

ORF307 – Optimization

8. Piecewise linear optimization

Ed Forum

- What are exactly x_i^+ and x_i^- ?
- What does it mean to eliminate free/unconstrained variables?

Recap

Standard form

Definition

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

- Minimization
- Equality constraints
- Nonnegative variables

- Matrix notation for **theory**
- Standard form for **algorithms**

Standard form

Transformation tricks

Change objective

If “maximize”, use $-c$ instead of c and change to “minimize”.

Eliminate inequality constraints

If $Ax \leq b$, define s and write $Ax + s = b$, $s \geq 0$.

If $Ax \geq b$, define s and write $Ax - s = b$, $s \geq 0$.

s are the **slack variables**

Change variable signs

If $x_i \leq 0$, define $y_i = -x_i$.

Eliminate “free” variables

If x_i unconstrained, define $x_i = x_i^+ - x_i^-$, with $x_i^+ \geq 0$ and $x_i^- \geq 0$.

Standard form

Transformation example

$$\begin{array}{ll} \text{minimize} & 2x_1 + 4x_2 \\ \text{subject to} & x_1 + x_2 \geq 3 \\ & 3x_1 + 2x_2 = 14 \\ & x_1 \geq 0 \end{array}$$



$$\begin{array}{ll} \text{minimize} & 2x_1 + 4x_2^+ - 4x_2^- \\ \text{subject to} & x_1 + x_2^+ - x_2^- - x_3 = 3 \\ & 3x_1 + 2x_2^+ - 2x_2^- = 14 \\ & x_1, x_2^+, x_2^-, x_3 \geq 0. \end{array}$$

Today's lecture

Piecewise linear optimization

- Vector norms
- Piecewise linear optimization
- Turning vector norm problems into LPs
- Sparse signal recovery
- Support vector machines

Vector norms

Vector norms

Euclidean norm

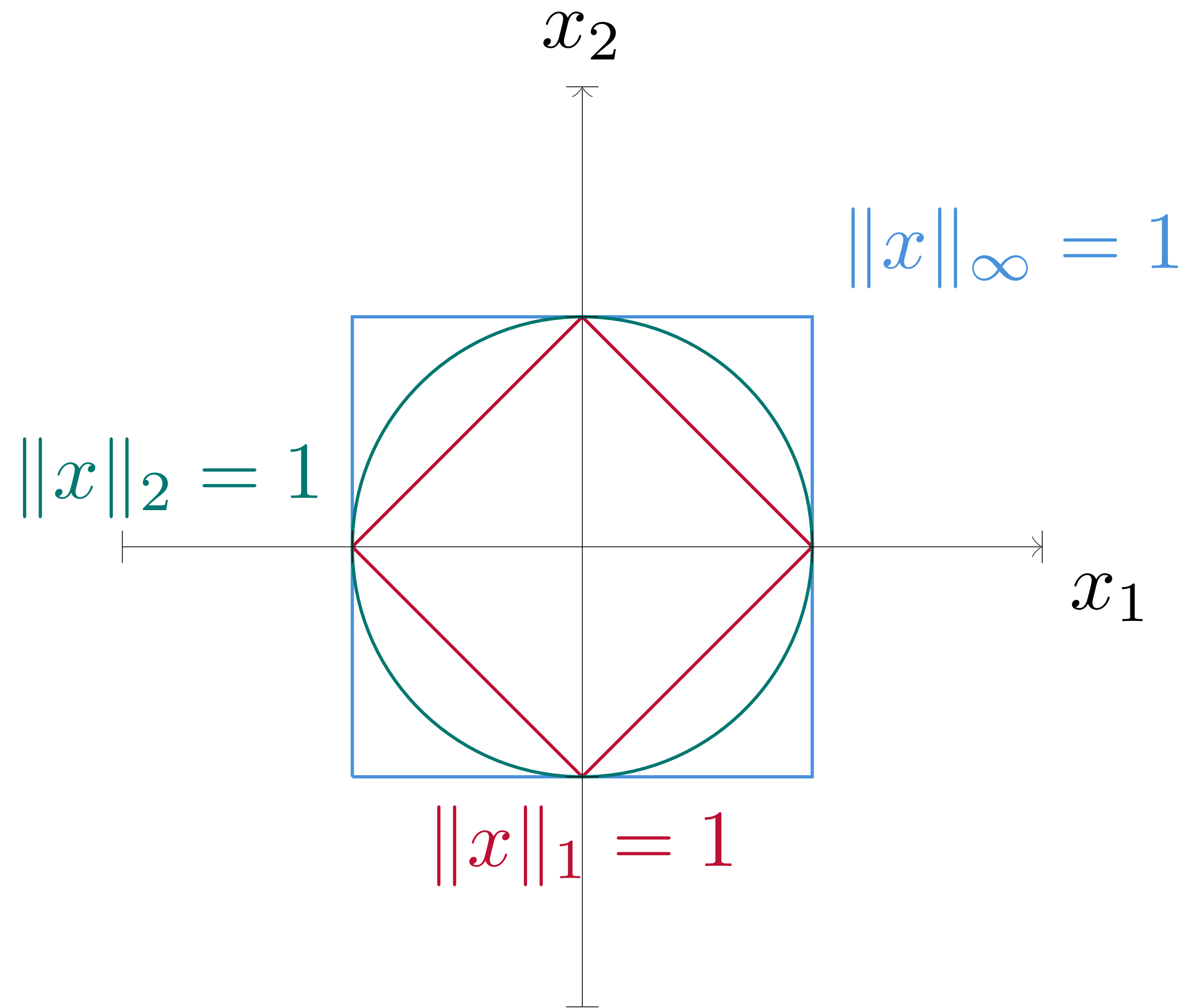
$$\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$$

1-norm (Manhattan norm)

$$\|x\|_1 = \sum_{i=1}^n |x_i|$$

∞ -norm (max-norm)

$$\|x\|_\infty = \max_i |x_i|$$

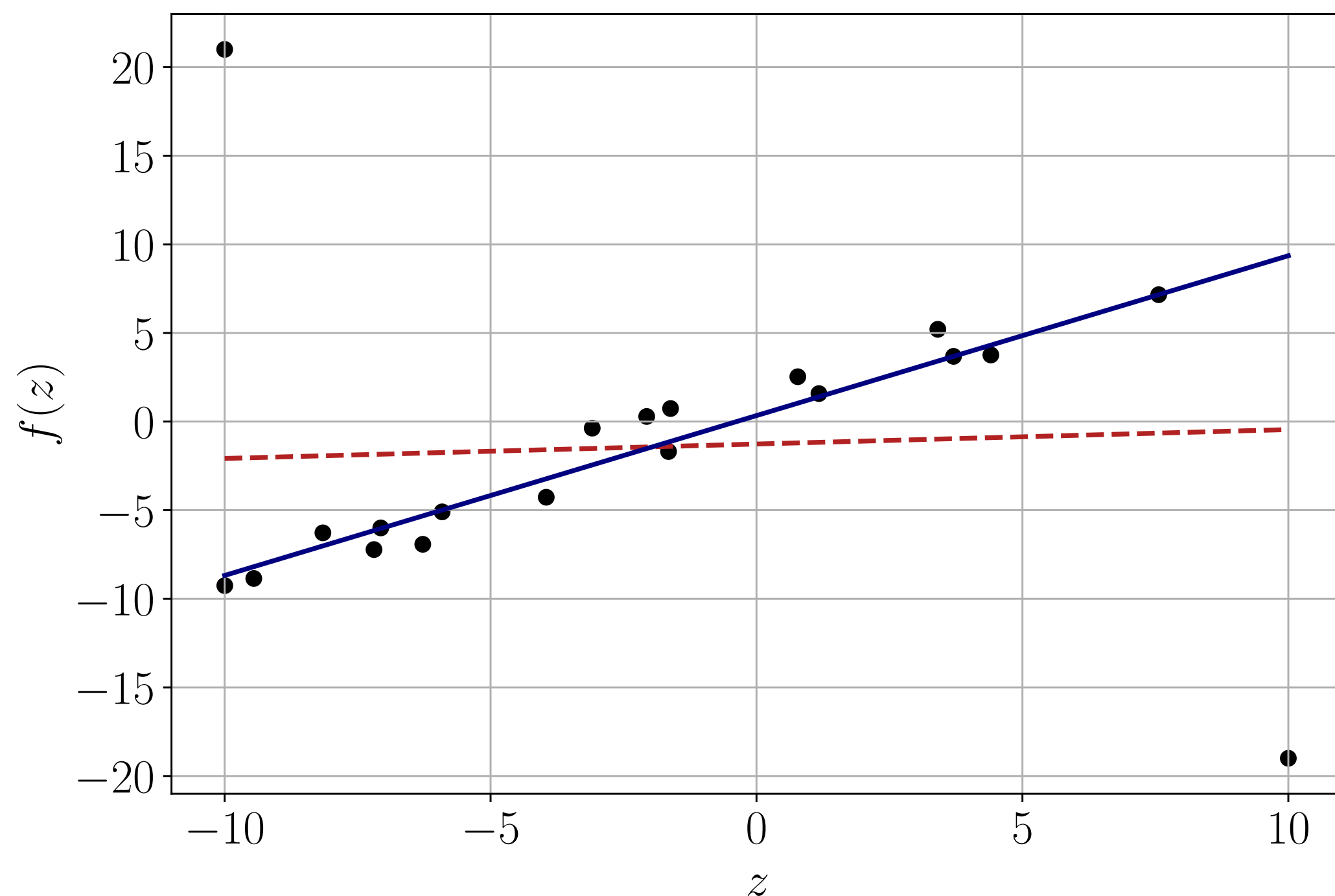


Data-fitting example

Fit a linear function $f(z) = x_1 + x_2 z$ to m data points (z_i, f_i) :

Approximation problem $Ax \approx b$ where

$$\underbrace{\begin{bmatrix} 1 & z_1 \\ \vdots & \vdots \\ 1 & z_m \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_x \approx \underbrace{\begin{bmatrix} f_1 \\ \vdots \\ f_m \end{bmatrix}}_b$$



Recall our regression problem:

$$\text{minimize } \sum_{i=1}^m |Ax - b|_i = \|Ax - b\|_1$$

Why is it a linear program?

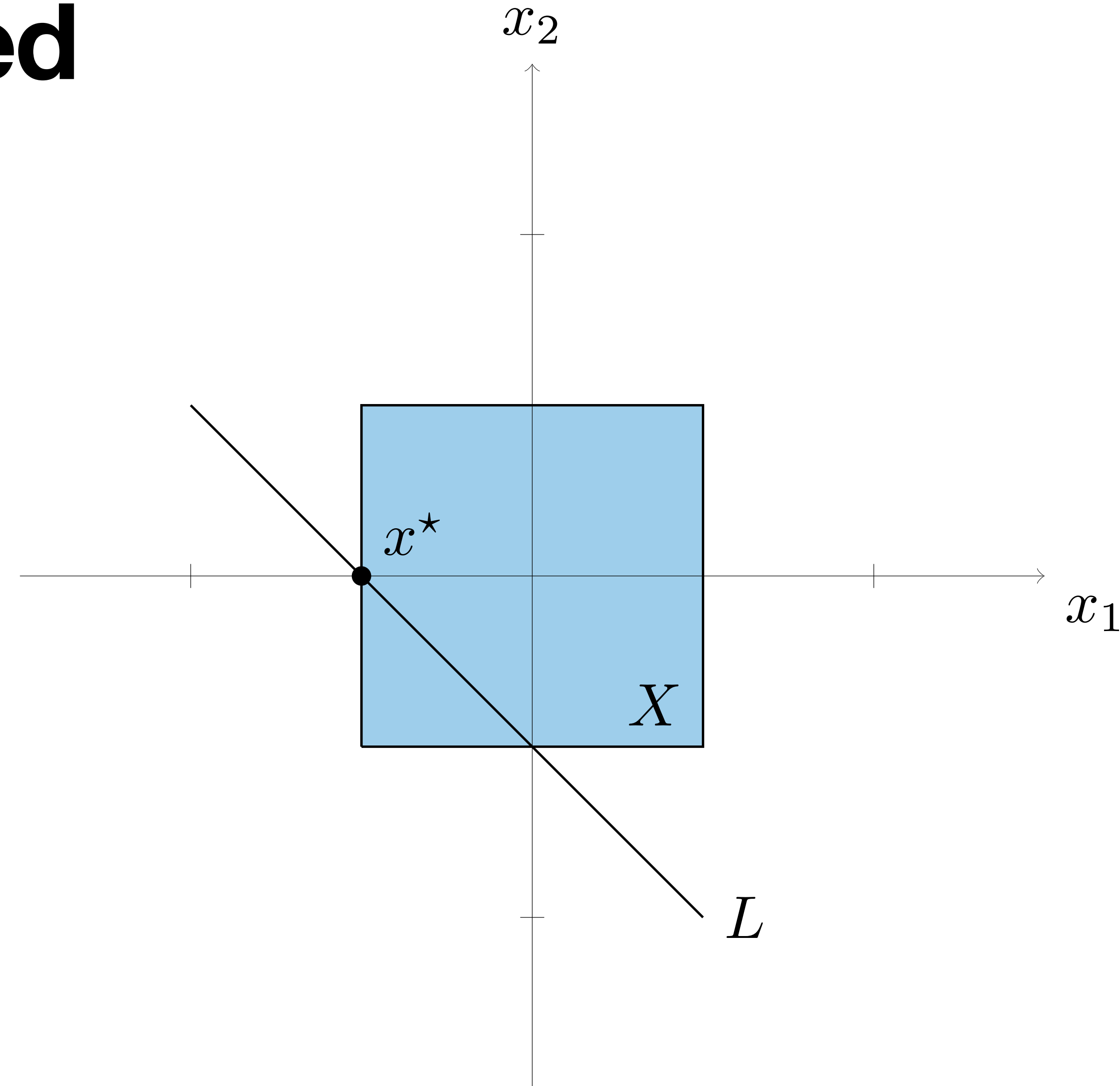
Simple example revisited

Goal find point as far left as possible,
in the unit box X ,
and restricted to the line L

$$\begin{array}{ll} \text{minimize} & x_1 \\ \text{subject to} & \|x\|_\infty \leq 1 \\ & x_1 + x_2 = -1 \end{array}$$

The (nonlinear) norm function
appears in the constraints

Why is it a linear program?

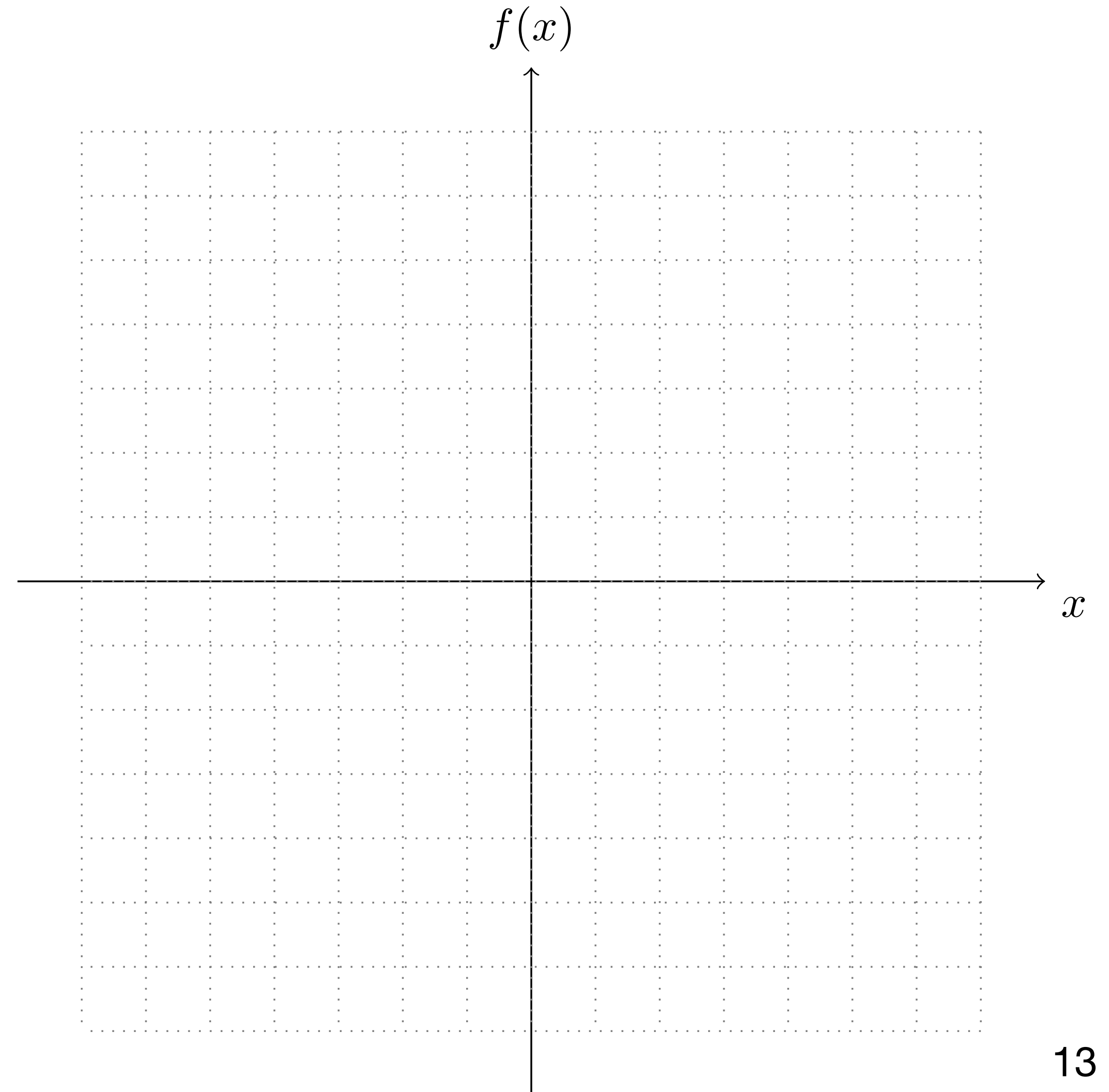


Piecewise linear optimization

Linear, affine and convex functions

Linear function: $f(x) = a^T x$

Affine function: $f(x) = a^T x + b$

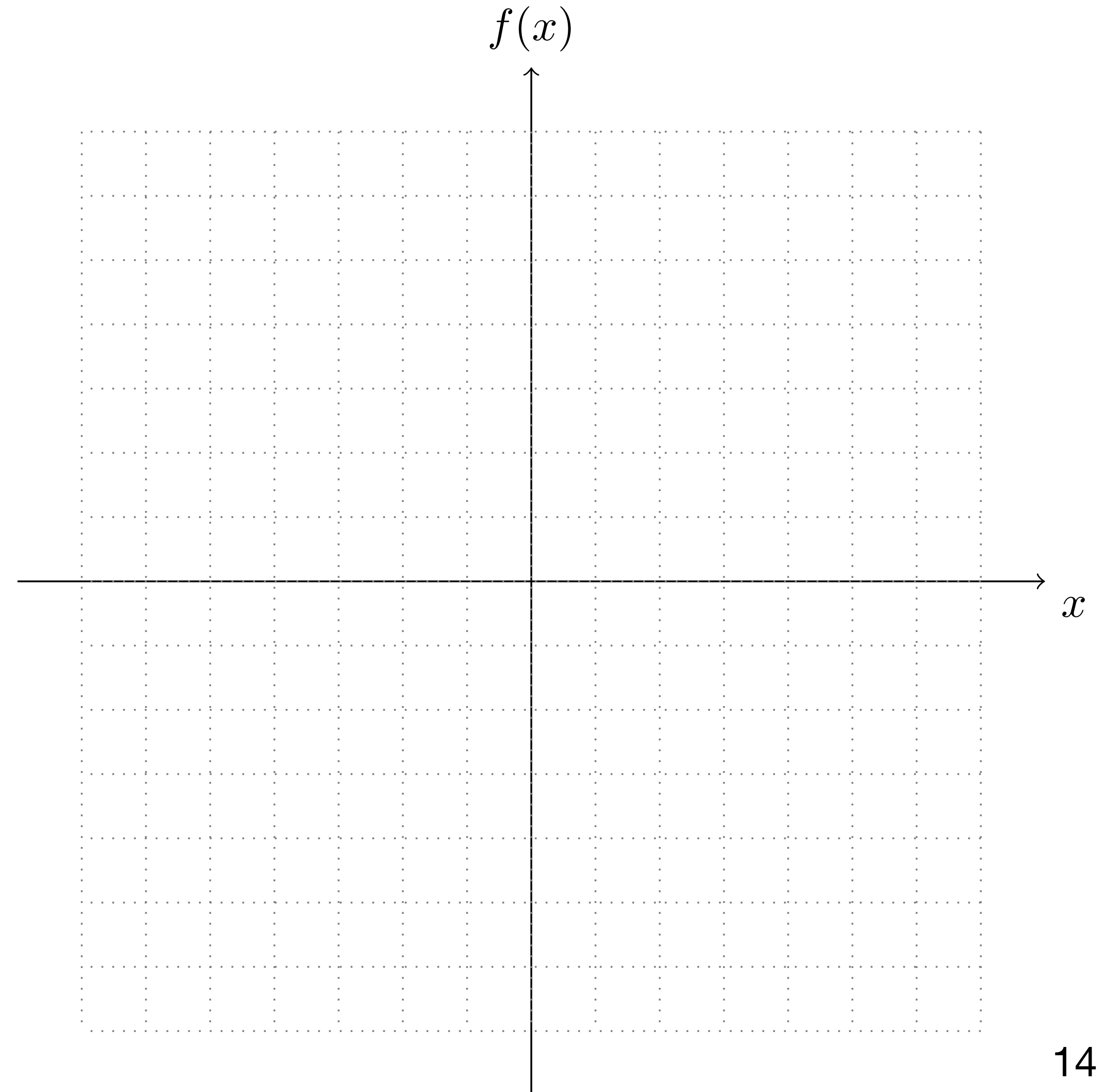


Linear, affine and convex functions

Convex function:

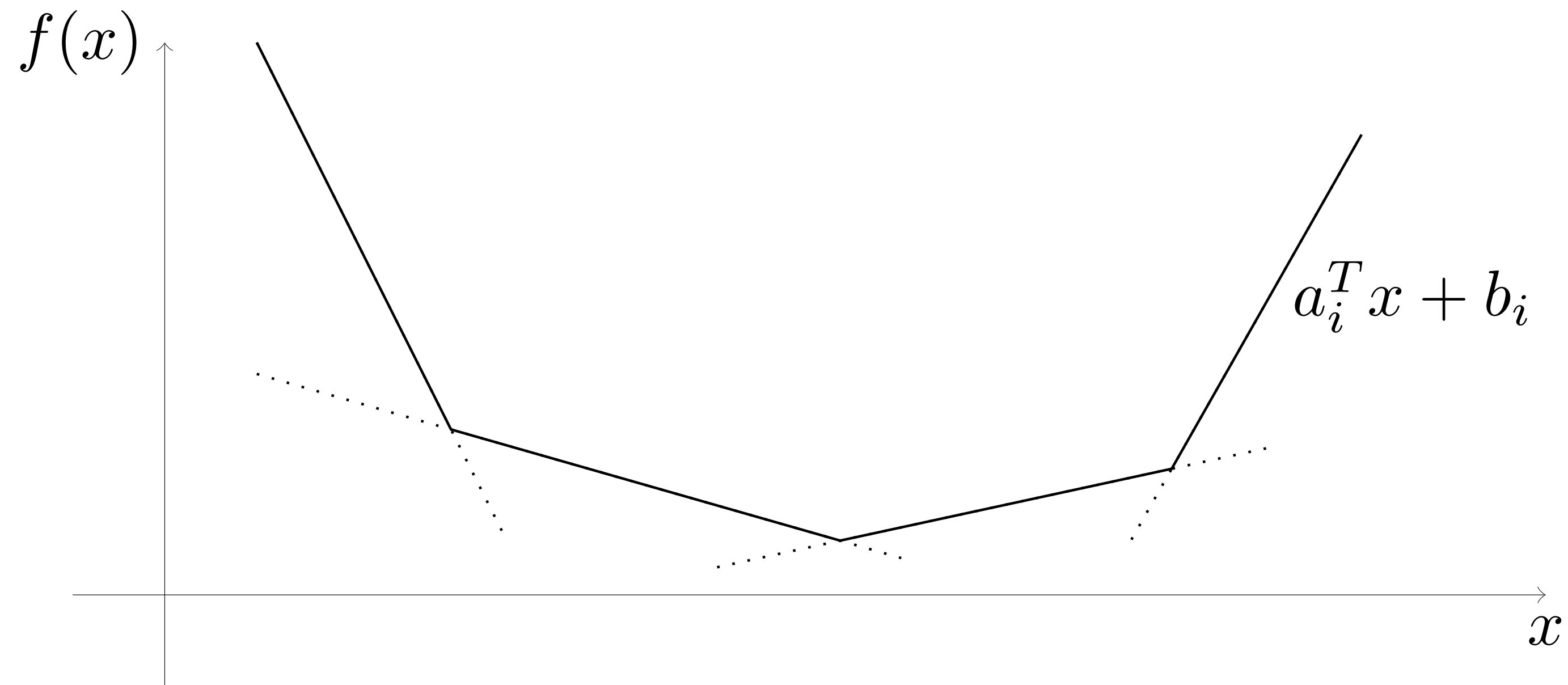
$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y),$$

$$\forall x, y \in \mathbf{R}^n, \alpha \in [0, 1]$$



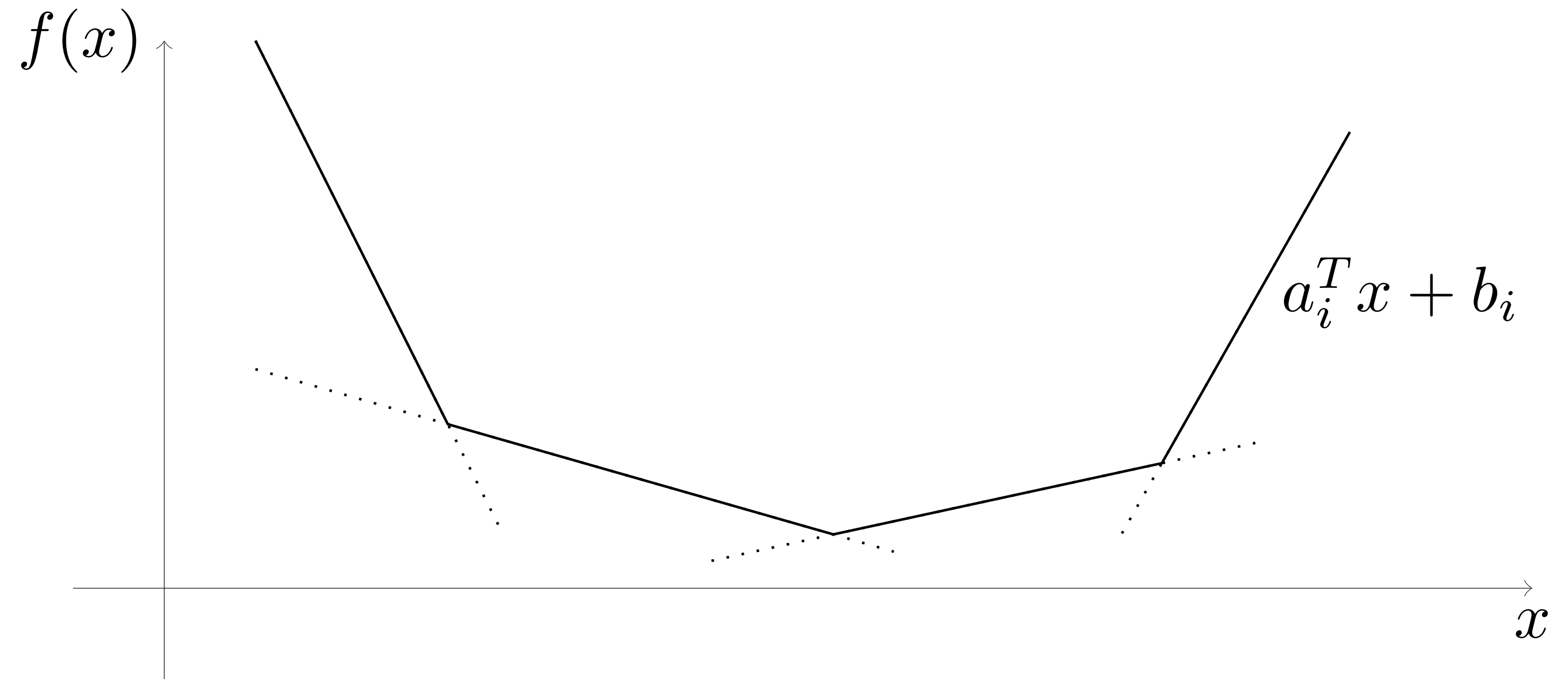
Convex piecewise-linear functions

$$f(x) = \max_{i=1, \dots, m} (a_i^T x + b_i)$$



Convex piecewise-linear minimization

$$\text{minimize} \quad \max_{i=1, \dots, m} (a_i^T x + b_i)$$



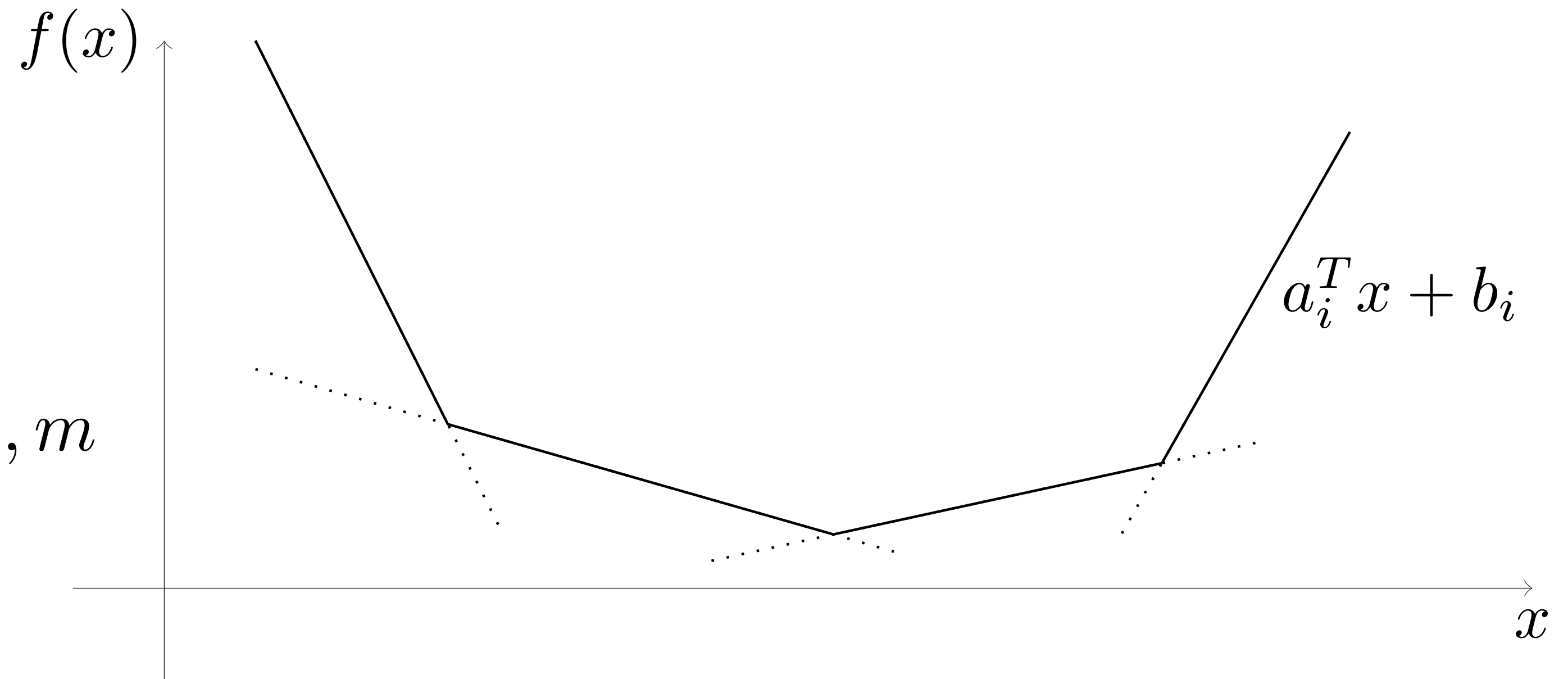
Equivalent linear optimization

$$\begin{aligned} &\text{minimize} && t \\ &\text{subject to} && a_i^T x + b_i \leq t, \quad i = 1, \dots, m \end{aligned}$$

Convex piecewise-linear minimization

Equivalent linear optimization

$$\begin{aligned} &\text{minimize} && t \\ &\text{subject to} && a_i^T x + b_i \leq t, \quad i = 1, \dots, m \end{aligned}$$



Matrix notation

$$\begin{aligned} &\text{minimize} && \tilde{c}^T \tilde{x} \\ &\text{subject to} && \tilde{A} \tilde{x} \leq \tilde{b} \end{aligned} \quad \tilde{x} = \begin{bmatrix} x \\ t \end{bmatrix}, \quad \tilde{c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} a_1^T & -1 \\ \vdots & \vdots \\ a_m^T & -1 \end{bmatrix}, \quad \tilde{b} = \begin{bmatrix} -b_1 \\ \vdots \\ -b_m \end{bmatrix}$$

Vector norm problems as linear optimization

∞ -norm regression

$$\text{minimize } \|Ax - b\|_\infty$$

The ∞ -norm of m -vector y is

$$\|y\|_\infty = \max_{i=1,\dots,m} |y_i| = \max_{i=1,\dots,m} \max\{y_i, -y_i\}$$

Equivalent problem

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & (Ax - b)_i \leq t, \quad i = 1, \dots, m \\ & -(Ax - b)_i \leq t, \quad i = 1, \dots, m \end{array}$$



$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & Ax - b \leq t\mathbf{1} \\ & -(Ax - b) \leq t\mathbf{1} \end{array}$$

∞ -norm regression

$$\text{minimize } \|Ax - b\|_\infty$$

The ∞ -norm of m -vector y is

$$\|y\|_\infty = \max_{i=1,\dots,m} |y_i| = \max_{i=1,\dots,m} \max\{y_i, -y_i\}$$

Equivalent problem

$$\begin{aligned} &\text{minimize } t \\ &\text{subject to } Ax - b \leq t\mathbf{1} \\ &\quad \quad \quad -(Ax - b) \leq t\mathbf{1} \end{aligned}$$

Matrix notation

$$\begin{aligned} &\text{minimize } \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} x \\ t \end{bmatrix} \\ &\text{subject to } \begin{bmatrix} A & -\mathbf{1} \\ -A & -\mathbf{1} \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \leq \begin{bmatrix} b \\ -b \end{bmatrix} \end{aligned}$$

Sum of piecewise-linear functions

$$\text{minimize } f(x) + g(x) = \max_{i=1,\dots,m} (a_i^T x + b_i) + \max_{i=1,\dots,p} (c_i^T x + d_i)$$

Equivalent linear optimization

$$\begin{aligned} \text{minimize } & t_1 + t_2 \\ \text{subject to } & a_i^T x + b_i \leq t_1, \quad i = 1, \dots, m \\ & c_i^T x + d_i \leq t_2, \quad i = 1, \dots, p \end{aligned}$$

1-norm regression

$$\text{minimize } \|Ax - b\|_1$$

The 1-norm of m -vector y is

$$\|y\|_1 = \sum_{i=1}^m |y_i| = \sum_{i=1}^m \max\{y_i, -y_i\}$$

Equivalent problem

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^m u_i \\ \text{subject to} & (Ax - b)_i \leq u_i, \quad i = 1, \dots, m \\ & -(Ax - b)_i \leq u_i, \quad i = 1, \dots, m \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & \mathbf{1}^T u \\ \text{subject to} & Ax - b \leq u \\ & -(Ax - b) \leq u \end{array}$$

1-norm regression

$$\text{minimize } \|Ax - b\|_1$$

The 1-norm of m -vector y is

$$\|y\|_1 = \sum_{i=1}^m |y_i| = \sum_{i=1}^m \max\{y_i, -y_i\}$$

Equivalent problem

$$\begin{aligned} &\text{minimize } \mathbf{1}^T u \\ &\text{subject to } Ax - b \leq u \\ &\quad -(Ax - b) \leq u \end{aligned}$$

Matrix notation

$$\begin{aligned} &\text{minimize } \begin{bmatrix} 0 \\ \mathbf{1} \end{bmatrix}^T \begin{bmatrix} x \\ u \end{bmatrix} \\ &\text{subject to } \begin{bmatrix} A & -I \\ -A & -I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \leq \begin{bmatrix} b \\ -b \end{bmatrix} \end{aligned}$$

Summary: 1 and ∞ -norm regression

∞ -norm

$$\text{minimize } \|Ax - b\|_\infty$$

Equivalent to

$$\begin{aligned} \text{minimize } & t \\ \text{subject to } & Ax - b \leq t\mathbf{1} \\ & -(Ax - b) \leq t\mathbf{1} \end{aligned}$$

Absolute value of every element $(Ax - b)_i$ is bounded by the same **scalar** t

1-norm

$$\text{minimize } \|Ax - b\|_1$$

Equivalent to

$$\begin{aligned} \text{minimize } & \mathbf{1}^T u \\ \text{subject to } & Ax - b \leq u \\ & -(Ax - b) \leq u \end{aligned}$$

Absolute value of every element $(Ax - b)_i$ is bounded by a component of the **vector** u

Example : converting to an LP

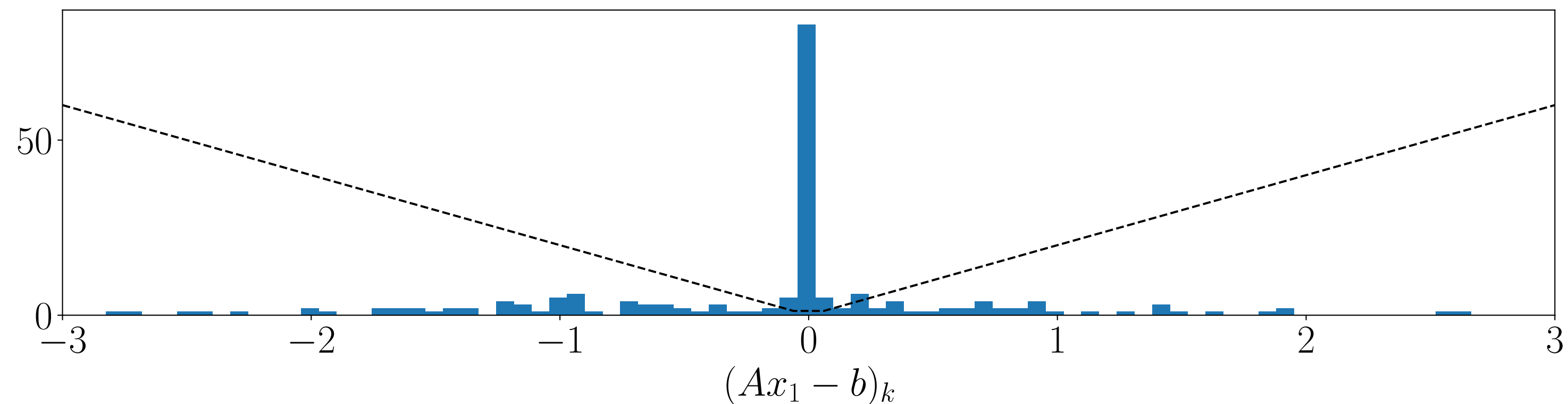
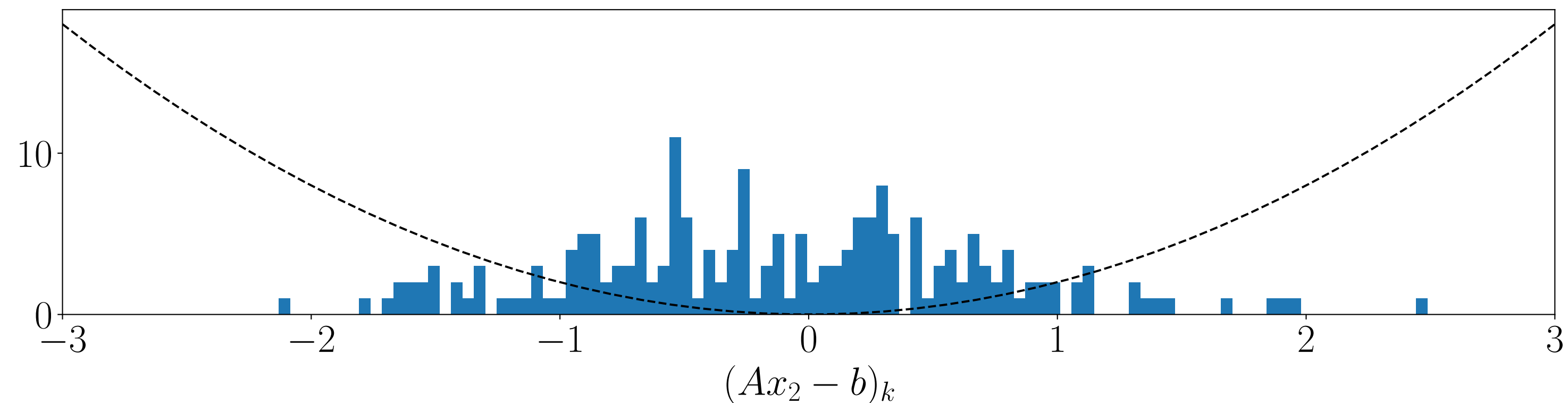
minimize $\|Ax - b\|_\infty$

subject to $\|x\|_1 \leq k$

Comparison with least-squares

Histogram of residuals $Ax - b$ with randomly generated $A \in \mathbf{R}^{200 \times 80}$

$$x_2 = \operatorname{argmin} \|Ax - b\|_2^2, \quad x_1 = \operatorname{argmin} \|Ax - b\|_1$$



1-norm distribution is **wider** with a **high peak at zero**

Modeling software does most of this for you

∞ -norm

minimize $\|Ax - b\|_\infty$

```
import numpy as np
import cvxpy as cp

m = 200; n = 80

A = np.random.randn(200, 80)
b = np.random.randn(200)
x = cp.Variable(80)

objective = cp.norm(A @ x - b, np.inf)
problem = cp.Problem(cp.Minimize(objective))
problem.solve()
```

1-norm

minimize $\|Ax - b\|_1$

```
import numpy as np
import cvxpy as cp

m = 200; n = 80

A = np.random.randn(200, 80)
b = np.random.randn(200)
x = cp.Variable(80)

objective = cp.norm(A @ x - b, 1)
problem = cp.Problem(cp.Minimize(objective))
problem.solve()
```

Sparse signal recovery

Sparse signal recovery via ℓ_1 -norm minimization

$\hat{x} \in \mathbf{R}^n$ is unknown signal, known to be sparse

We make linear measurements $y = A\hat{x}$ with $A \in \mathbf{R}^{m \times n}$, $m < n$

Estimate signal with smallest ℓ_1 -norm, consistent with measurements

$$\begin{aligned} &\text{minimize} && \|x\|_1 \\ &\text{subject to} && Ax = y \end{aligned}$$

Equivalent linear optimization

$$\begin{aligned} &\text{minimize} && \mathbf{1}^T u \\ &\text{subject to} && -u \leq x \leq u \\ &&& Ax = y \end{aligned}$$

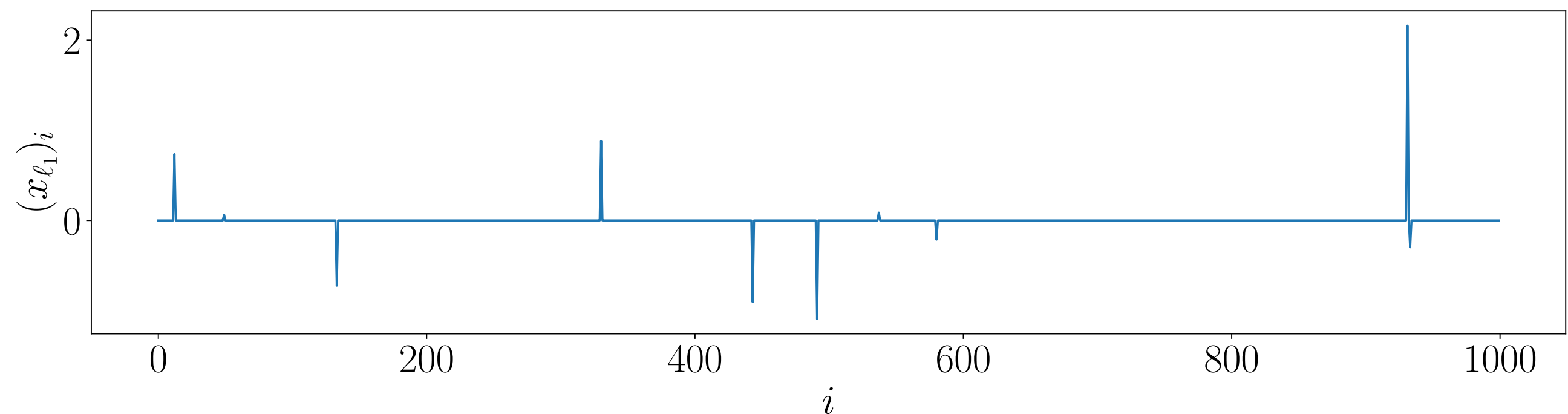
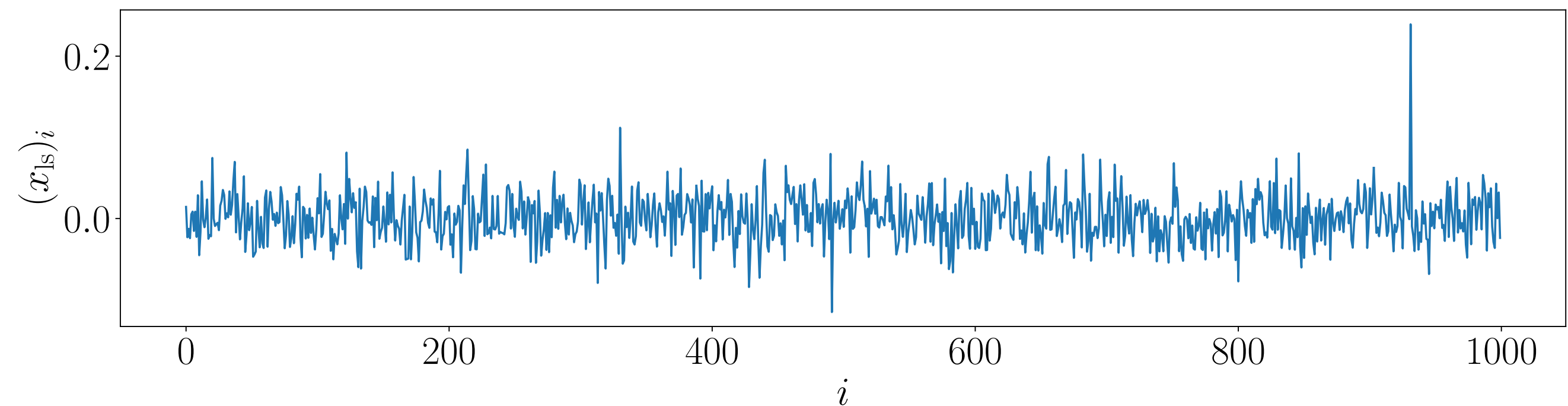
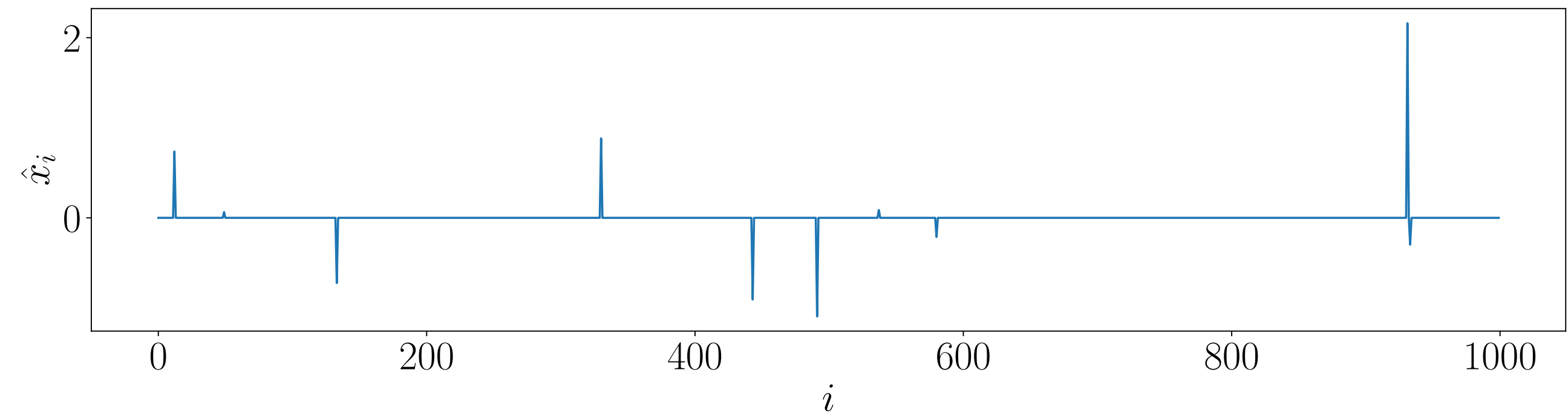
Sparse signal recovery via 1-norm minimization

Example

Exact signal $\hat{x} \in \mathbf{R}^{1000}$
10 nonzero components
Random $A \in \mathbf{R}^{100 \times 1000}$

The least squares estimate
cannot recover the sparse signal

The 1-norm estimate is **exact**



Support vector machines

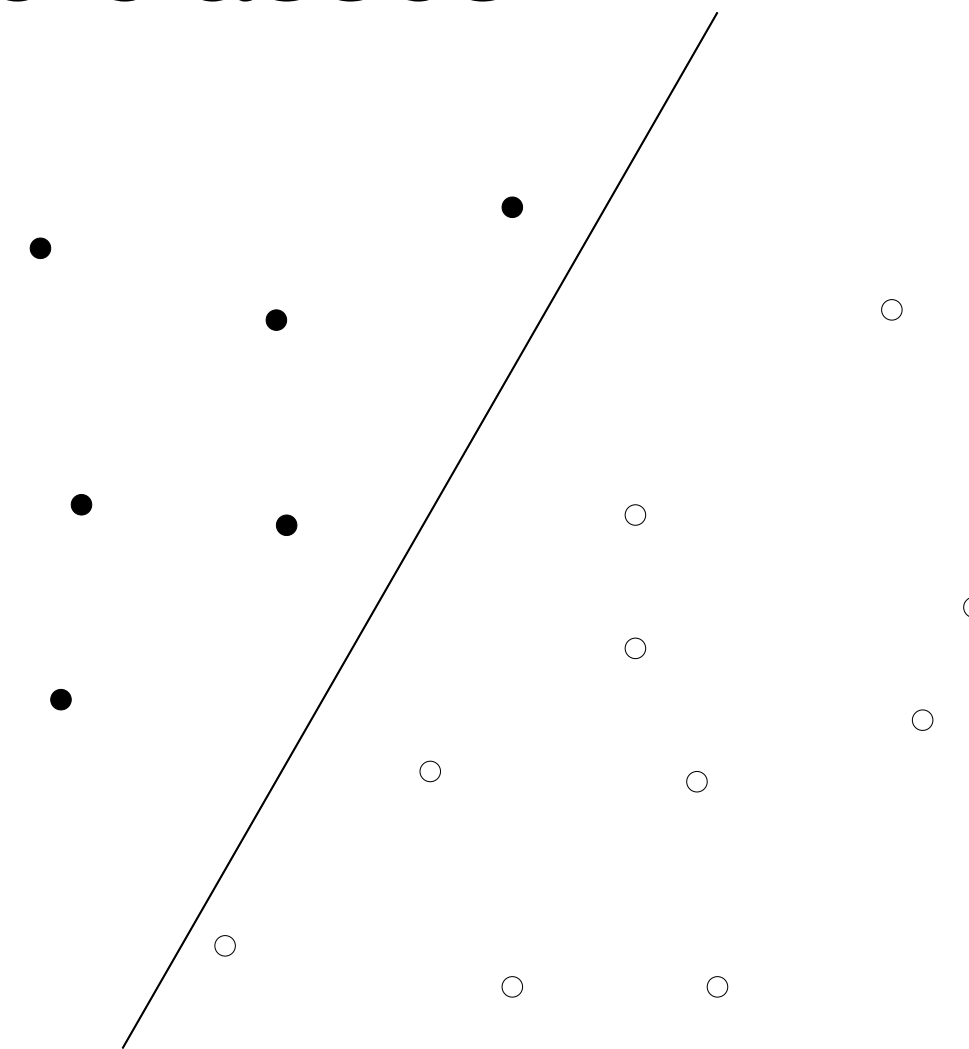
Linear classification

Support vector machine (linear separation)

Given a set of points $\{v_1, \dots, v_N\}$ with binary labels $s_i \in \{-1, 1\}$

Find hyperplane that strictly separates the two classes

$$\begin{aligned} a^T v_i + b &> 0 & \text{if } s_i = 1 \\ a^T v_i + b &< 0 & \text{if } s_i = -1 \end{aligned}$$



Homogeneous in (a, b) , hence equivalent to the linear inequalities (in a, b)

$$s_i(a^T v_i + b) \geq 1$$

Linear classification

Separable case

Feasibility problem

$$\begin{array}{ll} \text{find} & a, b \\ \text{subject to} & s_i(a^T v_i + b) \geq 1, \quad i = 1, \dots, N \end{array}$$

Which can be seen as a special case of LP with

$$\begin{array}{ll} \text{minimize} & 0 \\ \text{subject to} & s_i(a^T v_i + b) \geq 1, \quad i = 1, \dots, N \end{array}$$

$p^* = 0$ if problem feasible (points separable)

$p^* = \infty$ if problem infeasible (points not separable) \longrightarrow **What then?**

Linear classification

Approximate linear separation of non-separable points

Each of our constraints is

$$s_i(a^T v_i + b) \geq 1$$



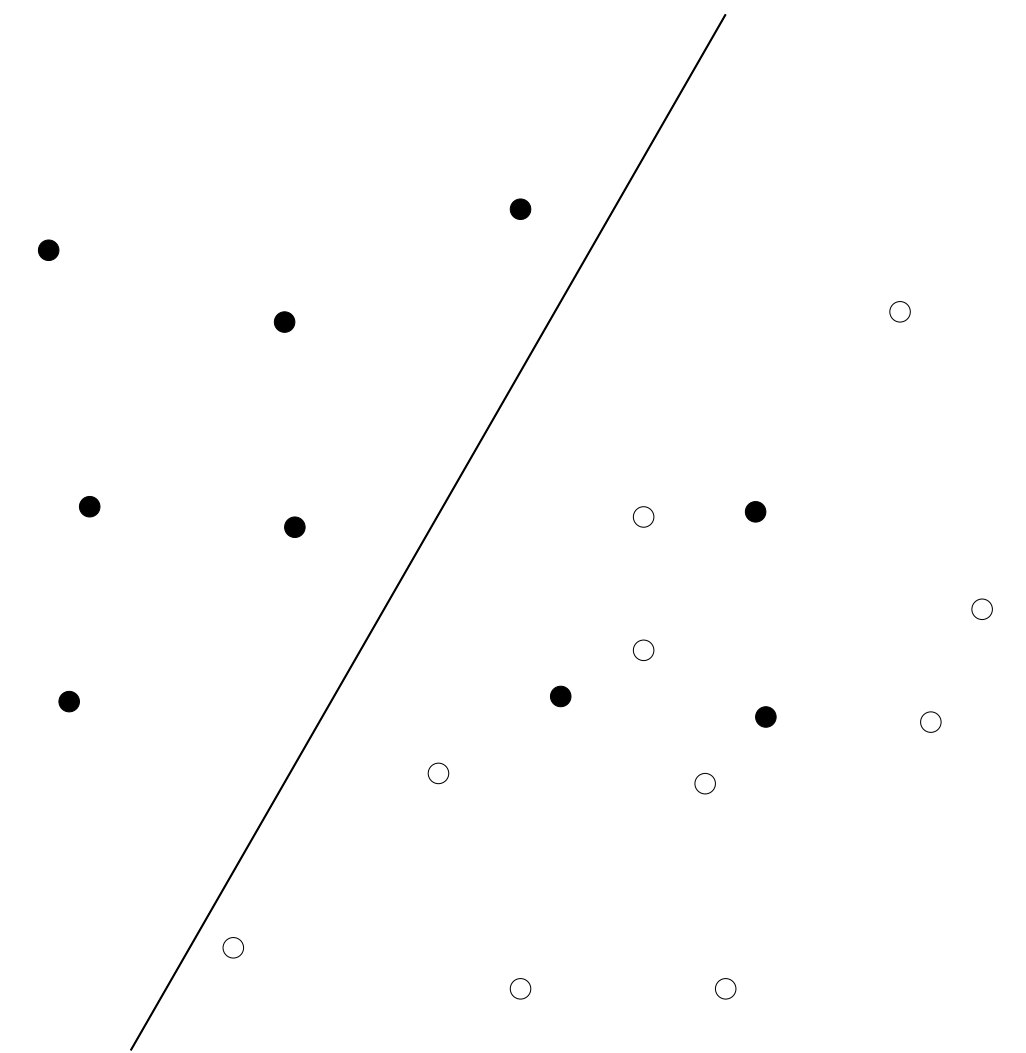
Violation

$$\max\{0, 1 - s_i(a^T v_i + b)\}$$

Goal

Minimize sum of the violations

$$\text{minimize } \sum_{i=1}^N \max\{0, 1 - s_i(a^T v_i + b)\}$$



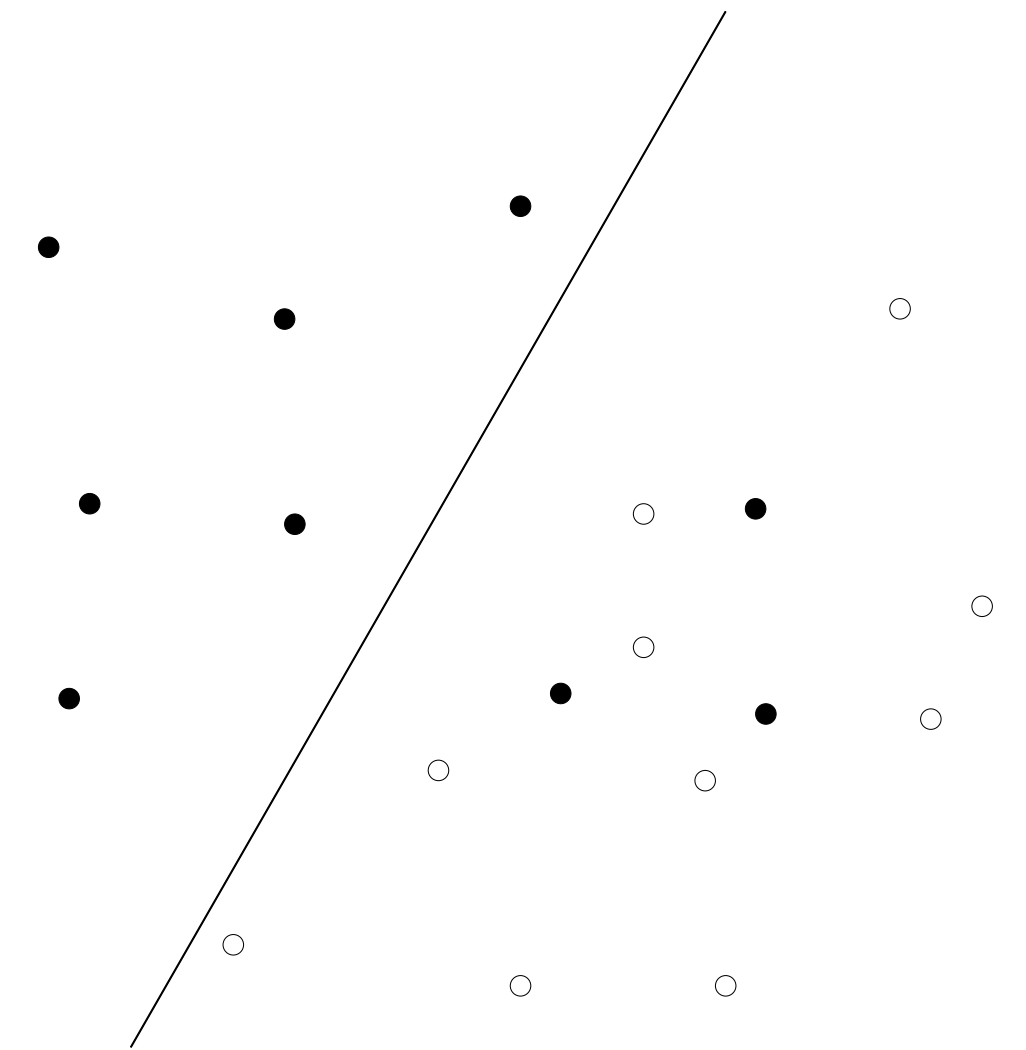
Piecewise-linear minimization problem with variables a, b

Linear classification

Approximate linear separation of non-separable points

$$\text{minimize } \sum_{i=1}^N \max\{0, 1 - s_i(a^T v_i + b)\}$$

As a linear optimization problem



Piecewise-linear optimization

Today, we learned to:

- **Understand** the differences between vector norms
- **Reformulate** convex piecewise linear minimization as linear optimization
- **Apply** these techniques to sparse signal recovery and classification problems

References

- Bertsimas, Tsitsiklis: Introduction to Linear Optimization
 - Chapter 1.3: piecewise linear optimization
- R. Vanderbei: Linear Programming — Foundations and Extensions
 - Chapter 12.4, 12.7: 1-norm regression and SVMs

Next time

- Linear optimization geometry
- Optimality conditions