ORF307 – Optimization

13. Duality

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Course feedback survey

URL

https://forms.gle/2zTGHmKaQrksDCAU9

QR Code

Ed Forum

- Midterm solutions will be posted today
- When implementing the simplex method in practice, is there a way to know whether you have an example of worst-case behavior? If so, how should you pick the best pivot rule to avoid these worst cases?

Recap

Linear optimization formulations

Standard form LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Inequality form LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

Today's agenda
Duality

- Obtaining lower bounds
- The dual problem
- Weak and strong duality

Obtaining lower bounds

Obtaining lower bounds

A simple example

$$\begin{aligned} \text{minimize} \quad & x_1 + 3x_2 \\ \text{subject to} \quad & x_1 + 3x_2 \geq 2 \end{aligned}$$

What is a **lower bound** on the optimal cost?

A lower bound is 2 because $x_1 + 3x_2 \geq 2$

Obtaining lower bounds

Another example

$$\begin{aligned} \text{minimize} \quad & x_1 + 3x_2 \\ \text{subject to} \quad & x_1 + x_2 \geq 2 \\ & x_2 \geq 1 \end{aligned}$$

What is a **lower bound** on the optimal cost?

Let's sum the constraints

$$\begin{aligned} & 1 \cdot (x_1 + x_2 \geq 2) \\ + \, & 2 \cdot (x_2 \geq 1) \\ = \, & x_1 + 3x_2 \geq 4 \end{aligned}$$

A lower bound is 4

Obtaining lower bounds

A more interesting example

$$
\begin{aligned}
\text{minimize} \quad & x_1 + 3x_2 \\
\text{subject to} \quad & x_1 + x_2 \geq 2 \\
& x_2 \geq 1 \\
& x_1 - x_2 \geq 3
\end{aligned}
$$

How can we obtain a lower bound?

Add constraints

$$
\begin{aligned}
& y_1 \cdot (x_1 + x_2 \geq 2) \\
+ & y_2 \cdot (x_2 \geq 1) \\
+ & y_3 \cdot (x_1 - x_2 \geq 3) \\
\Rightarrow & (y_1 + y_3)x_1 + (y_1 + y_2 - y_3)x_2 \geq 2y_1 + y_2 + 3y_3
\end{aligned}
$$

Bound

Match cost coefficients

$$
\begin{aligned}
y_1 + y_3 &= 1 \\
y_1 + y_2 - y_3 &= 3 \\
y_1, y_2, y_3 &\geq 0
\end{aligned}
$$

Many options

$y = (1, 2, 0) \Rightarrow$ Bound 4

$y = (0, 4, 1) \Rightarrow$ Bound 7

How can we get the **best one**?

Obtaining lower bounds

A more interesting example — Best lower bound

We can obtain the **best lower bound** by solving the following problem

$$
\begin{array}{ll}
\text{maximize} & 2y_1 + y_2 + 3y_3 \\
\text{subject to} & y_1 + y_3 = 1 \\
& y_1 + y_2 - y_3 = 3 \\
& y_1, y_2, y_3 \geq 0
\end{array}
$$

This linear optimization problem is called the **dual problem**

The dual problem

Lagrange multipliers

Consider the LP in standard form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Relax the constraint

$$g(y) = \begin{array}{ll} \underset{x}{\text{minimize}} & c^T x + y^T (Ax - b) \\ \text{subject to} & x \geq 0 \end{array}$$

Lower bound

$$g(y) \leq c^T x^\star + y^T (Ax^\star - b) = c^T x^\star$$

Best lower bound

$$\underset{y}{\text{maximize}} \quad g(y)$$

The dual

Dual function

$$
\begin{aligned}
g(y) &= \underset{x \geq 0}{\text{minimize}} \left(c^T x + y^T (Ax - b)\right) \\
&= -b^T y + \underset{x \geq 0}{\text{minimize}} \left(c + A^T y\right)^T x
\end{aligned}
$$

$$
g(y) = \begin{cases} -b^T y & \text{if } c + A^T y \geq 0 \\ -\infty & \text{otherwise} \end{cases}
$$

Dual problem (find the best bound)

$$
\underset{y}{\text{maximize}} \quad g(y) \quad = \quad \begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c \geq 0 \end{array}
$$

Primal and dual problems

Primal problem

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Dual problem

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c \geq 0 \end{array}$$

Primal variable $x \in \mathbf{R}^n$

Dual variable $y \in \mathbf{R}^m$

The dual problem carries **useful information** for the primal problem

Duality is useful also to **solve** optimization problems

Dual of inequality form LP

What if you find an LP with inequalities?

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

1. We could first transform it to standard form
2. We can compute the dual function (same procedure as before)

Relax the constraint

$$g(y) = \underset{x}{\text{minimize}} \quad c^T x + y^T(Ax - b)$$

Lower bound

$$g(y) \leq c^T x^\star + y^T(Ax^\star - b) \leq c^T x^\star$$

we must have $y \geq 0$

Dual of LP with inequalities

Derivation

Dual function

$$g(y) = \underset{x}{\text{minimize}} \left(c^T x + y^T (Ax - b)\right)$$

$$= -b^T y + \underset{x}{\text{minimize}} \left(c + A^T y\right)^T x$$

$$g(y) = \begin{cases} -b^T y & \text{if } c + A^T y = 0 \quad \text{(and } y \geq 0\text{)} \\ -\infty & \text{otherwise} \end{cases}$$

Dual problem (find the best bound)

$$\begin{array}{llll} \underset{y}{\text{maximize}} \quad g(y) & = & \text{maximize} & -b^T y \\ & & \text{subject to} & A^T y + c = 0 \\ & & & y \geq 0 \end{array}$$

General forms

Standard form LP

Primal

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Dual

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c \geq 0 \end{array}$$

Inequality form LP

Primal

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

Dual

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c = 0 \\ & y \geq 0 \end{array}$$

LP with inequalities and equalities

Primal

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \\ & Cx = d \end{array}$$

Dual

$$\begin{array}{ll} \text{maximize} & -b^T y - d^T z \\ \text{subject to} & A^T y + C^T z + c = 0 \\ & y \geq 0 \end{array}$$

Example from before

minimize $x_1 + 3x_2$

subject to $x_1 + x_2 \geq 2$

$x_2 \geq 1$

$x_1 - x_2 \geq 3$

→

Inequality form LP

minimize $c^T x$

subject to $Ax \leq b$

$$c = (1, 3)$$

$$A = \begin{bmatrix} -1 & -1 \\ 0 & -1 \\ -1 & 1 \end{bmatrix}$$

$$b = (-2, -1, -3)$$

Dual

maximize $-b^T y$

subject to $A^T y + c = 0$

$y \geq 0$

→

maximize $2y_1 + y_2 + 3y_3$

subject to $-y_1 - y_3 = -1$

$-y_1 - y_2 + y_3 = -3$

$y_1, y_2, y_3 \geq 0$

To memorize

Ways to get the dual

- Derive dual function directly
- Transform the problem in inequality form LP and dualize

Sanity-checks and signs convention

- Consider constraints as $Ax - b \leq 0$ or $Ax - b = 0$ (not ≥ 0)
- Each dual variable is associated to a primal constraint
- y free for primal equalities and $y \geq 0$ for primal inequalities

Dual of the dual

Theorem

If we transform the primal into its dual and then transform the dual to its dual, we obtain a problem equivalent to the original problem. In other words, the **dual of the dual is the primal**.

Exercise

Derive dual and dualize again

Primal

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax \leq b \\
& Cx = d
\end{array}
$$

Dual

$$
\begin{array}{ll}
\text{maximize} & -b^T y - d^T z \\
\text{subject to} & A^T y + C^T z + c = 0 \\
& y \geq 0
\end{array}
$$

Theorem

If we **transform a linear optimization problem to another form** (inequality form, standard form, inequality and equality form), **the dual of the two problems will be equivalent**.

Weak and strong duality

Optimal objective values

Primal

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

Dual

$$\begin{array}{ll} \text{maximize} & -b^T y \\ \text{subject to} & A^T y + c = 0 \\ & y \geq 0 \end{array}$$

$p^\star$ is the primal optimal value

$d^\star$ is the dual optimal value

Primal infeasible: $p^\star = +\infty$
Primal unbounded: $p^\star = -\infty$

Dual infeasible: $d^\star = -\infty$
Dual unbounded: $d^\star = +\infty$

Weak duality

Theorem

If x, y satisfy:

- x is a feasible solution to the primal problem
- y is a feasible solution to the dual problem

$\longrightarrow$ $-b^T y \leq c^T x$

Proof

We know that $Ax \leq b$, $A^T y + c = 0$ and $y \geq 0$. Therefore,

$$0 \leq y^T(b - Ax) = b^T y - y^T A x = c^T x + b^T y \qquad \blacksquare$$

Remark

- Any dual feasible y gives a **lower bound** on the primal optimal value
- Any primal feasible x gives an **upper bound** on the dual optimal value
- $c^T x + b^T y$ is the **duality gap**

Weak duality

Corollaries

Unboundedness vs feasibility

- Primal unbounded ($p^\star = -\infty$) $\Rightarrow$ dual infeasible ($d^\star = -\infty$)
- Dual unbounded ($d^\star = +\infty$) $\Rightarrow$ primal infeasible ($p^\star = +\infty$)

Optimality condition

If x, y satisfy:

- x is a feasible solution to the primal problem
- y is a feasible solution to the dual problem
- The duality gap is zero, *i.e.*, $c^T x + b^T y = 0$

Then x and y are **optimal solutions** to the primal and dual problem respectively

Strong duality

Theorem
If a linear optimization problem has an optimal solution, so does its dual, and the optimal value of primal and dual are equal

$$d^\star = p^\star$$

Strong duality

Constructive proof

Given a primal optimal solution $x^\star$ we will construct a dual optimal solution $y^\star$

Apply simplex to problem in **standard form**

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax = b \\
 & x \geq 0
\end{array}
\longrightarrow
$$

- optimal basis B
- optimal solution $x^\star$ with $A_B x_B^\star = b$
- reduced costs $\bar{c} = c - A^T A_B^{-T} c_B \geq 0$

Define $y^\star$ such that $y^\star = -A_B^{-T} c_B$. Therefore, $A^T y^\star + c \geq 0$ ($y^\star$ dual feasible).

$$-b^T y^\star = -b^T(-A_B^{-T} c_B) = c_B^T(A_B^{-1} b) = c_B^T x_B^\star = c^T x^\star$$

By weak duality theorem corollary, $y^\star$ is an optimal solution of the dual.
Therefore, $d^\star = p^\star$.

■

Exception to strong duality

Primal

minimize x

subject to $0 \cdot x \leq -1$

Dual

maximize y

subject to $0 \cdot y + 1 = 0$

$y \geq 0$

Optimal value is $p^\star = +\infty$

Optimal value is $d^\star = -\infty$

Both **primal** and **dual infeasible**

Relationship between primal and dual

	$p^\star = +\infty$	$p^\star$ finite	$p^\star = -\infty$
$d^\star = +\infty$	primal inf. dual unb.		
$d^\star$ finite		optimal values equal	
$d^\star = -\infty$	exception		primal unb. dual inf

- Upper-right excluded by **weak duality**
- $(1, 1)$ and $(3, 3)$ proven by **weak duality**
- $(3, 1)$ and $(2, 2)$ proven by **strong duality**

Example

Production problem

maximize $x_1 + 2x_2$ ← Profits

subject to

$$x_1 \leq 100$$
$$2x_2 \leq 200$$
$$x_1 + x_2 \leq 150$$

← Resources

$$x_1, x_2 \geq 0$$

$$c = (-1, -2)$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$b = (100, 200, 150, 0, 0)$$

Dualize

1. Transform in inequality form

minimize $c^T x$

subject to $Ax \leq b$

2. Derive dual

maximize $-b^T y$

subject to $A^T y + c = 0$

$y \geq 0$

Production problem

Dualized

$$
\begin{aligned}
\text{maximize} \quad & -b^T y \\
\text{subject to} \quad & A^T y + c = 0 \\
& y \geq 0
\end{aligned}
$$

$$c = (-1, -2)$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$b = (100, 200, 150, 0, 0)$$

Fill-in data

$$
\begin{aligned}
\text{minimize} \quad & 100y_1 + 200y_2 + 150y_3 \\
\text{subject to} \quad & y_1 + y_3 - \cancel{y_4} = 1 \\
& 2y_2 + y_3 - \cancel{y_5} = 2 \\
& y_1, y_2, y_3, y_4, y_5 \geq 0
\end{aligned}
$$

$\longrightarrow$

Eliminate variables

$$
\begin{aligned}
\text{minimize} \quad & 100y_1 + 200y_2 + 150y_3 \\
\text{subject to} \quad & y_1 + y_3 \geq 1 \\
& 2y_2 + y_3 \geq 2 \\
& y_1, y_2, y_3 \geq 0
\end{aligned}
$$

Production problem

The dual

$$\begin{array}{ll} \text{minimize} & 100y_1 + 200y_2 + 150y_3 \\ \text{subject to} & y_1 + y_3 \geq 1 \\ & 2y_2 + y_3 \geq 2 \\ & y_1, y_2, y_3 \geq 0 \end{array}$$

Interpretation

- **Sell all your resources** at a fair (minimum) price
- Selling must be **more convenient than producing**:
  - Product 1 (price 1, needs $1\times$ resource 1 and 3): $y_1 + y_3 \geq 1$
  - Product 2 (price 2, needs $2\times$ resource 2 and $1\times$ resource 3): $2y_2 + y_3 \geq 2$

Linear optimization duality

Today, we learned to:

- **Dualize** linear optimization problems
- **Prove** weak and strong duality conditions
- **Interpret** simple dual optimization problems

References

- Bertsimas and Tsitsiklis: Introduction to Linear Optimization
  - Chapter 4: Duality theory
- R. Vanderbei: Linear Programming — Foundations and Extensions
  - Chapter 5: Duality theory

Next lecture

More on duality:

- Game theory
- Complementary slackness
- Farkas lemma