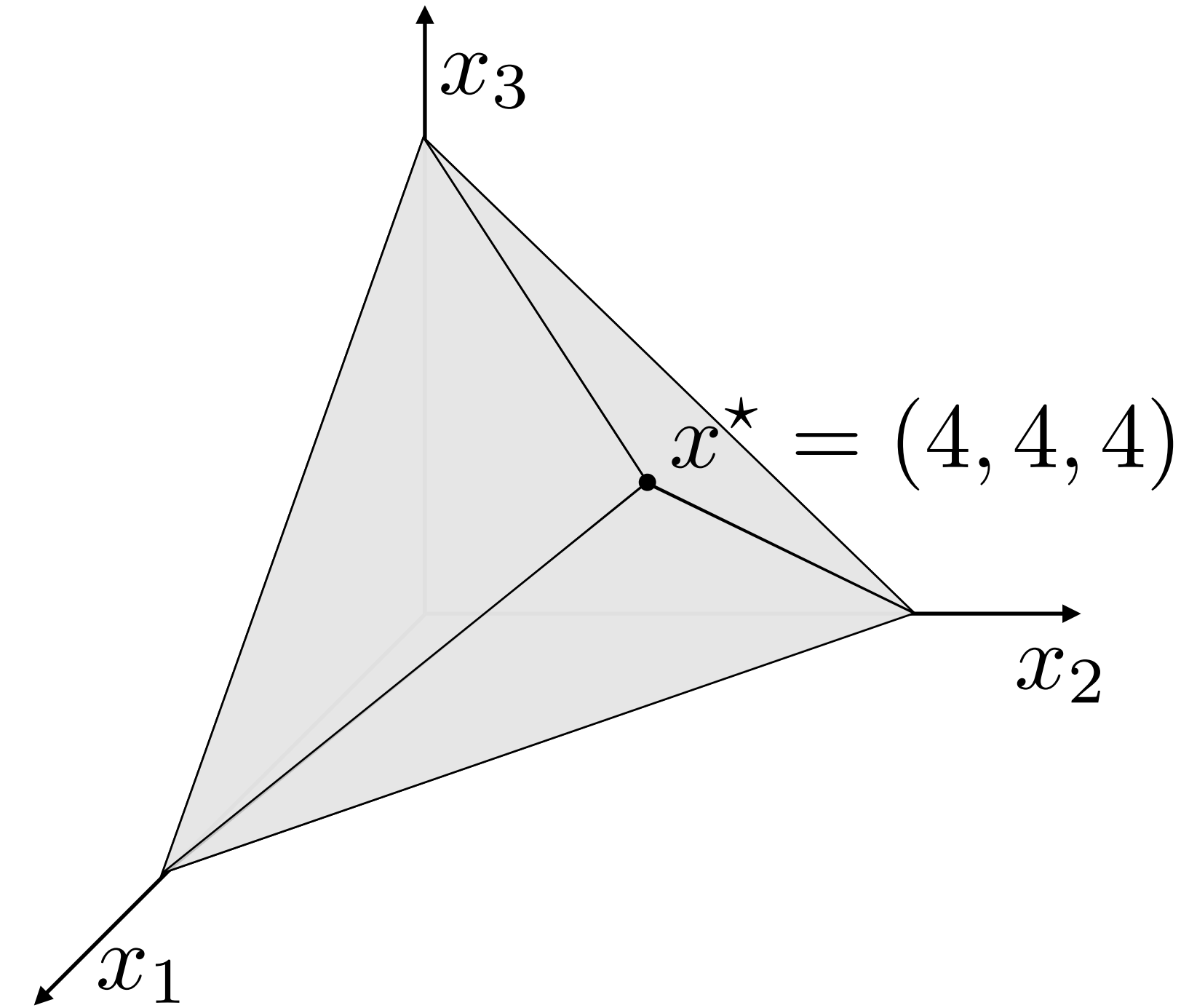
- Assume Bland's rule is applied and there exists a cycle with different bases.
- Obtain contradiction.

Example

Example

Inequality form

$$\begin{aligned} \text{minimize} \quad & -10x_1 - 12x_2 - 12x_3 \\ \text{subject to} \quad & x_1 + 2x_2 + 2x_3 \leq 20 \\ & 2x_1 + x_2 + x_3 \leq 20 \\ & 2x_1 + 2x_2 + x_3 \leq 20 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$



Standard form

$$\text{minimize} \quad -10x_1 - 12x_2 - 12x_3$$

$$\text{subject to} \quad \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \\ 20 \end{bmatrix}$$

$$x \geq 0$$

Example

Start

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

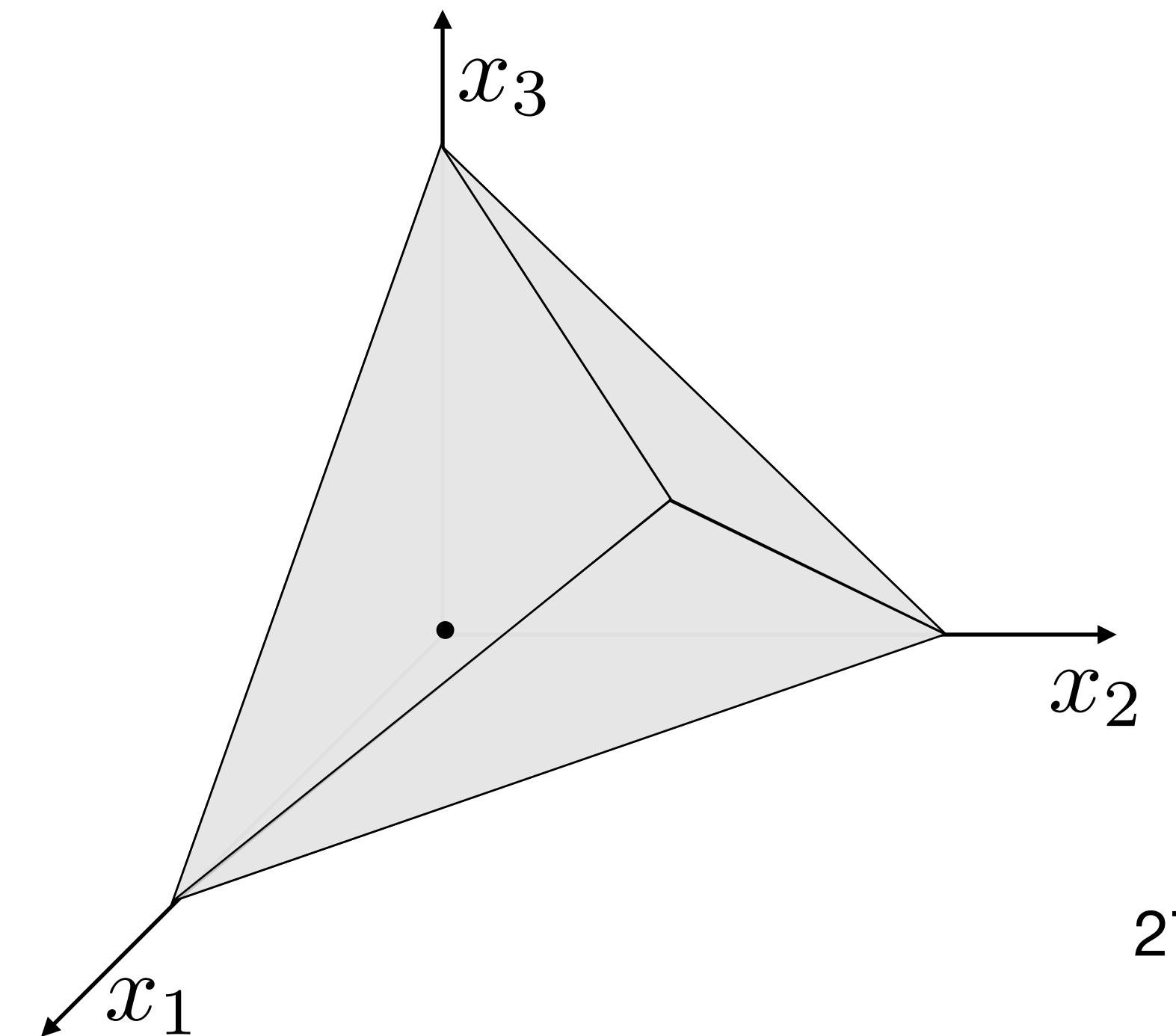
$$c = (-10, -12, -12, 0, 0, 0)$$

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$b = (20, 20, 20)$$

Initialize

$$x = (0, 0, 0, 20, 20, 20) \quad A_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Example

Iteration 1

Current point

$$x = (0, 0, 0, 20, 20, 20)$$

$$c^T x = 0$$

Basis: $\{4, 5, 6\}$

$$A_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$c = (-10, -12, -12, 0, 0, 0)$$

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$b = (20, 20, 20)$$

Reduced costs $\bar{c} = c$

$$\text{Solve } A_B^T p = c_B \Rightarrow p = c_B = 0$$

$$\bar{c} = c - A^T p = c$$

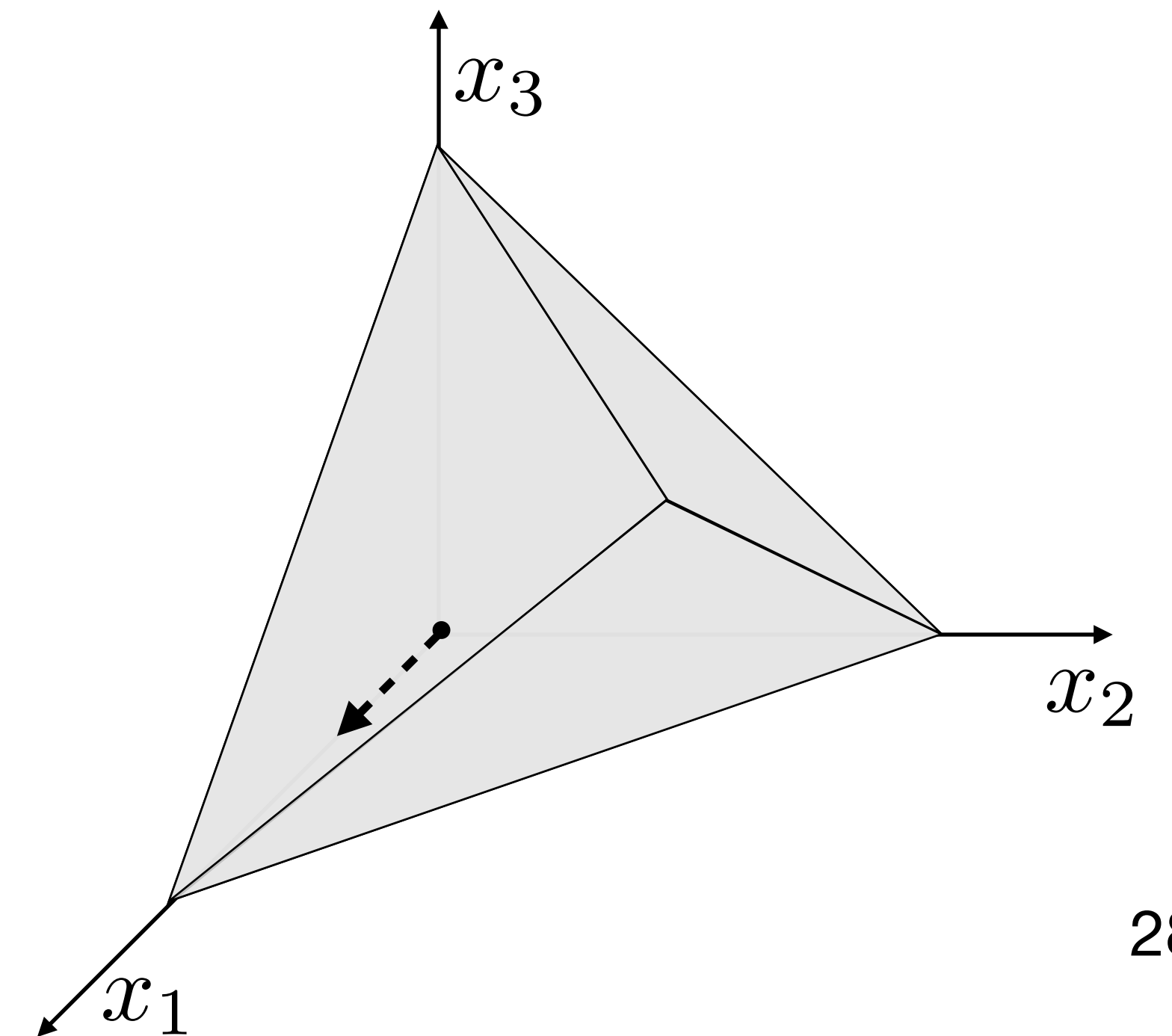
Direction $d = (1, 0, 0, -1, -2, -2), \quad j = 1$

$$\text{Solve } A_B d_B = -A_j \Rightarrow d_B = (-1, -2, -2)$$

Step $\theta^* = 10, \quad i = 5$

$$\theta^* = \min_{\{i | d_i < 0\}} (-x_i / d_i) = \min\{20, 10, 10\}$$

$$\text{New } x \leftarrow x + \theta^* d = (10, 0, 0, 10, 0, 0)$$



Example

Iteration 2

Current point

$$x = (10, 0, 0, 10, 0, 0)$$

$$c^T x = -100$$

Basis: $\{4, 1, 6\}$

$$A_B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$c = (-10, -12, -12, 0, 0, 0)$$

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$b = (20, 20, 20)$$

Reduced costs $\bar{c} = (0, -7, -2, 0, 5, 0)$

Solve $A_B^T p = c_B \Rightarrow p = (0, -5, 0)$

$$\bar{c} = c - A^T p = (0, -7, -2, 0, 5, 0)$$

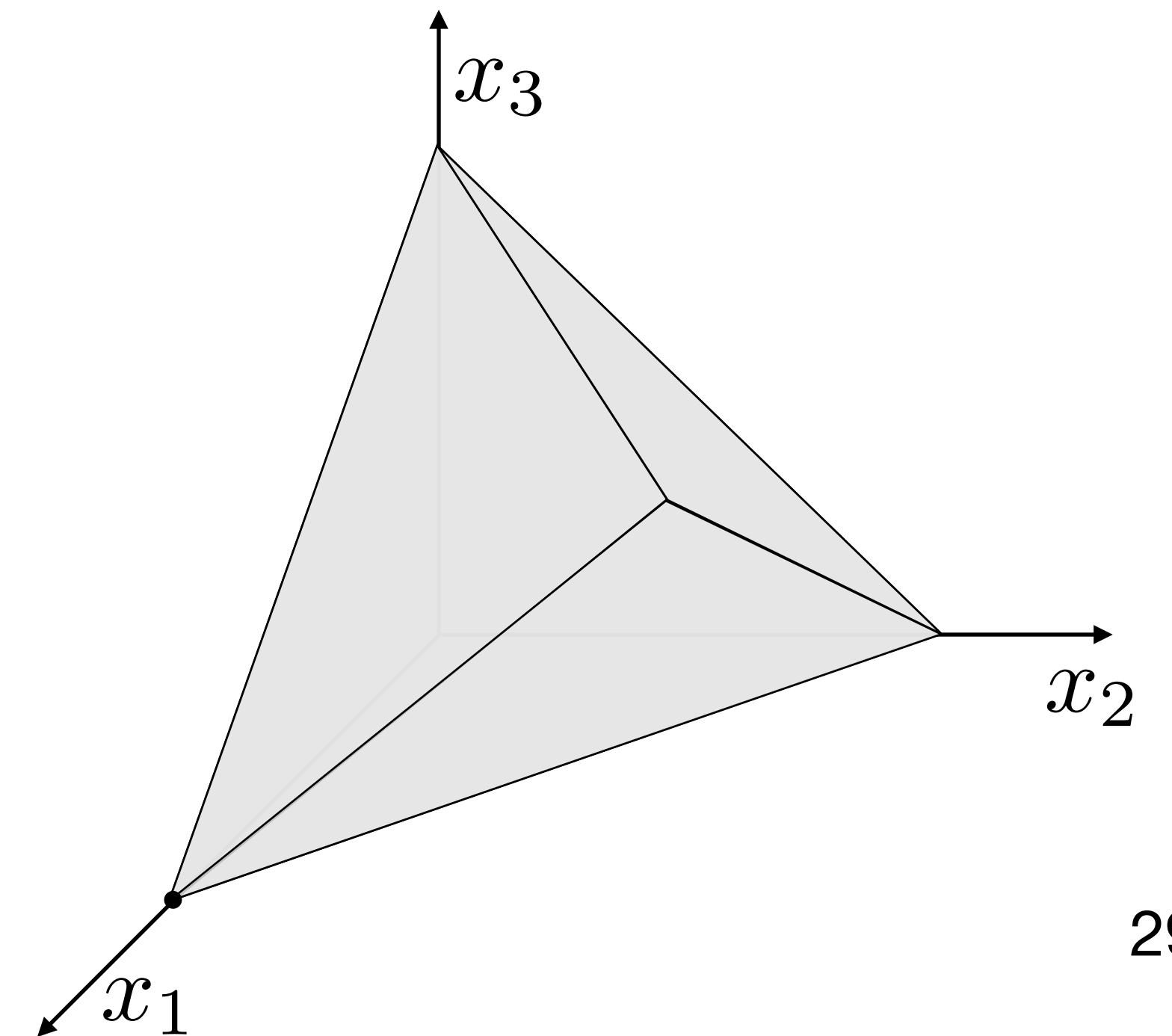
Direction $d = (-0.5, 1, 0, -1.5, 0, -1), \quad j = 2$

Solve $A_B d_B = -A_j \Rightarrow d_B = (-1.5, -0.5, -1)$

Step $\theta^* = 0, \quad i = 6$

$$\theta^* = \min_{\{i|d_i < 0\}} (-x_i/d_i) = \min\{6.66, 20, 0\}$$

New $x \leftarrow x + \theta^* d = (10, 0, 0, 10, 0, 0)$



Example

Iteration 3

Current point

$$x = (10, 0, 0, 10, 0, 0)$$

$$c^T x = -100$$

Basis: $\{4, 1, 2\}$

$$A_B = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

$$c = (-10, -12, -12, 0, 0, 0)$$

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$b = (20, 20, 20)$$

Reduced costs $\bar{c} = (0, 0, -9, 0, -2, 7)$

Solve $A_B^T p = c_B \Rightarrow p = (0, 2, -7)$

$$\bar{c} = c - A^T p = (0, 0, -9, 0, -2, 7)$$

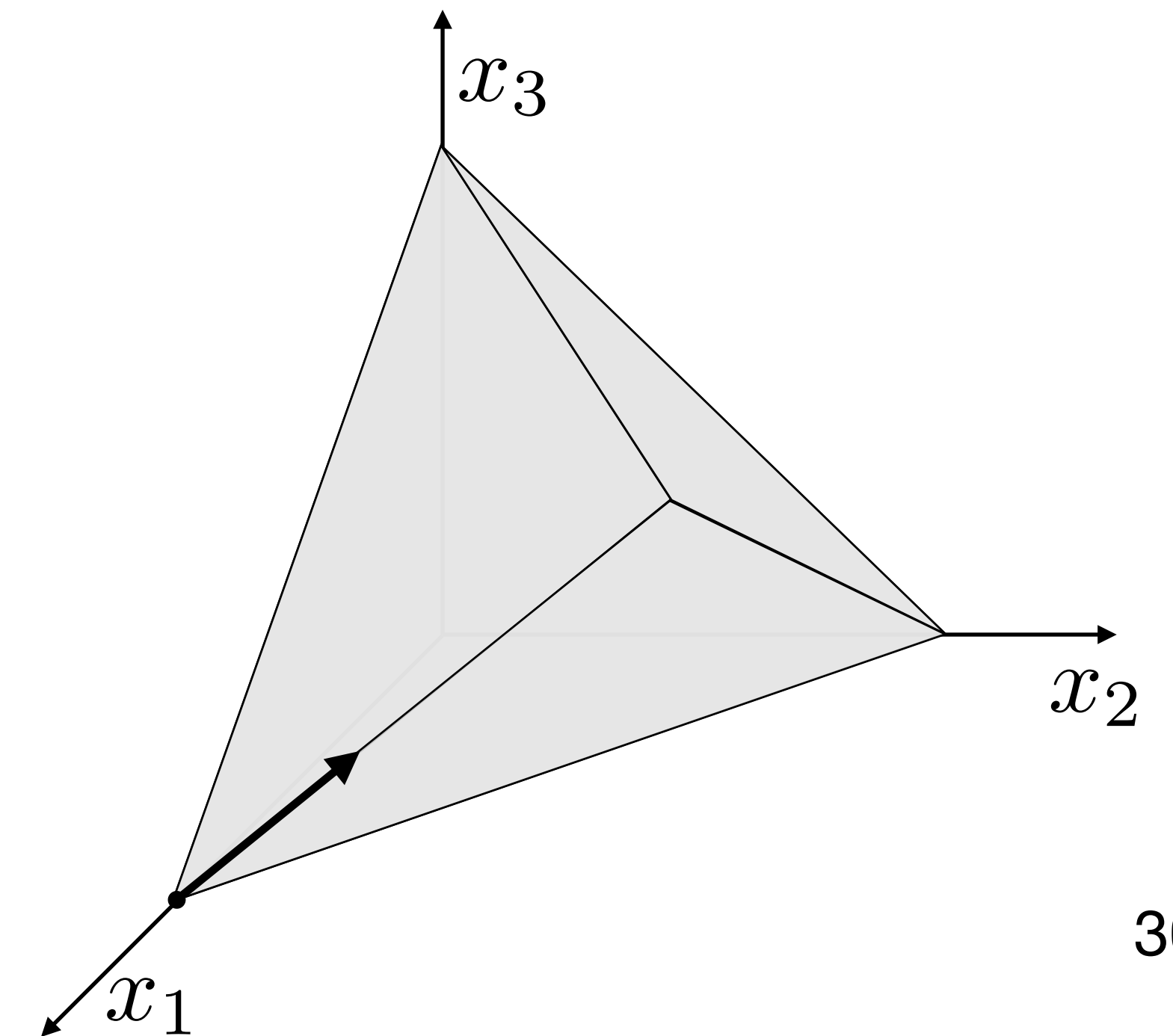
Direction $d = (-1.5, 1, 1, -2.5, 0, 0), \quad j = 3$

Solve $A_B d_B = -A_j \Rightarrow d_B = (-2.5, -1.5, 1)$

Step $\theta^* = 4, \quad i = 4$

$$\theta^* = \min_{\{i|d_i < 0\}} (-x_i/d_i) = \min\{4, 6.67\}$$

New $x \leftarrow x + \theta^* d = (4, 4, 4, 0, 0, 0)$



Example

Iteration 4

Current point

$$x = (4, 4, 4, 0, 0, 0)$$

$$c^T x = -136$$

Basis: $\{3, 1, 2\}$

$$A_B = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$c = (-10, -12, -12, 0, 0, 0)$$

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{bmatrix}$$

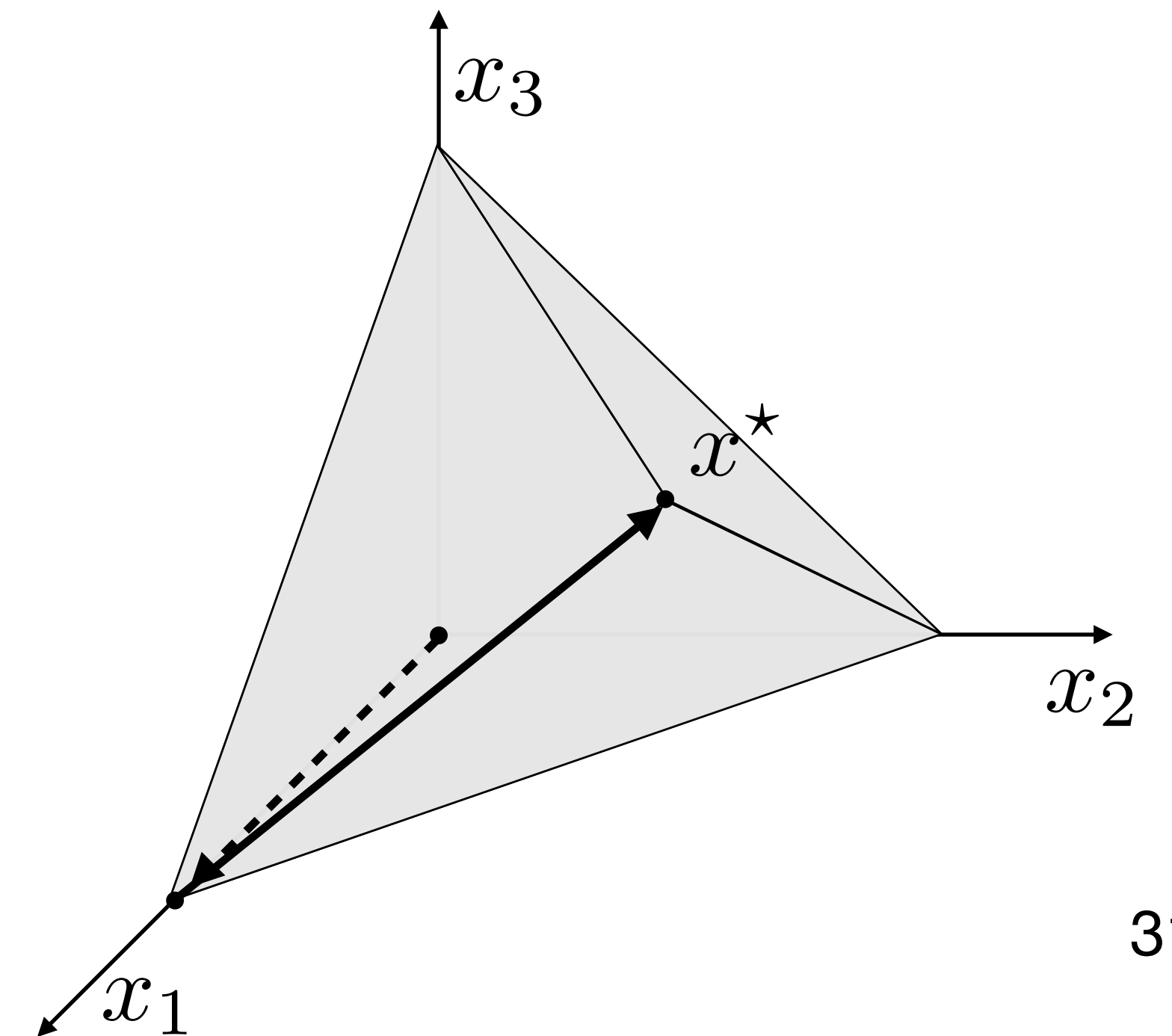
$$b = (20, 20, 20)$$

Reduced costs $\bar{c} = (0, 0, 0, 3.6, 1.6, 1.6)$

Solve $A_B^T p = c_B \Rightarrow p = (-3.6, -1.6, -1.6)$

$$\bar{c} = c - A^T p = (0, 0, 0, 3.6, 1.6, 1.6)$$

$$\bar{c} \geq 0 \quad \longrightarrow \quad \text{Optimal} \quad x^* = (4, 4, 4, 0, 0, 0)$$



Complexity

Complexity of a single simplex iteration

1. Compute the reduced costs \bar{c}
 - Solve $A_B^T p = c_B$
 - $\bar{c} = c - A^T p$
2. If $\bar{c} \geq 0$, x **optimal. break**
3. Choose j such that $\bar{c}_j < 0$
4. Compute search direction d with $d_j = 1$ and $A_B d_B = -A_j$
5. If $d_B \geq 0$, the problem is **unbounded** and the optimal value is $-\infty$. **break**
6. Compute step length $\theta^* = \min_{\{i \in B \mid d_i < 0\}} \left(-\frac{x_i}{d_i} \right)$
7. Define y such that $y = x + \theta^* d$
8. Get new basis \bar{B} (i exits and j enters)

Bottleneck
Two linear systems

Linear system solutions

Very similar linear systems

$$\begin{aligned} A_B^T p &= c_B \\ A_B d_B &= -A_j \end{aligned}$$

LU factorization
 $(2/3)n^3$ flops

$$A_B = PLU$$

Easy linear systems

$4n^2$ flops

$$\begin{aligned} U^T L^T P^T p &= c_B \\ PLU d_B &= -A_j \end{aligned}$$

Factorization is expensive

Do we need to recompute it at every iteration?

Basis update

Index update

- j enters (x_j becomes θ^*)
- $i = B(\ell)$ exists (x_i becomes 0)



Basis matrix change

$$A_{\bar{B}} = A_B + (A_i - A_j)e_\ell^T$$

Example

$$B = \{4, 1, 6\} \rightarrow \bar{B} = \{4, 1, 2\}$$

- 2 enters
- $6 = B(3)$ exists

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$A_{\bar{B}} = \begin{matrix} & A_B & & A_2 e_3^T & & A_6 e_3^T & \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix} & + & \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} & - & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} & = & \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 2 & 2 \end{bmatrix} \end{matrix}$$

Smarter linear system solution

Basis matrix change

$$A_{\bar{B}} = A_B + \overbrace{(A_i - A_j)}^v e_\ell^T \longrightarrow (A_B + v e_\ell^T)^{-1} = \left(I - \frac{1}{1 + e_\ell^T A_B^{-1} v} A_B^{-1} v e_\ell^T \right) A_B^{-1}$$

Matrix inversion lemma
(from homework 2)

Solve $A_{\bar{B}} d_{\bar{B}} = -A_j$

1. Solve $A_B z^1 = e_\ell$ ($2n^2$ flops)
2. Solve $A_B z^2 = -A_j$ ($2n^2$ flops)
3. Solve $d_{\bar{B}} = z^2 - \frac{v^T z^2}{1 + v^T z^1} z^1$

Remarks

- Same complexity for $A_B^T p = c_B$ ($4n^2$ flops)
- k -th next iteration ($4kn^2$ flops, derive as exercise...)
- Once in a while (e.g., $k = 100$), better to refactor A_B

Complexity of a single simplex iteration

1. Compute the reduced costs \bar{c}
 - Solve $A_B^T p = c_B$
 - $\bar{c} = c - A^T p$
2. If $\bar{c} \geq 0$, x **optimal. break**
3. Choose j such that $\bar{c}_j < 0$
4. Compute search direction d with $d_j = 1$ and $A_B d_B = -A_j$
5. If $d_B \geq 0$, the problem is **unbounded** and the optimal value is $-\infty$. **break**
6. Compute step length $\theta^* = \min_{\{i \in B \mid d_i < 0\}} \left(-\frac{x_i}{d_i} \right)$
7. Define y such that $y = x + \theta^* d$
8. Get new basis \bar{B} (i exits and j enters)

Bottleneck
Two linear systems



Matrix inversion lemma trick
 $\approx n^2$ per iteration
(very cheap)

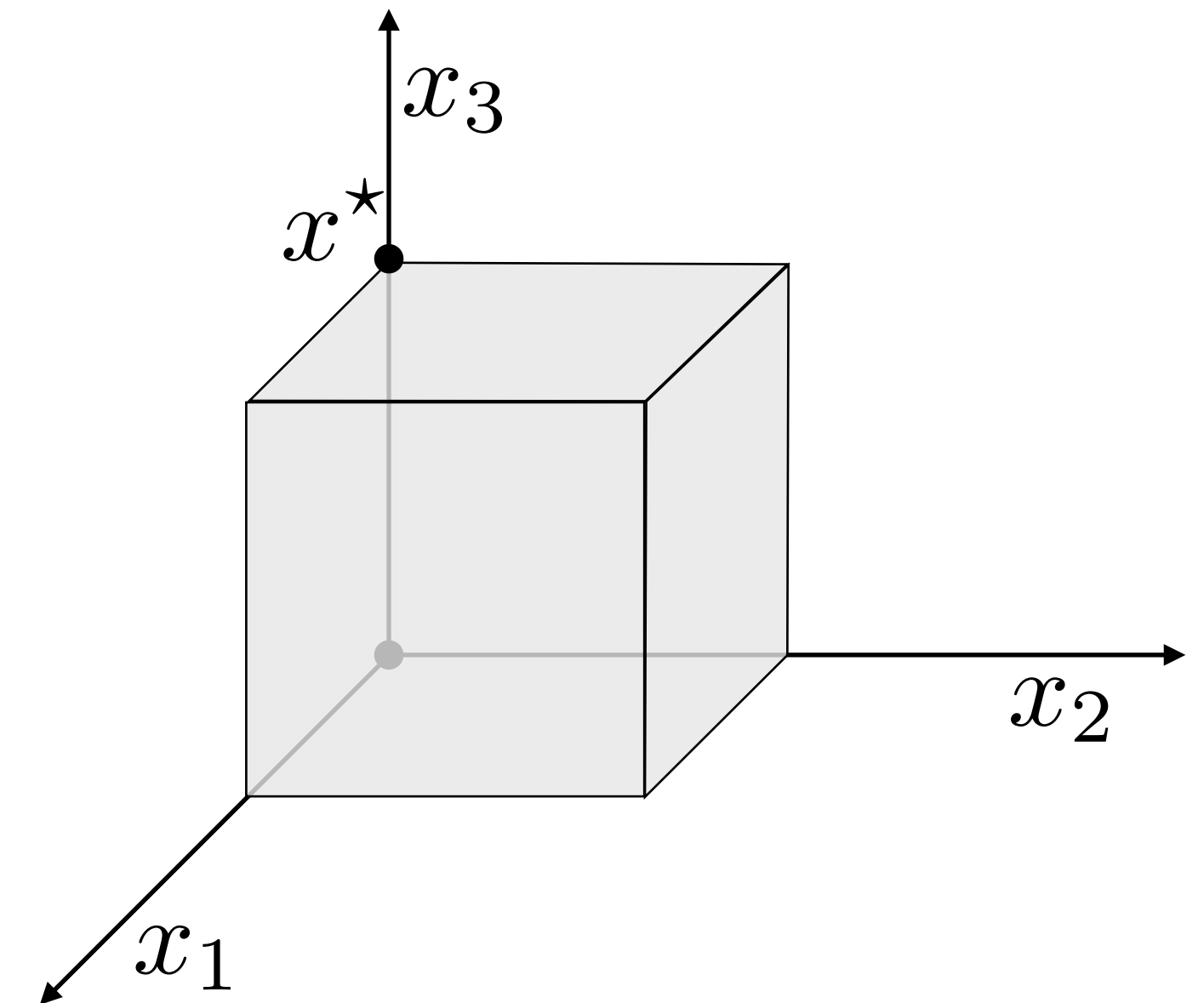
How many iterations do we need?

Complexity of the simplex method

Example of worst-case behavior

Innocent-looking problem

$$\begin{array}{ll} \text{minimize} & -x_n \\ \text{subject to} & 0 \leq x \leq 1 \end{array} \quad \begin{array}{l} 2^n \text{ vertices} \\ 2^n/2 \text{ vertices: cost} = 1 \\ 2^n/2 \text{ vertices: cost} = 0 \end{array}$$



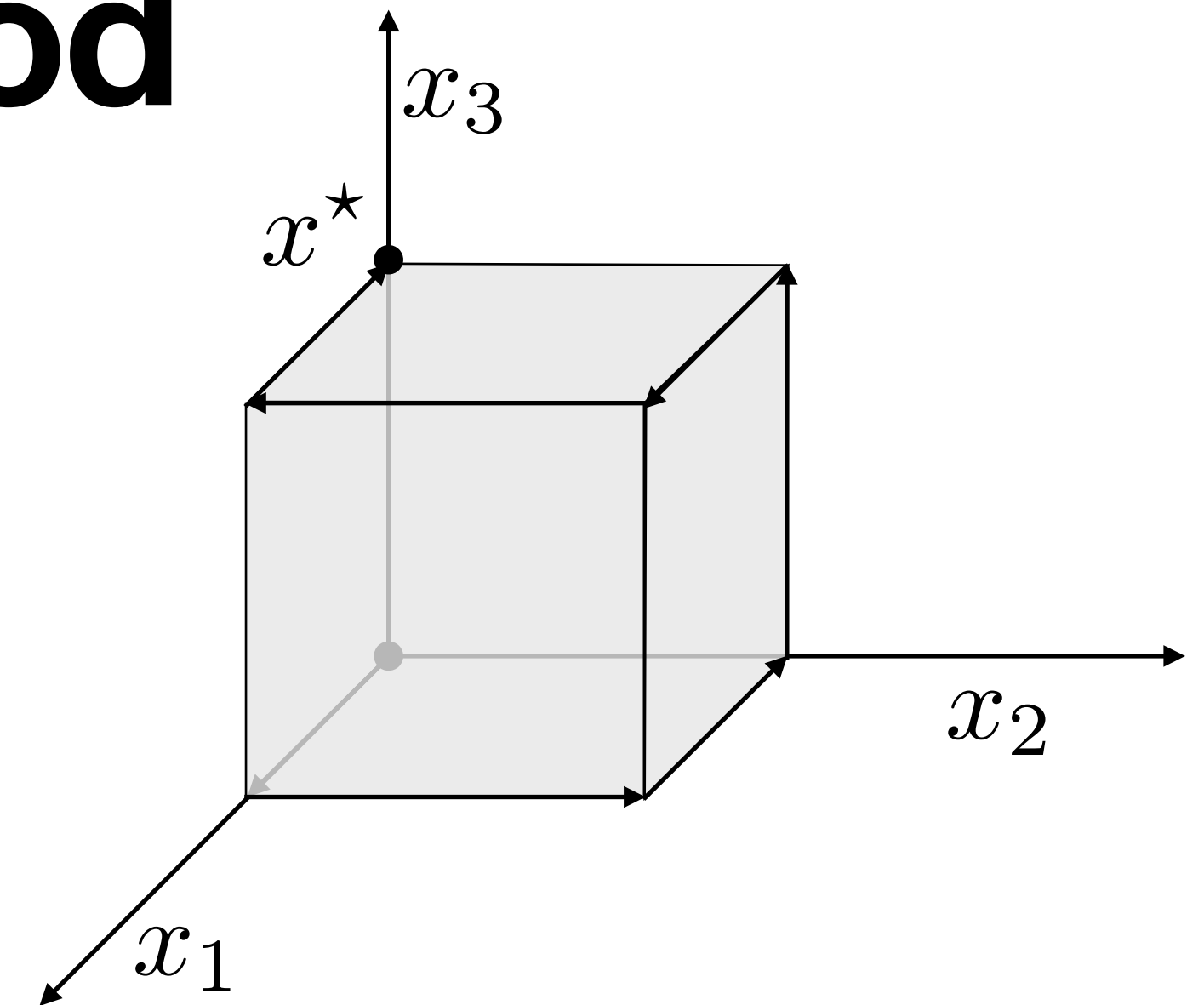
Perturb unit cube

$$\begin{array}{ll} \text{minimize} & -x_n \\ \text{subject to} & \epsilon \leq x_1 \leq 1 \\ & \epsilon x_{i-1} \leq x_i \leq 1 - \epsilon x_{i-1}, \quad i = 2, \dots, n \end{array}$$

Complexity of the simplex method

Example of worst-case behavior

$$\begin{aligned} &\text{minimize} && -x_n \\ &\text{subject to} && \epsilon \leq x_1 \leq 1 \\ & && \epsilon x_{i-1} \leq x_i \leq 1 - \epsilon x_{i-1}, \quad i = 2, \dots, n \end{aligned}$$



Theorem

- The vertices can be ordered so that each one is adjacent to and has a **lower cost than the previous one**
- There exists a pivoting rule under which the simplex method terminates after $2^n - 1$ **iterations**

Remark

- A **different pivot rule** would have converged in one iteration.
- We have a bad example for every pivot rule.

Complexity of the simplex method

We do not know any polynomial version of the simplex method, no matter which pivoting rule we pick.



Still open research question!

Worst-case

There are problem instances where the simplex method will run an **exponential number of iterations** in terms of the dimensions, e.g. 2^n

Good news: average-case

Practical performance is very good. On average, it stops in n iterations.

Average simplex complexity

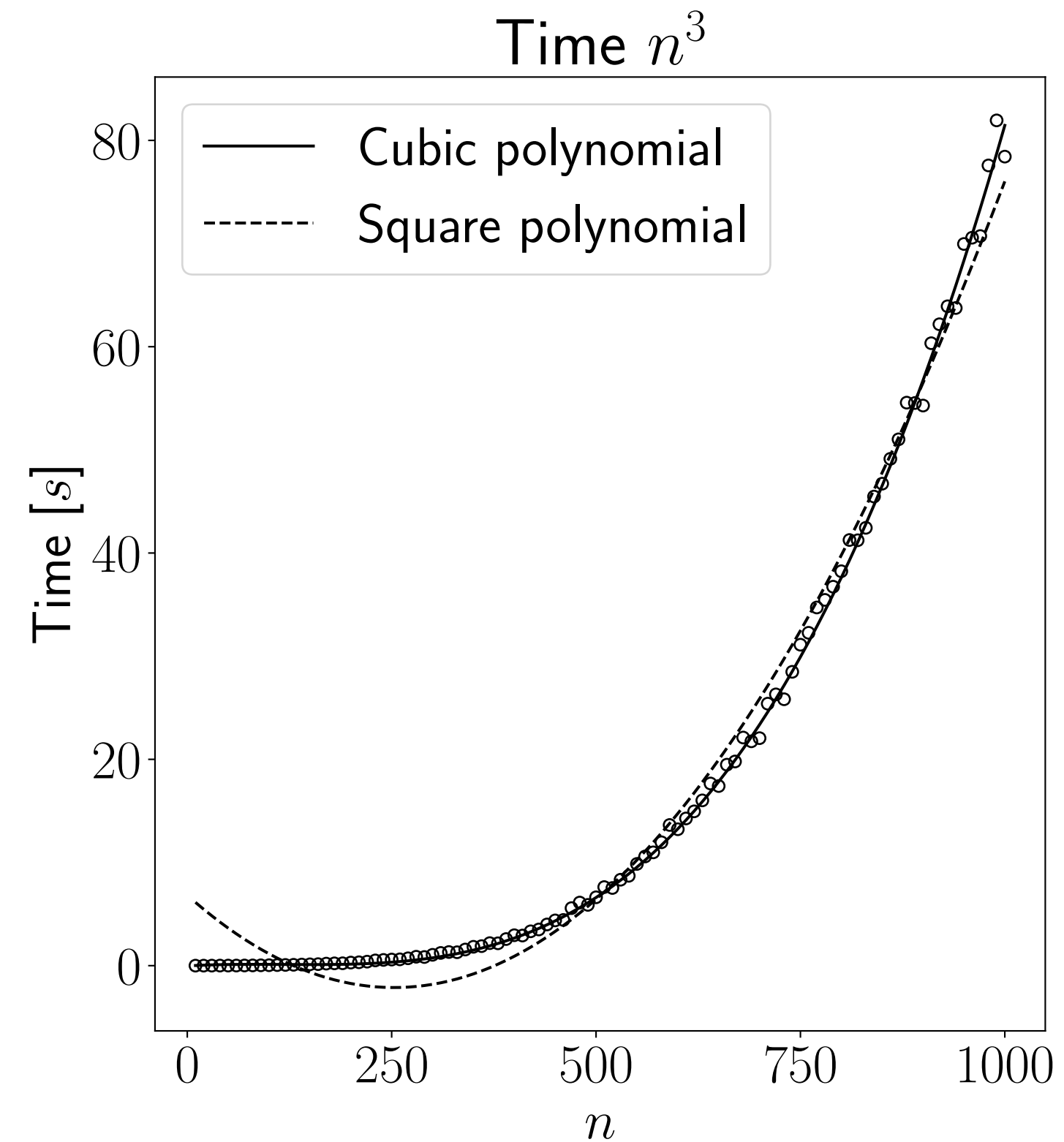
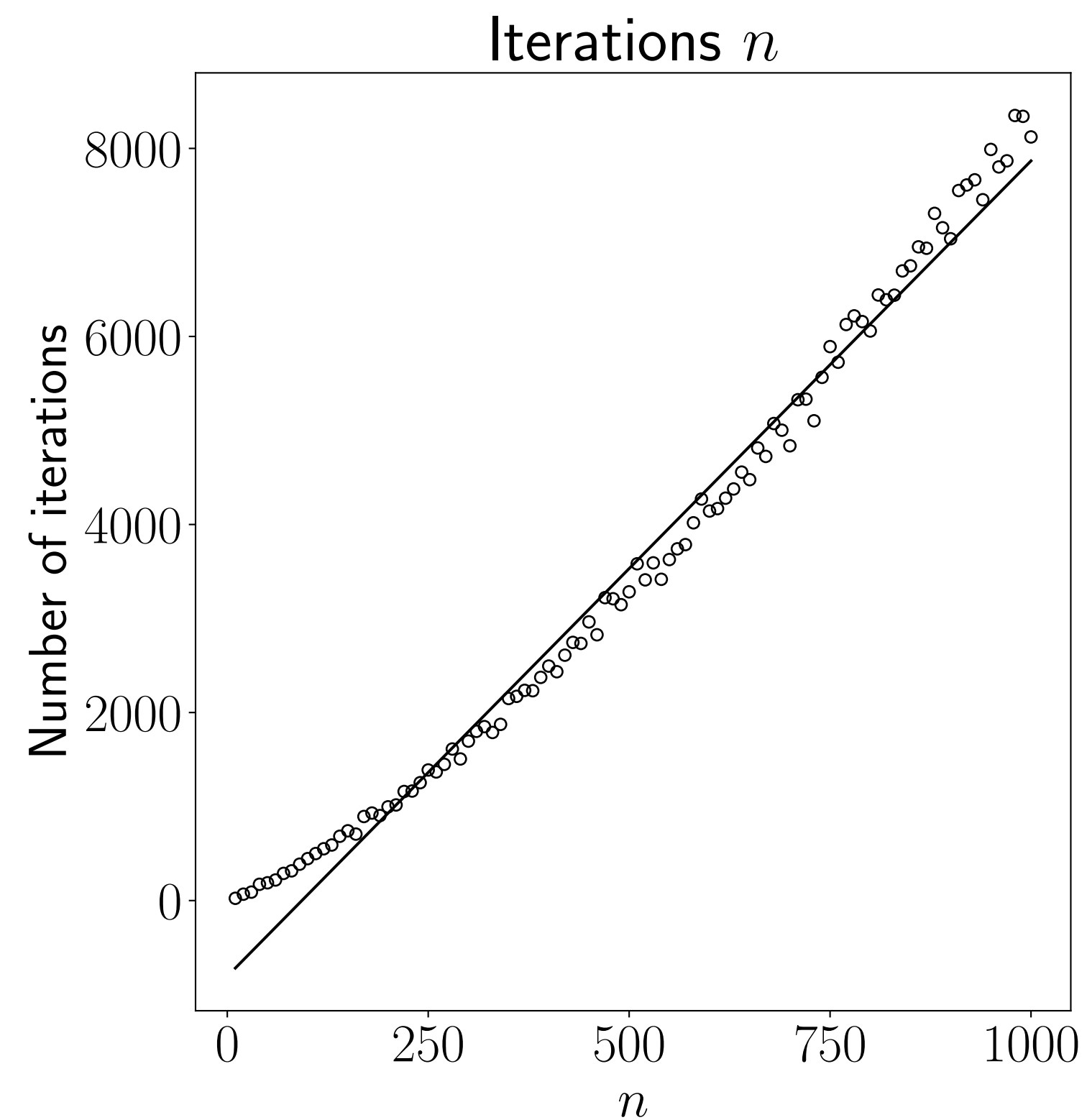
Random LPs

minimize $c^T x$

n variables

subject to $Ax \leq b$

$3n$ constraints



The simplex method implementation

Today, we learned to:

- **Find** an initial basic feasible solution (Phase-I/II Simplex)
- **Deal** with degenerate basic feasible solution (Bland's rule)
- **Compute** the simplex method complexity (**per iteration** and **overall**)

References

- Bertsimas and Tsitsiklis: Introduction to Linear Optimization
 - Chapter 3: The simplex method
- R. Vanderbei: Linear Programming — Foundations and Extensions
 - Chapter 3: Degeneracy
 - Chapter 4: Efficiency of the simplex method
 - Chapter 8: Implementation issues

Next lecture

- Duality