ORF307 – Optimization

9. Geometry and polyhedra

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Ed Forum

- **Questions**
    - Given that we have now seen the 1-norm (Manhattan norm), the 2-norm (Euclidian norm), and the infinity-norm (max norm), I was wondering if there were other norms commonly used (perhaps not as common as the previous three) in optimization and linear regression. Would they be used for more niche cases or are they just rarely used?
- **Midterm**
    - Next Thursday, Mar 3, lecture time. In class.
    - Past midterm exercises available
      (this year’s one will be shorter, only3 exercises)
- **Homeworks**
    - Always try to export with latex
    - If you export with colab “File -> Print”, you must check the plots. It is *your responsibility to make them visible*.

- On Colab: To export as pdf run the following commands:

```
# Install required packages
!apt-get install texlive texlive-xetex texlive-latex-extra pandoc cm-super dvipng
!pip install pypandoc
# Mount Google Drive
from google.colab import drive
drive.mount('/content/drive')
```

Then you can go to the notebook directory on your drive and export it, for example

```
%cd drive/MyDrive/orf307/homeworks/01_homework/
!jupyter nbconvert --to PDF "ORF307_HW1.ipynb"
```

Today's lecture

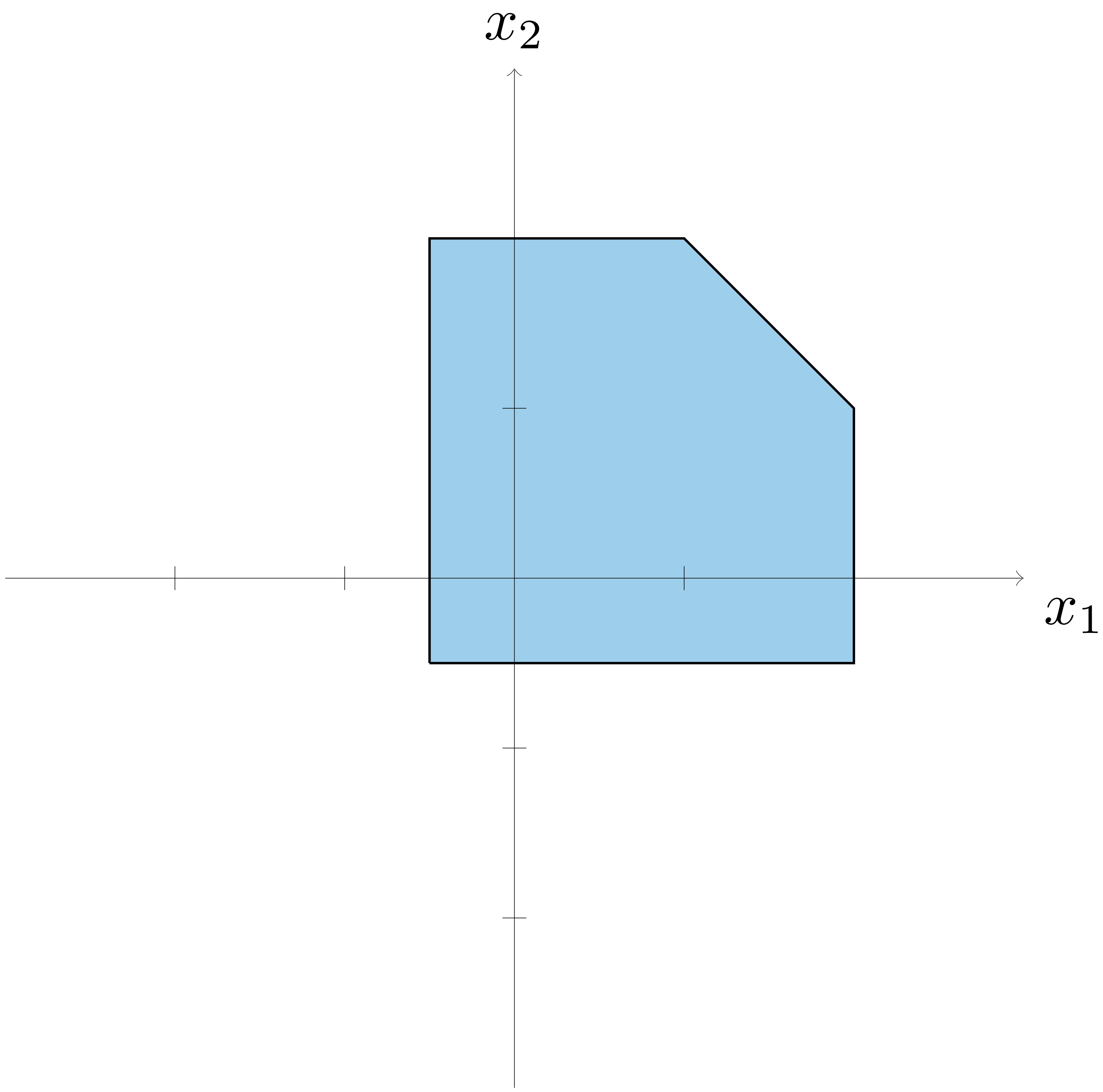
Geometry and polyhedra

- Simple example
- Polyhedra
- Corners: extreme points, vertices, basic feasible solutions
- Constructing basic solutions
- Existence and optimality of extreme points

A simple example

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & -1/2 \leq x_1 \leq 2 \\ & -1/2 \leq x_2 \leq 2 \\ & x_1 + x_2 \leq 2 \end{aligned}$$

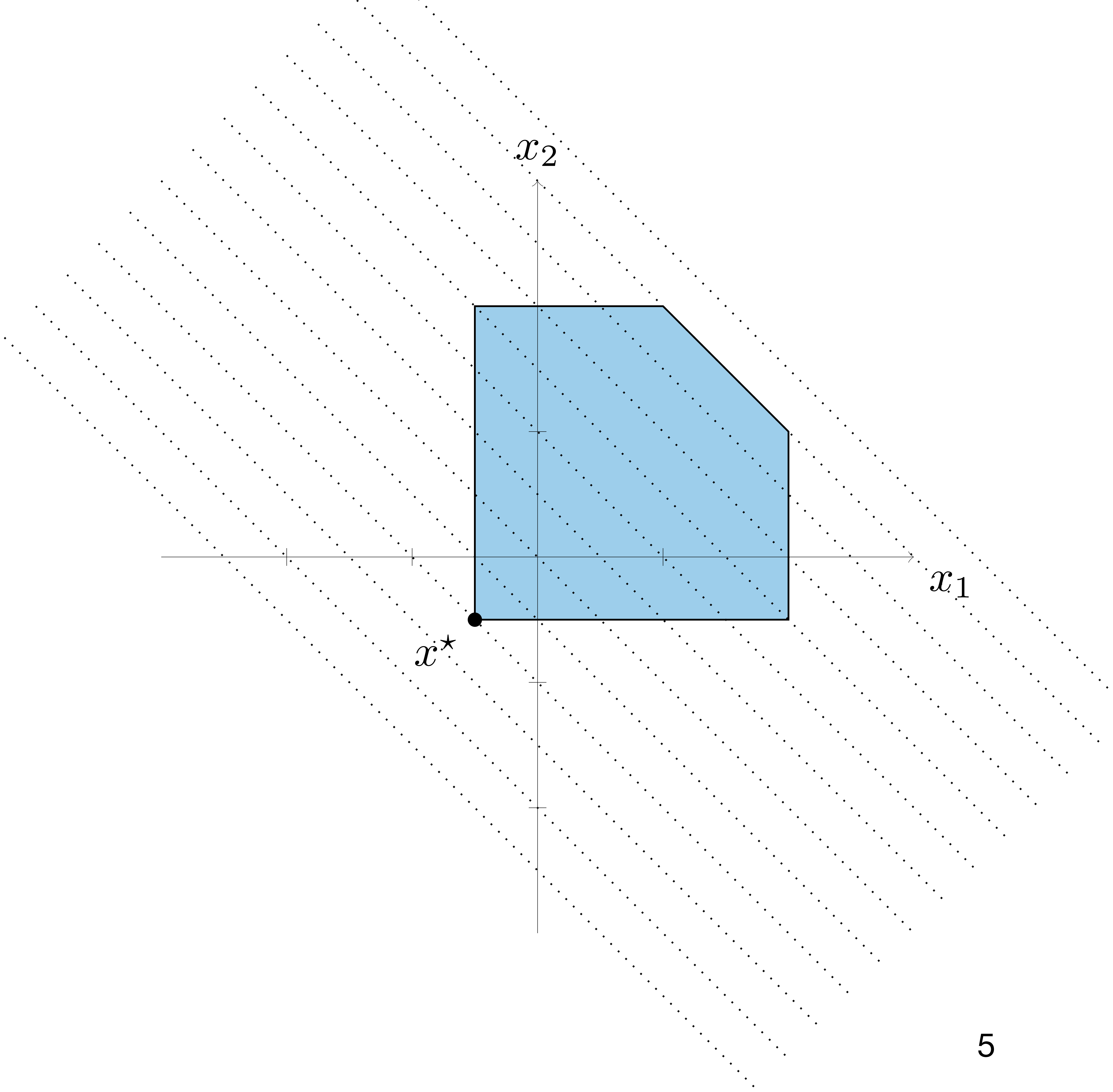
What kind of optimal solutions do we get?



A simple example

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & -1/2 \leq x_1 \leq 2 \\ & -1/2 \leq x_2 \leq 2 \\ & x_1 + x_2 \leq 2 \end{array}$$

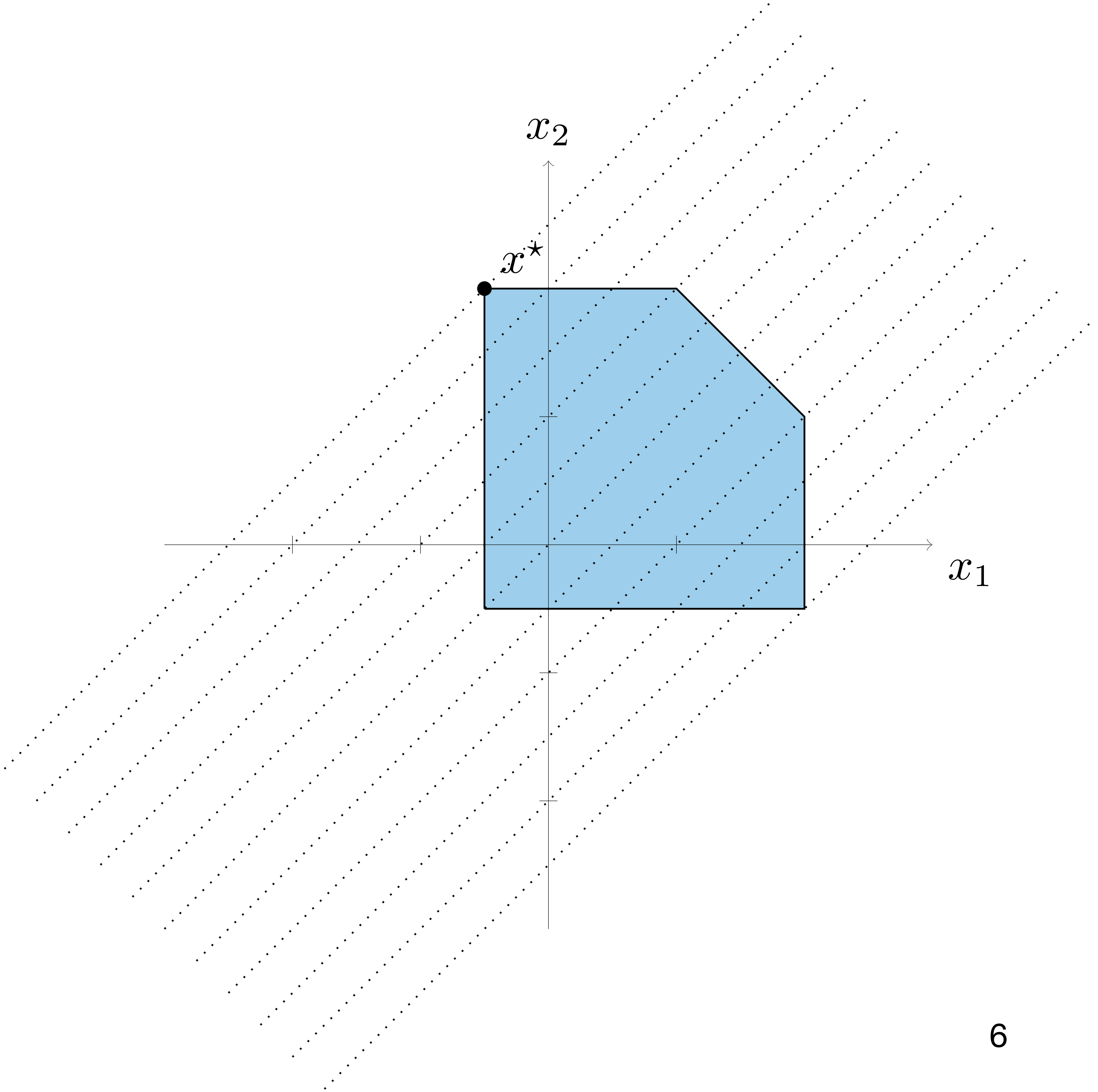
Suppose $c = (1, 1)$



A simple example

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & -1/2 \leq x_1 \leq 2 \\ & -1/2 \leq x_2 \leq 2 \\ & x_1 + x_2 \leq 2 \end{array}$$

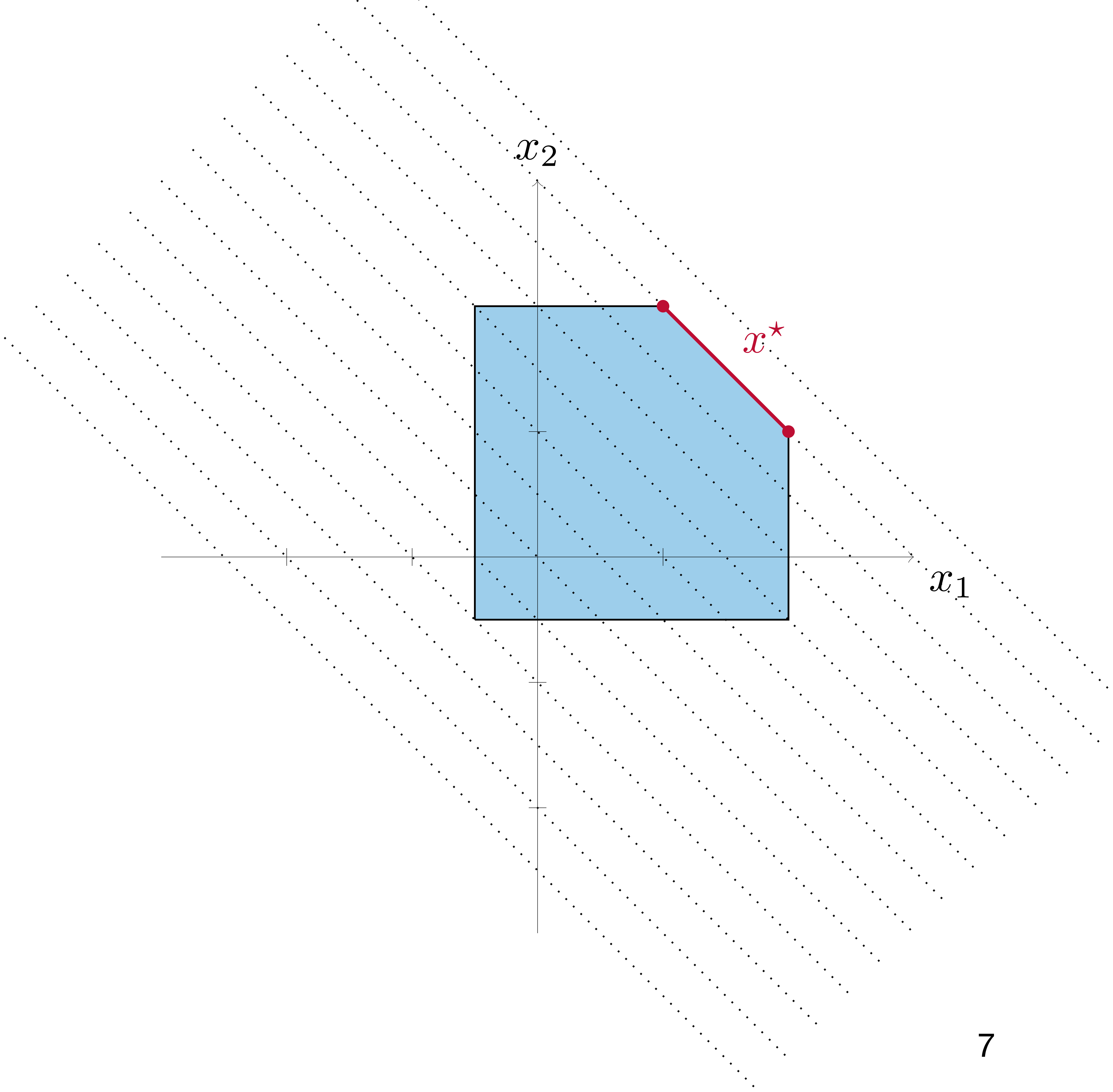
Suppose $c = (1, -1)$



A simple example

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & -1/2 \leq x_1 \leq 2 \\ & -1/2 \leq x_2 \leq 2 \\ & x_1 + x_2 \leq 2 \end{array}$$

Suppose $c = (-1, -1)$



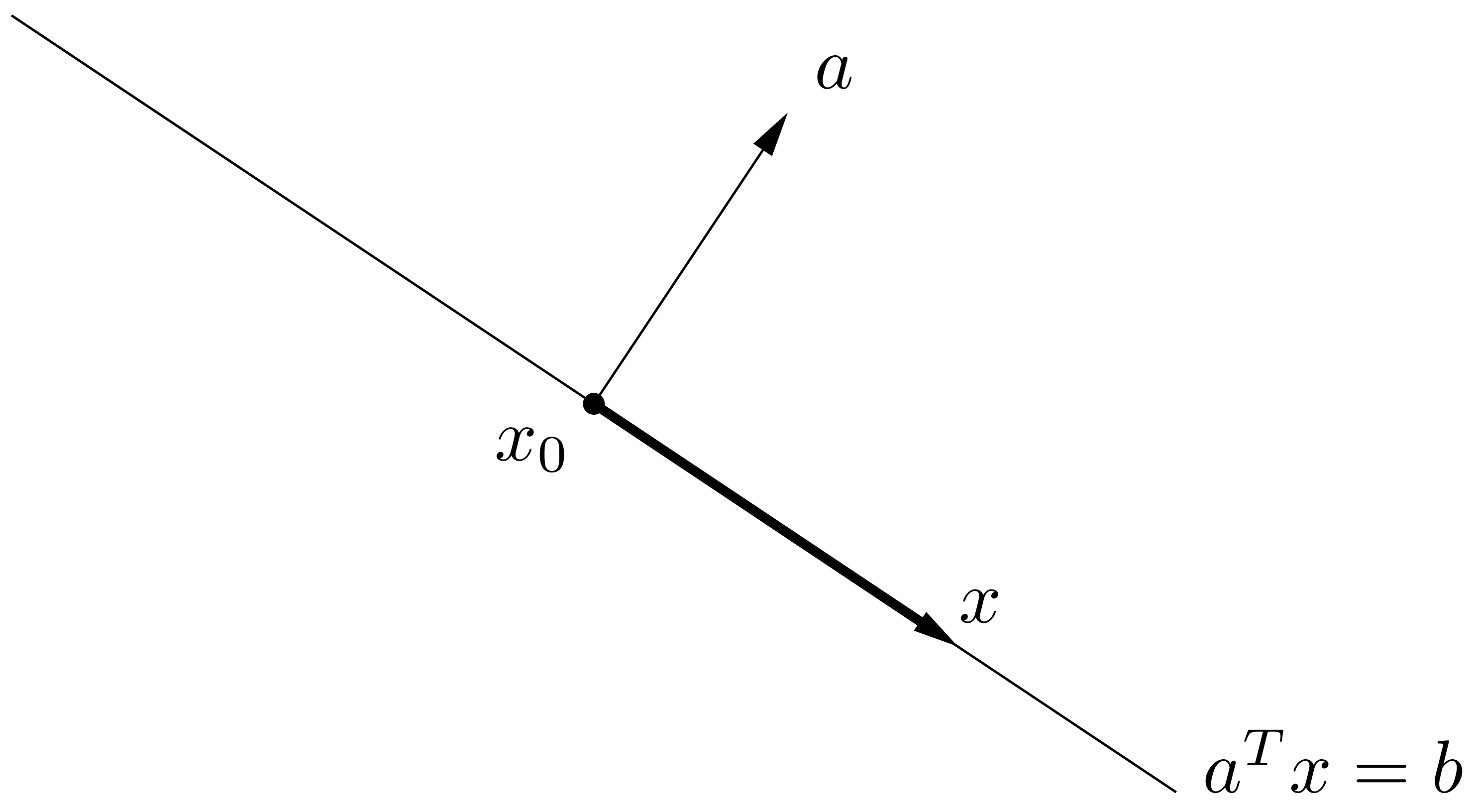
Polyhedra and linear algebra

Hyperplanes and halfspaces

Definitions

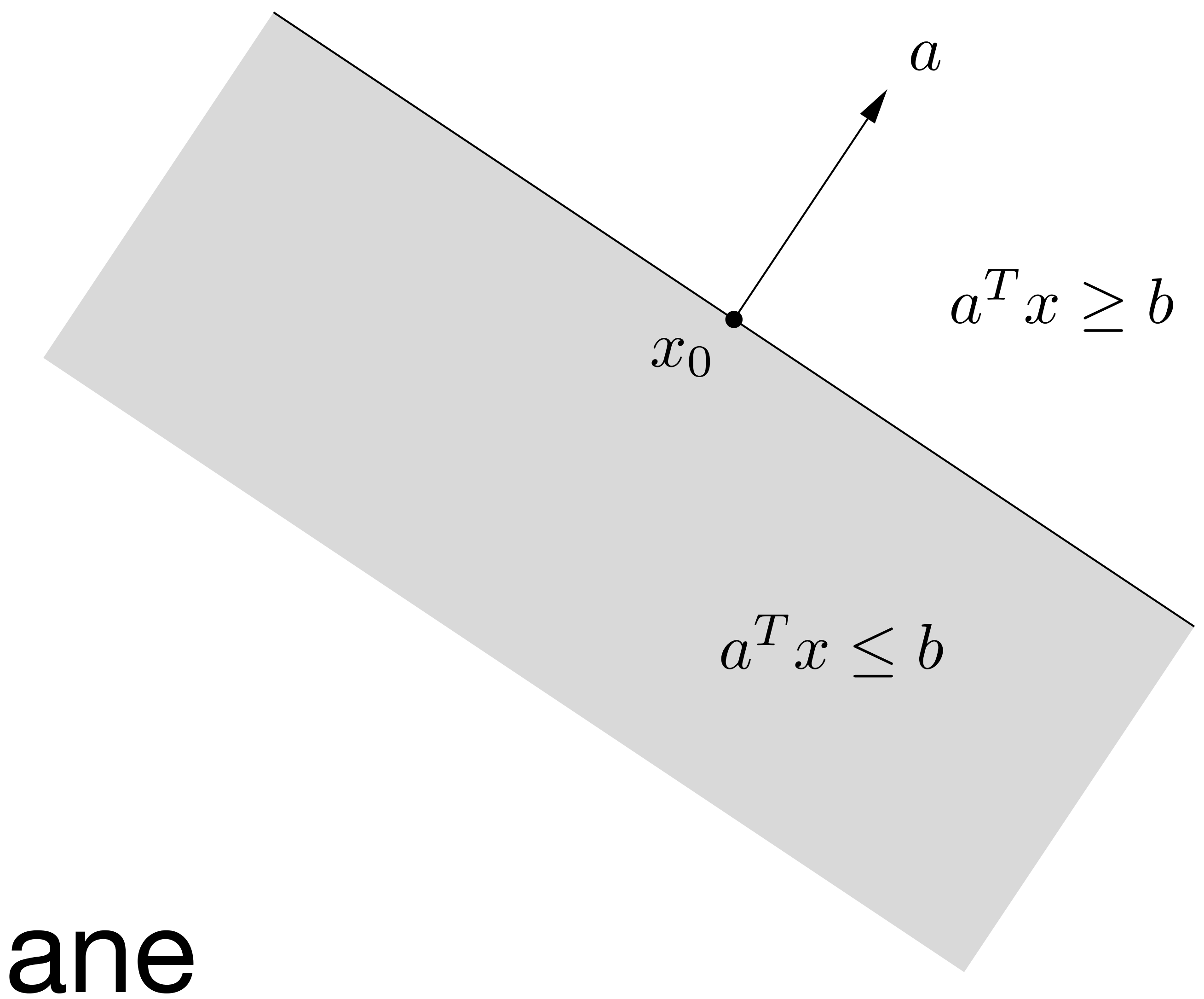
Hyperplane

$$\{x \mid a^T x = b\}$$



Halfspace

$$\{x \mid a^T x \leq b\}$$

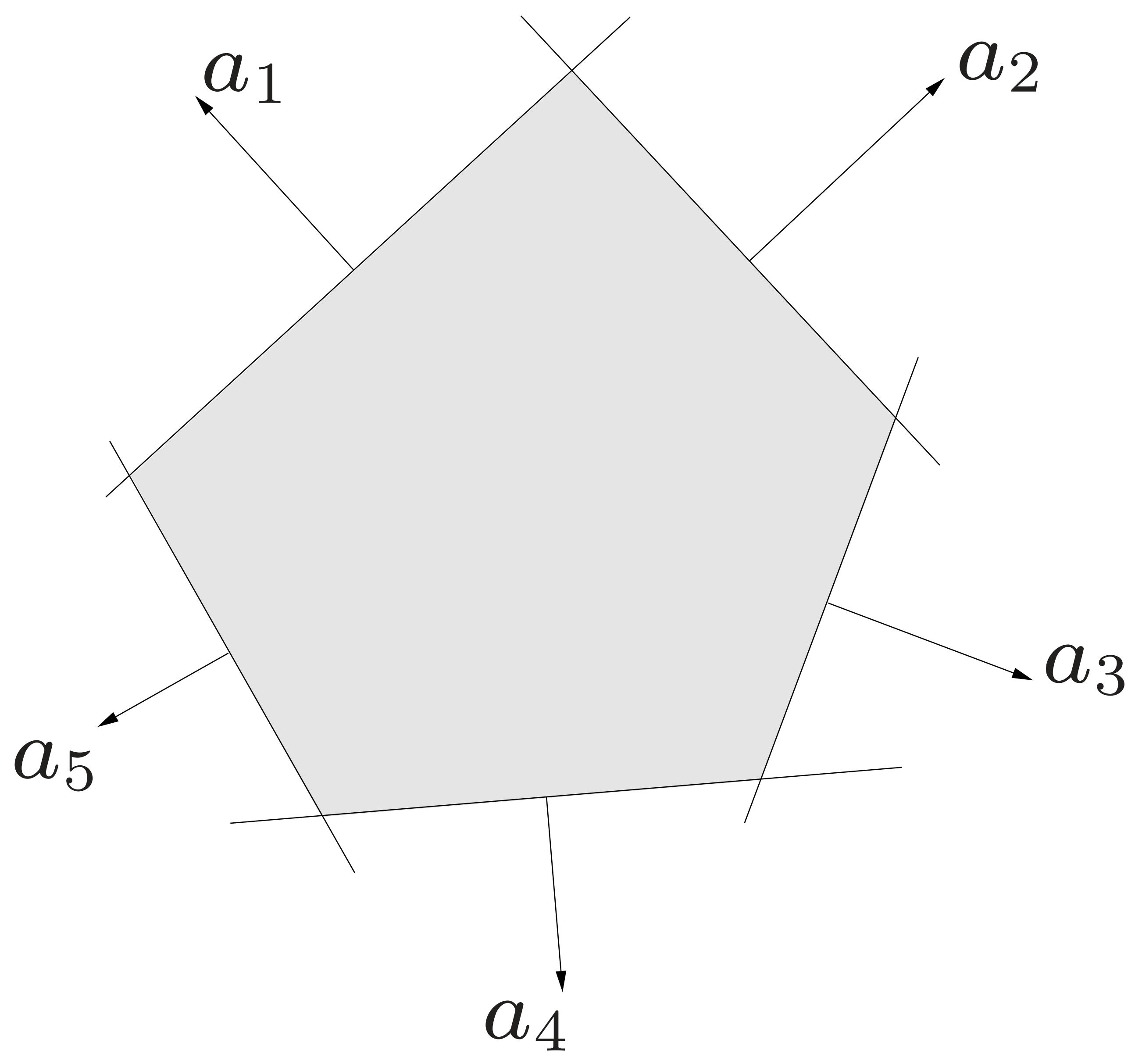


- x_0 is a specific point in the hyperplane
- For any x in the hyperplane defined by $a^T x = b$, $x - x_0 \perp a$
- The halfspace determined by $a^T x \leq b$ extends in the direction of $-a$

Polyhedron

Definition

$$P = \{x \mid a_i^T x \leq b_i, \quad i = 1, \ldots, m\} = \{x \mid Ax \leq b\}$$



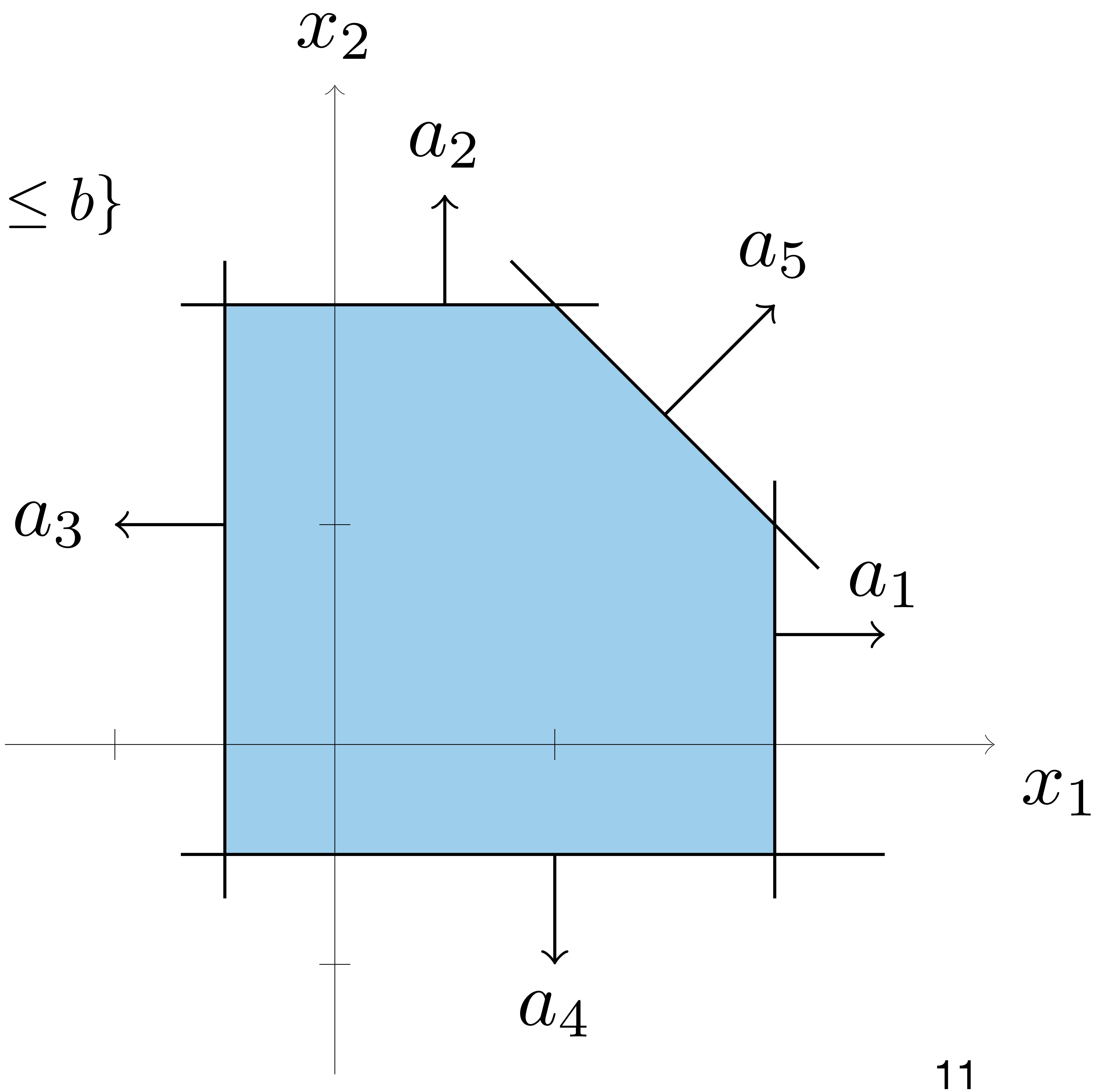
- Intersection of finite number of halfspaces
- Can include equalities

Polyhedron

Example

$$P = \{x \mid a_i^T x \le b_i, \quad i = 1, \dots, m\} = \{x \mid Ax \le b\}$$

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x_1 \le 2 \\ & x_2 \le 2 \\ & x_1 \ge -1/2 \\ & x_2 \ge -1/2 \\ & x_1 + x_2 \le 2 \end{array}$$

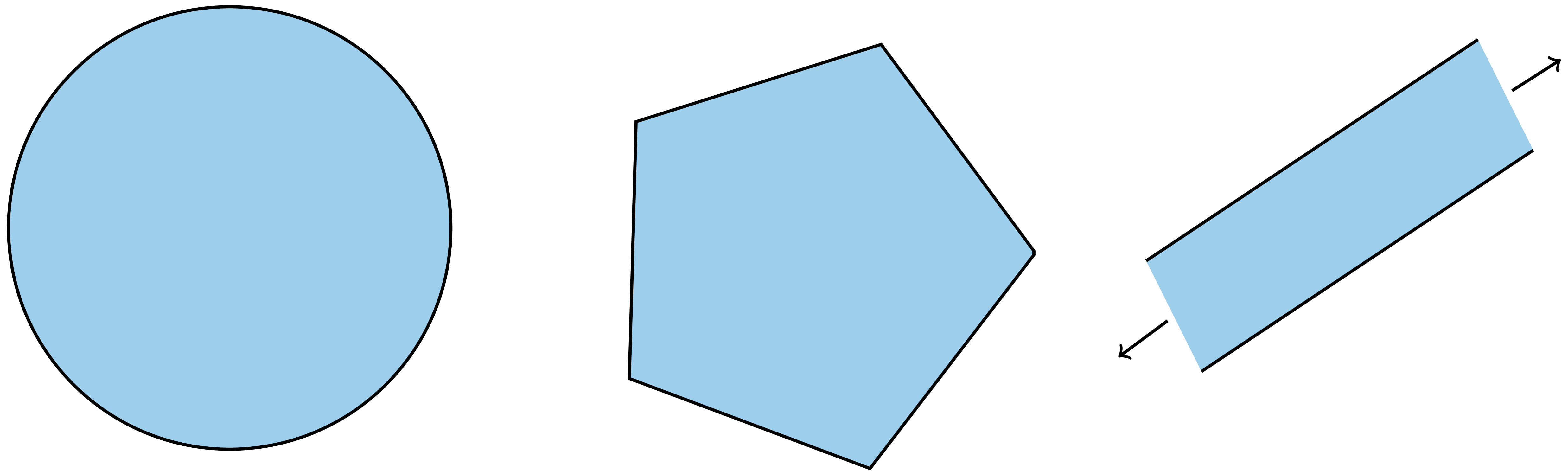


Convex set

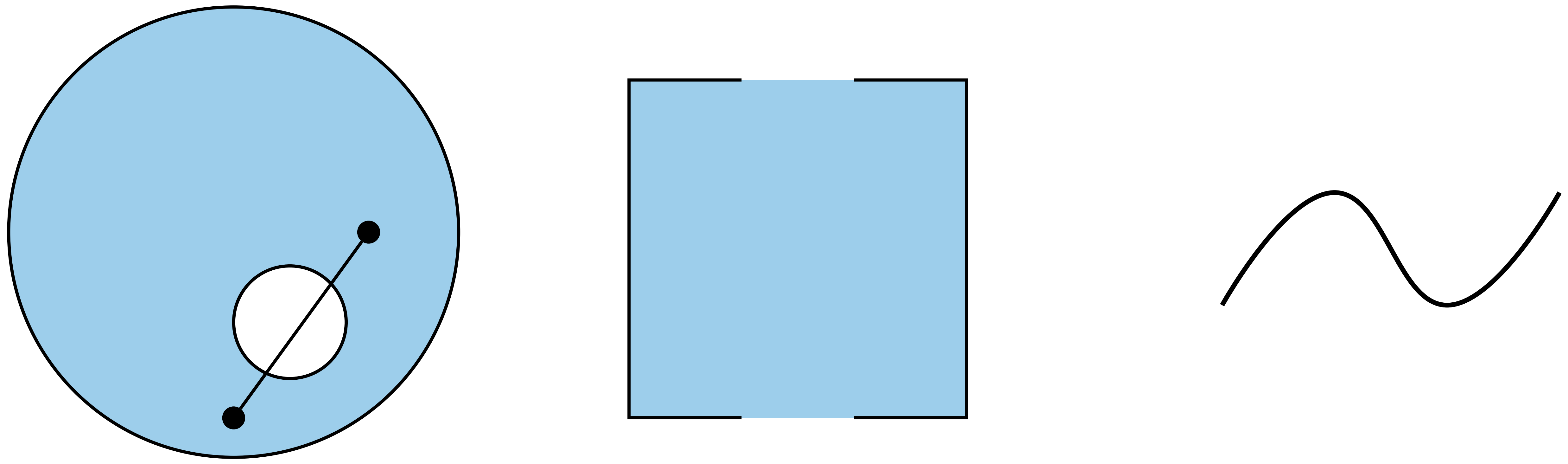
Definition

For any $x, y \in C$ and any $\alpha \in [0, 1]$ $\qquad \alpha x + (1 - \alpha) y \in C$

Convex



Nonconvex



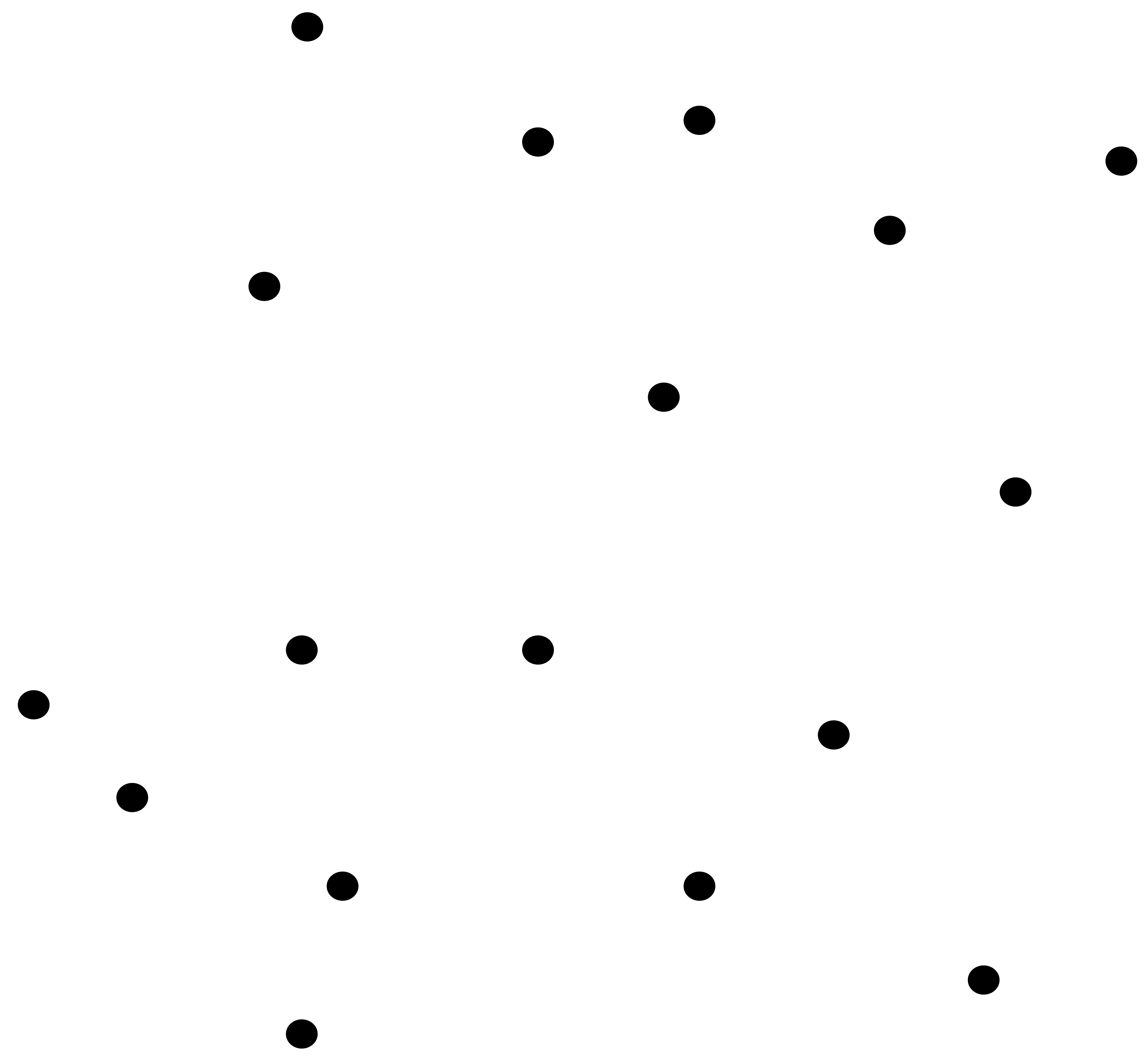
Examples

- $\mathbf{R}^n$
- Hyperplanes
- Halfspaces
- Polyhedra

Convex combinations

Ingredients :

- A collection of points $C = \{x_1, \ldots, x_k\}$
- A collection of non-negative weights α_i
- The weights α_i sum to 1

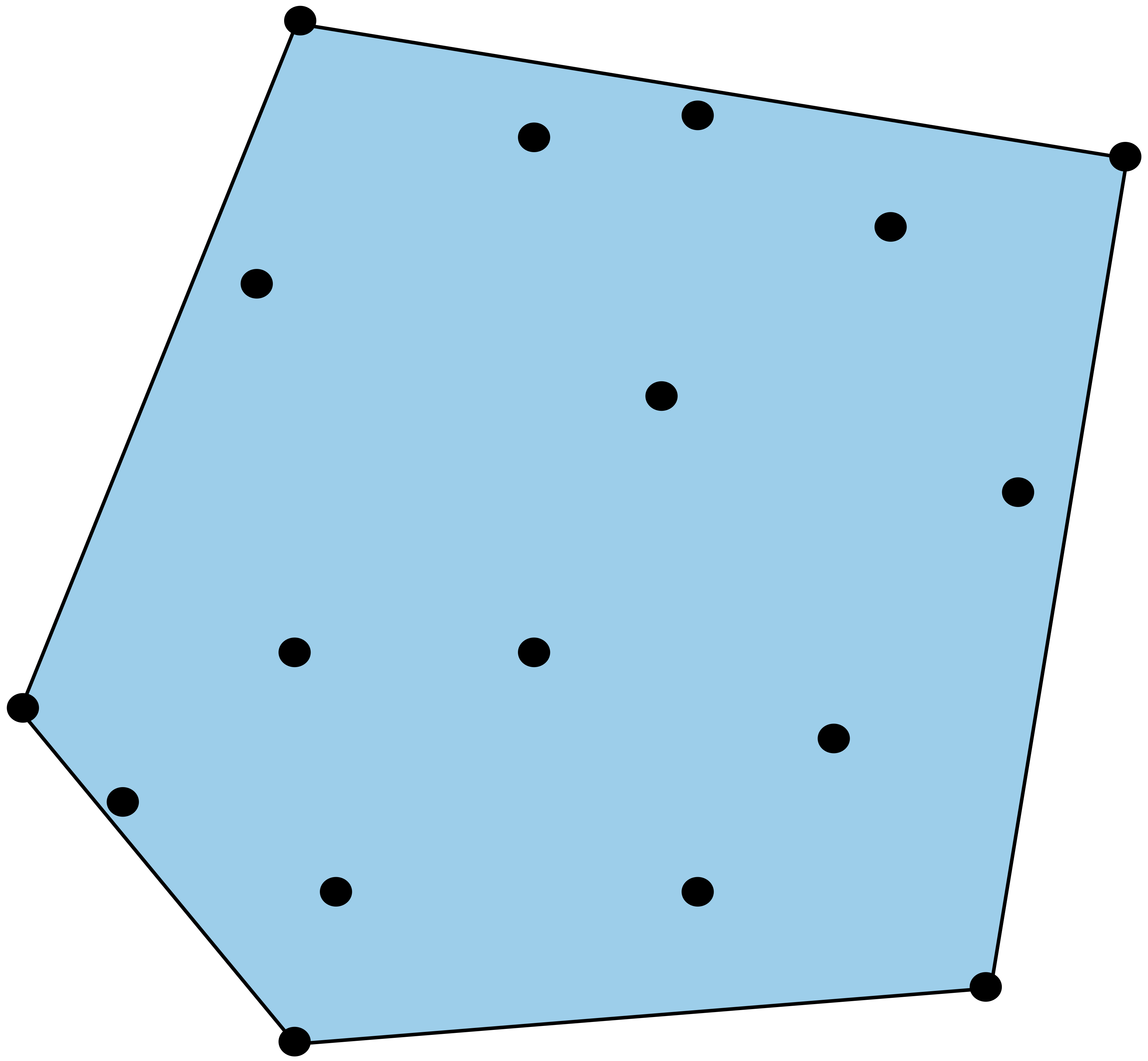


The vector $v = \alpha_1 x_1 + \cdots + \alpha_k x_k$ is a **convex combination** of the points.

Convex hull

The **convex hull** is the set of all possible convex combinations of the points.

$$\mathbf{conv}\, C = \left\{ \sum_{i=1}^{n} \alpha_i x_i \mid \alpha_i \geq 0,\ i = 1, \ldots, n,\ \mathbf{1}^T \alpha = 1 \right\}$$

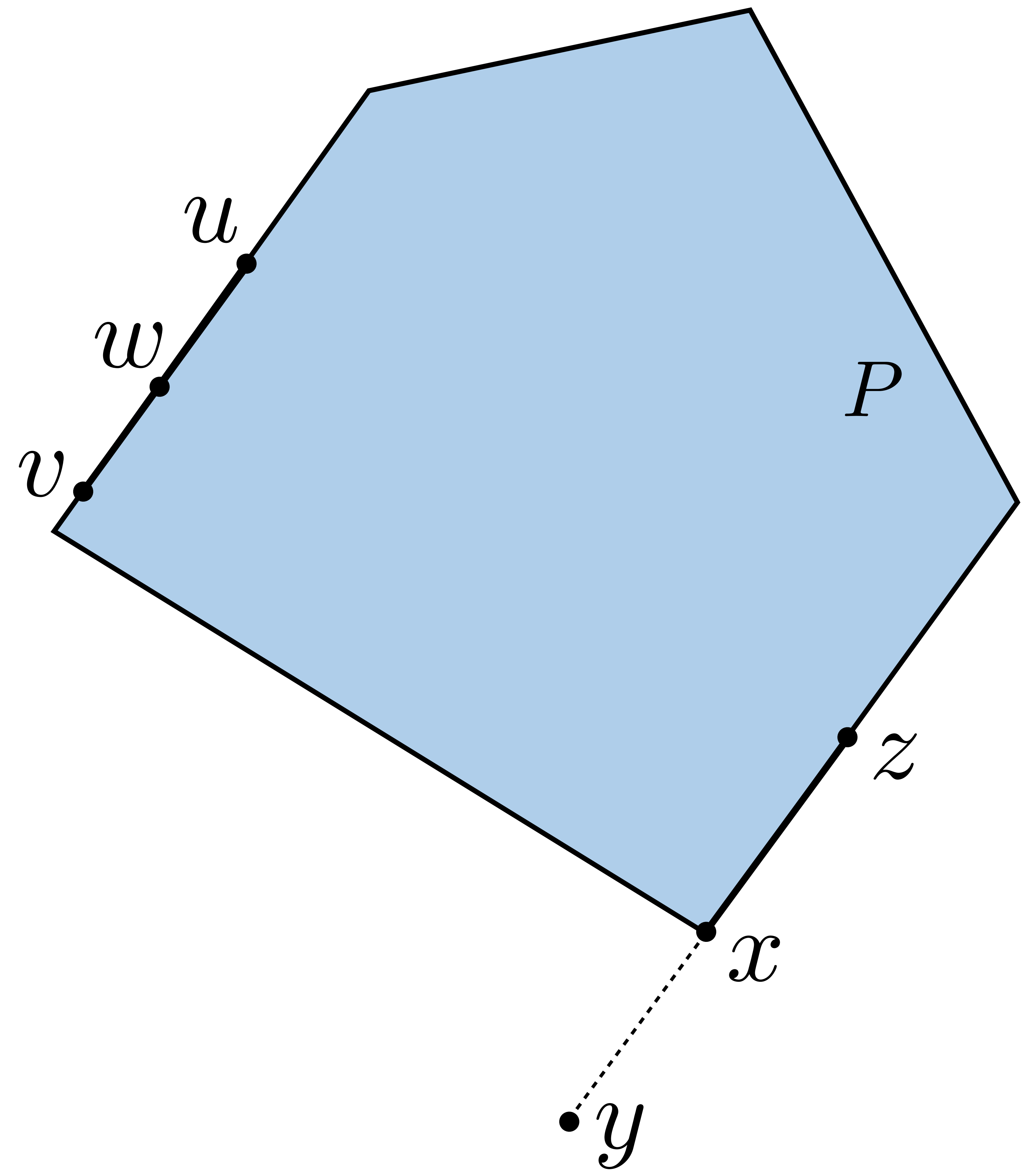


Corners

Extreme points

Definition:

An **extreme point** of a set is one not on a straight line between any other points in the set.

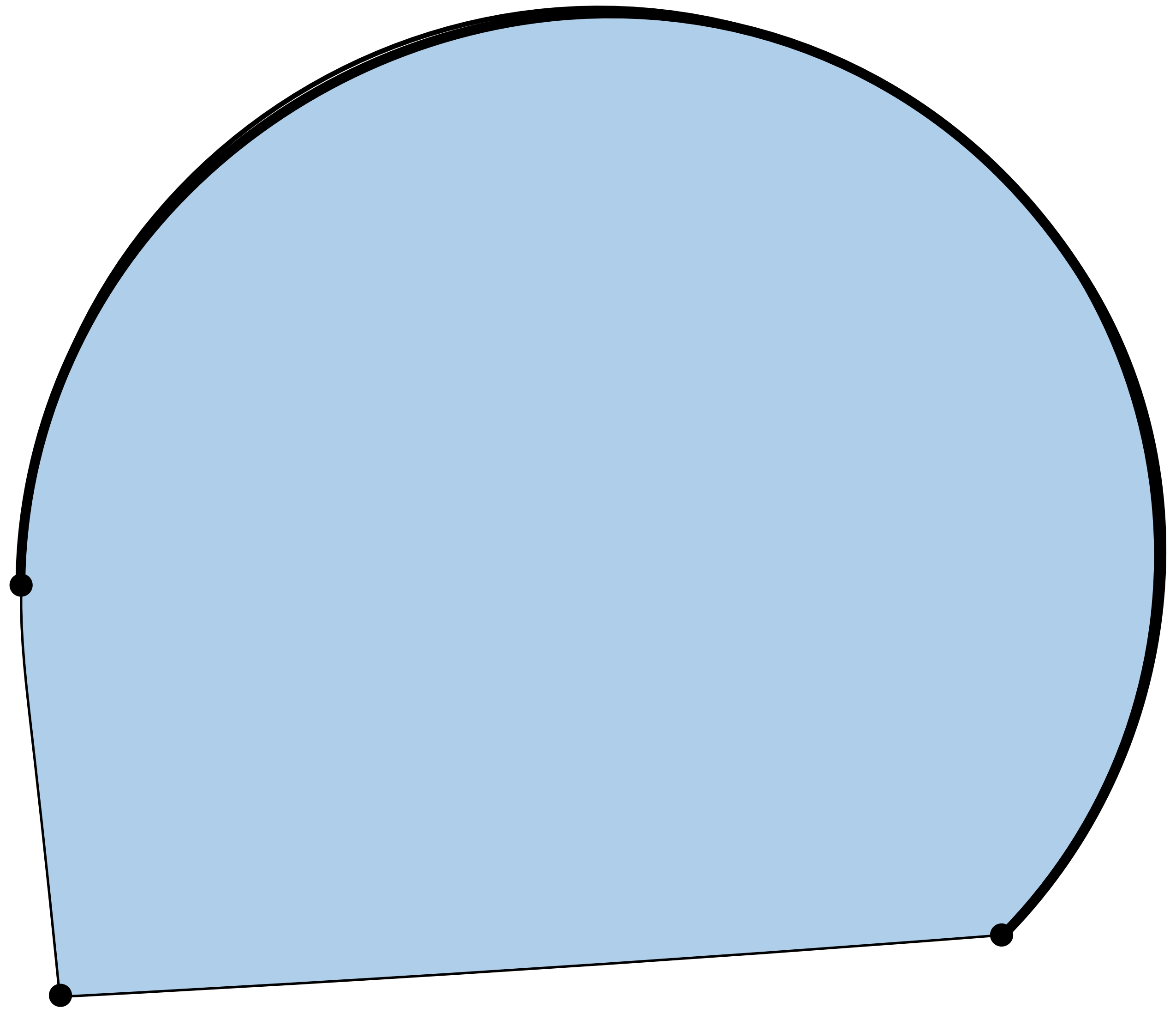


More formal definition:

The point $x \in P$ is an **extreme point** of P if

$\nexists y, z \in P$ $(y \neq x, z \neq x)$ and $\alpha \in [0,1]$ such that $x = \alpha y + (1-\alpha) z$

Extreme points

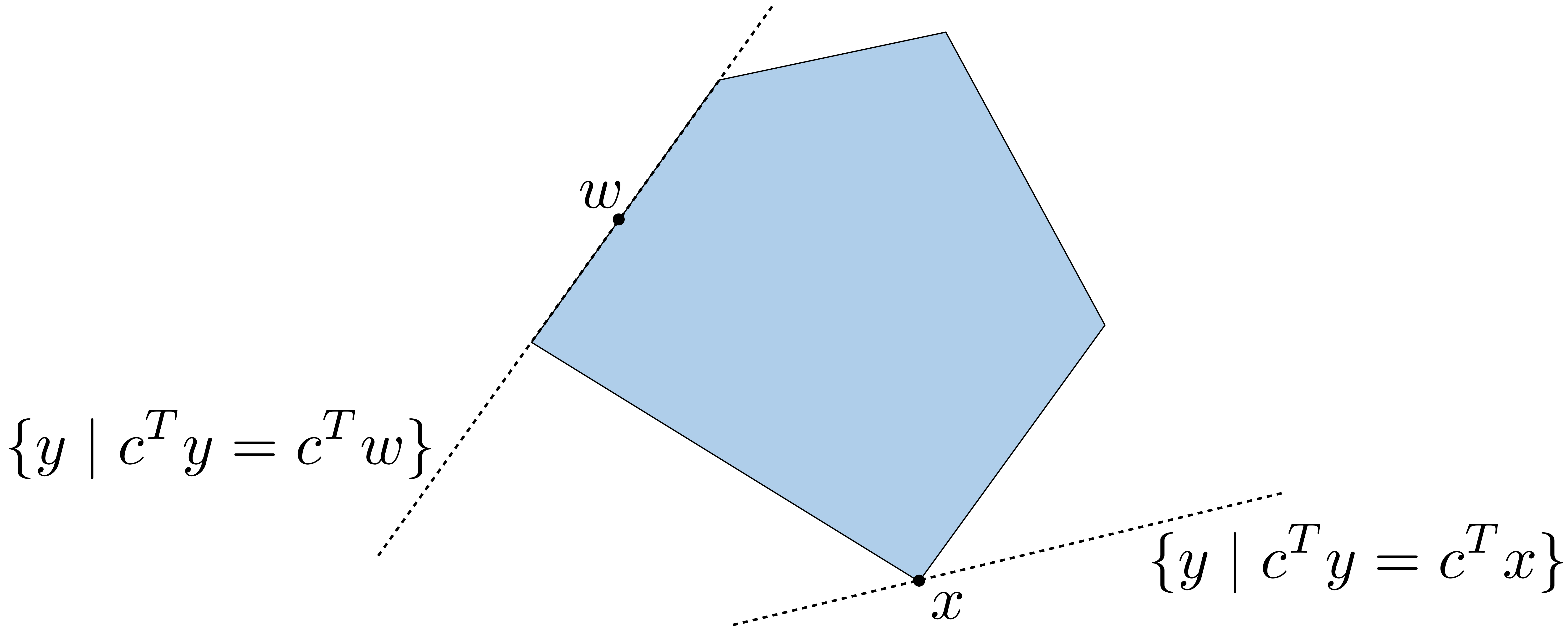


- General convex sets can have an infinite number of extreme points
- **Polyhedra** are convex sets with a finite number of extreme points

Vertices

The point $x \in P$ is a **vertex** if $\exists c$ such that x is the unique optimum of

$$\begin{array}{ll} \text{minimize} & c^T y \\ \text{subject to} & y \in P \end{array}$$

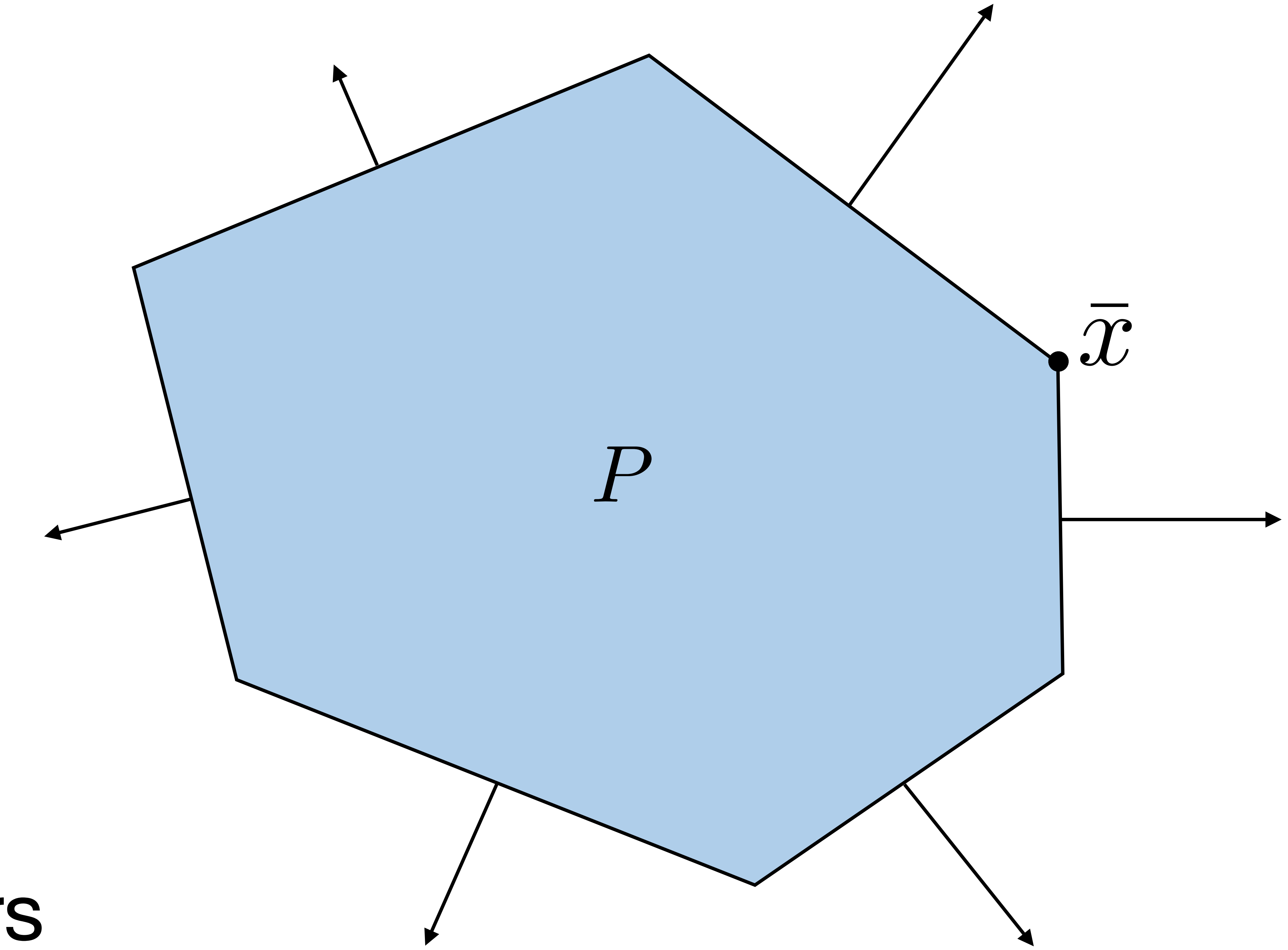


Basic feasible solution

Assume we have a polytope $P = \{x \mid a_i^T x \leq b_i, \quad i = 1, \ldots, m\}$

Active constraints at $\bar{x}$

$$\mathcal{I}(\bar{x}) = \{i \in \{1, \ldots, m\} \mid a_i^T \bar{x} = b_i\}$$



Basic feasible solution $\bar{x} \in P$

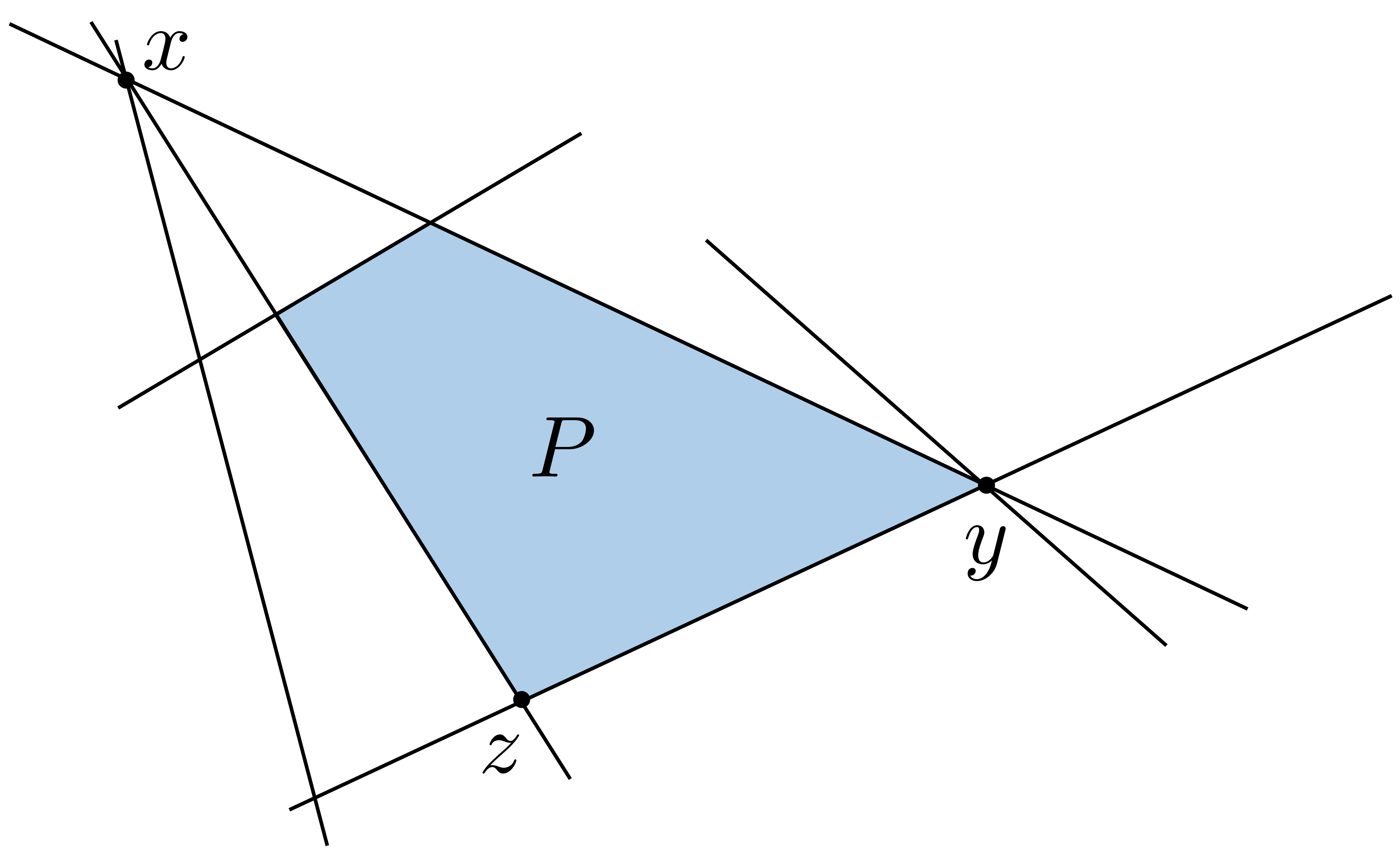
$\{a_i \mid i \in \mathcal{I}(\bar{x})\}$ has n linearly independent vectors

Degenerate basic feasible solutions

A solution $\bar{x}$ is **degenerate** if $|\mathcal{I}(\bar{x})| > n$

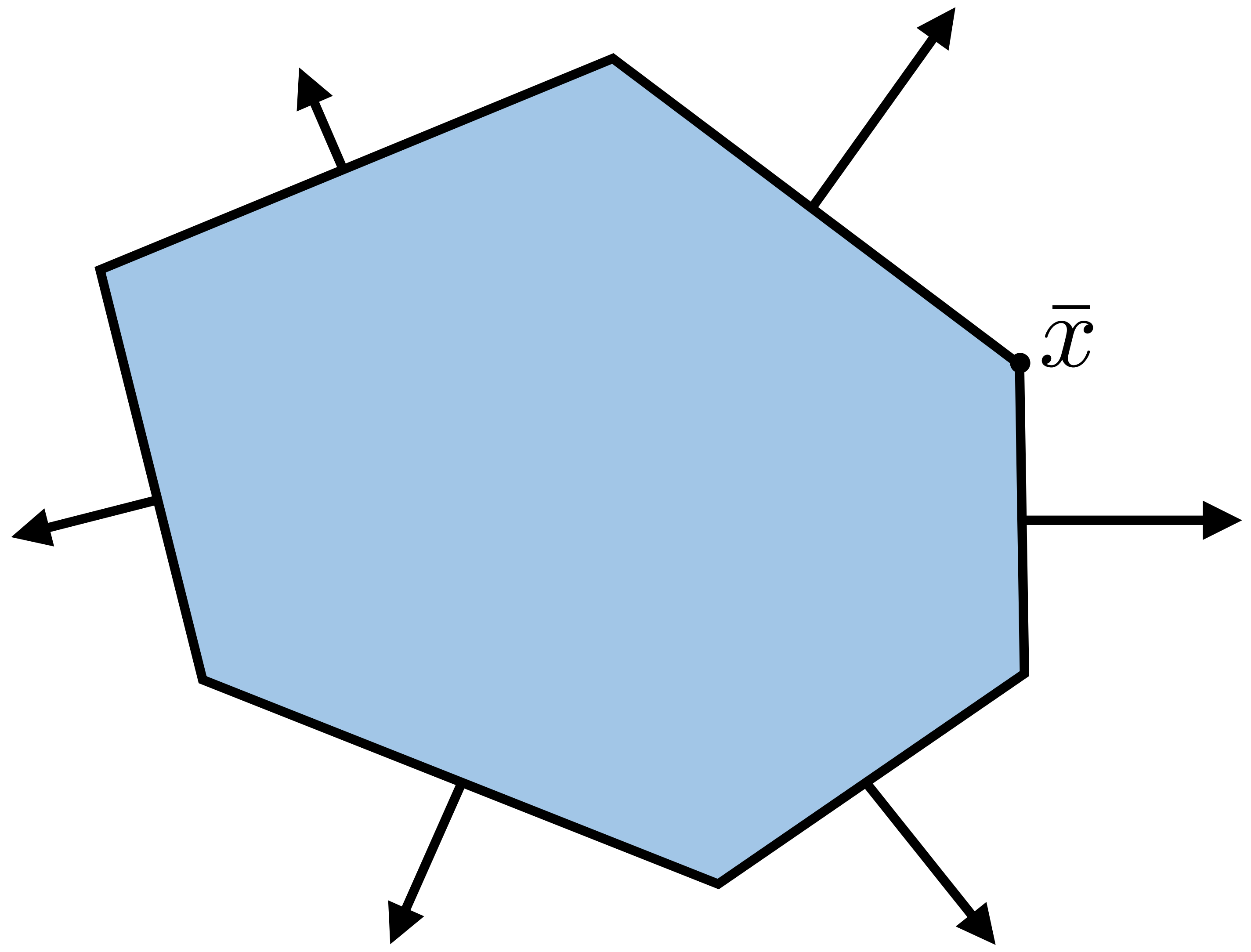
True or False?

	Basic	Feasible	Degenerate
x			
y			
z			



An Equivalence Theorem

Given a nonempty polyhedron $P = \{x \mid Ax \leq b\}$



x is a **vertex** $\iff$ x is an **extreme point** $\iff$ x is a **basic feasible solution**

Equivalent theorem proof

Vertex —> Extreme point

If x is a vertex, $\exists c$ such that $c^T x < c^T y, \quad \forall y \in P, y \neq x$

Let's assume x is not an extreme point:

$\exists y, z \neq x$ such that $x = \lambda y + (1 - \lambda) z$

Since x is a vertex, $c^T x < c^T y$ and $c^T x < c^T z$

Therefore, $c^T x = \lambda c^T y + (1 - \lambda) c^T z > \lambda c^T x + (1 - \lambda) c^T x = c^T x$

$\implies$ **contradiction** ■

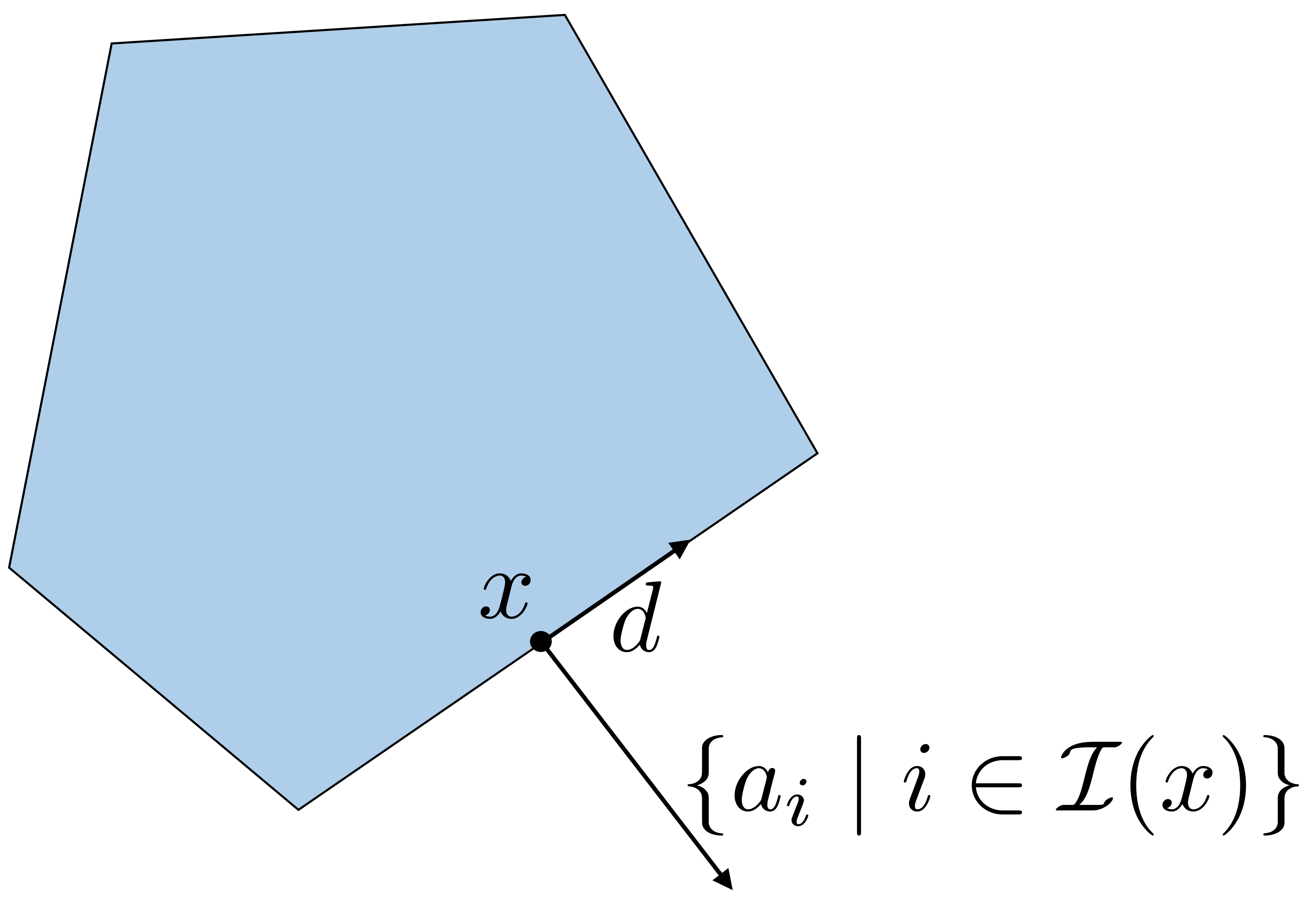
Equivalent theorem proof

Extreme point —> Basic feasible solution (proof by **contraposition**)

Suppose $x \in P$ is **not basic feasible solution**

$\{a_i \mid i \in \mathcal{I}(x)\}$ does not span $\mathbf{R}^n$

$\exists d \in \mathbf{R}^n$ perpendicular to all of them: $a_i^T d = 0, \quad \forall i \in \mathcal{I}(x)$



Equivalent theorem proof

Extreme point —> Basic feasible solution (proof by **contraposition**)

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$\exists d \in \mathbf{R}^n$ perpendicular to all of them: $a_i^T d = 0, \quad \forall i \in \mathcal{I}(x)$

Let $\epsilon > 0$ and define $y = x + \epsilon d$ and $z = x - \epsilon d$

For $i \in \mathcal{I}(x)$ we have $a_i^T y = b_i$ and $a_i^T z = b_i$

For $i \notin \mathcal{I}(x)$ we have $a_i^T x < b_i \quad \Rightarrow \quad a_i^T(x + \epsilon d) < b_i$ and $a_i^T(x - \epsilon d) < b_i$

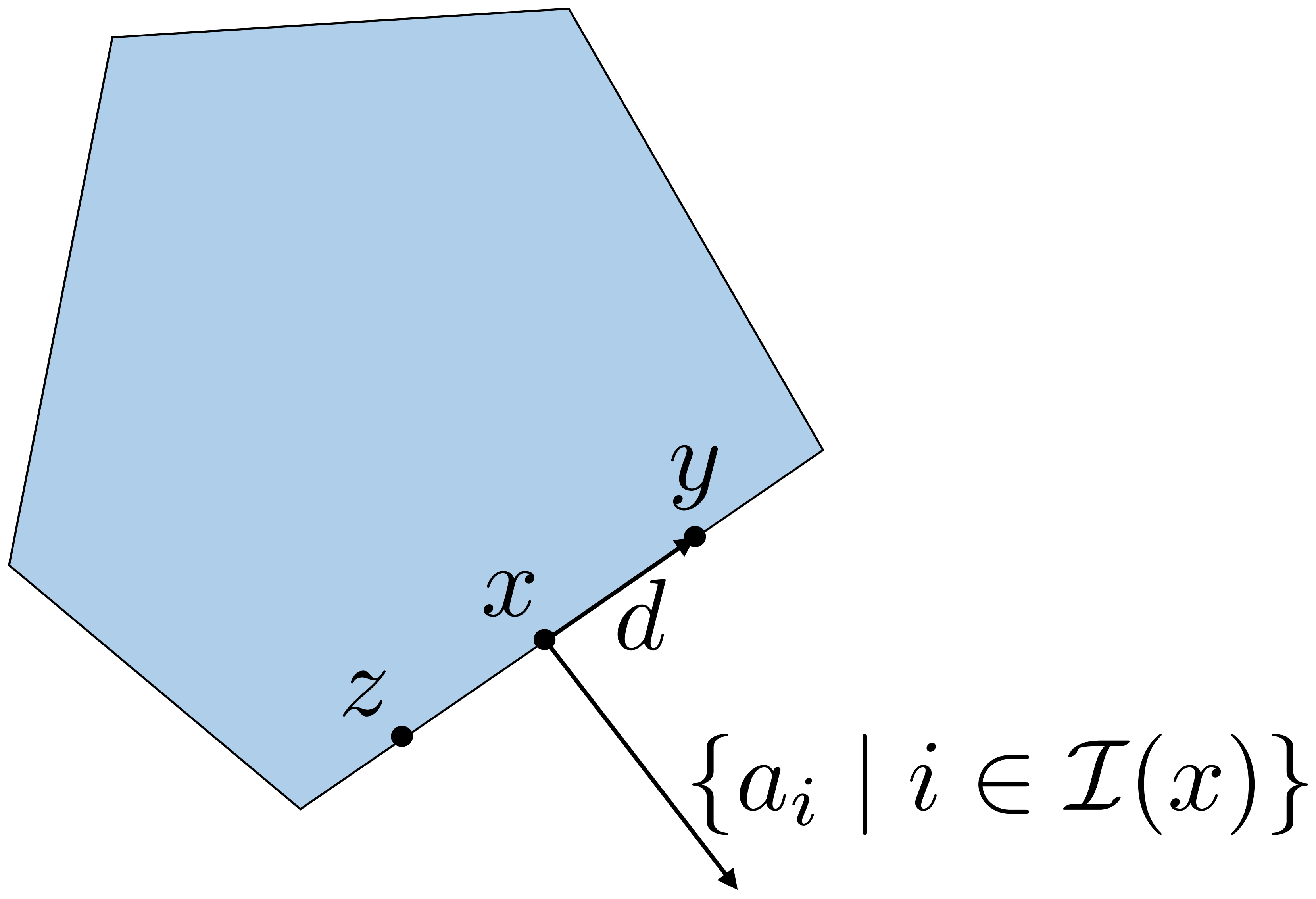
Hence, $y, z \in P$ and $x = \lambda y + (1 - \lambda) z$ with $\lambda = 0.5$.

$\Longrightarrow x$ is **not an extreme point** ∎

Equivalent theorem proof

Extreme point —> Basic feasible solution (proof by **contraposition**)

Suppose $x \in P$ is **not basic feasible solution**



Hence, $y, z \in P$ and $x = \lambda y + (1 - \lambda) z$ with $\lambda = 0.5$.

$\Longrightarrow x$ is **not an extreme point** ■

Equivalence theorem proof

Basic feasible solution —> Vertex

Left as exercise

Hint

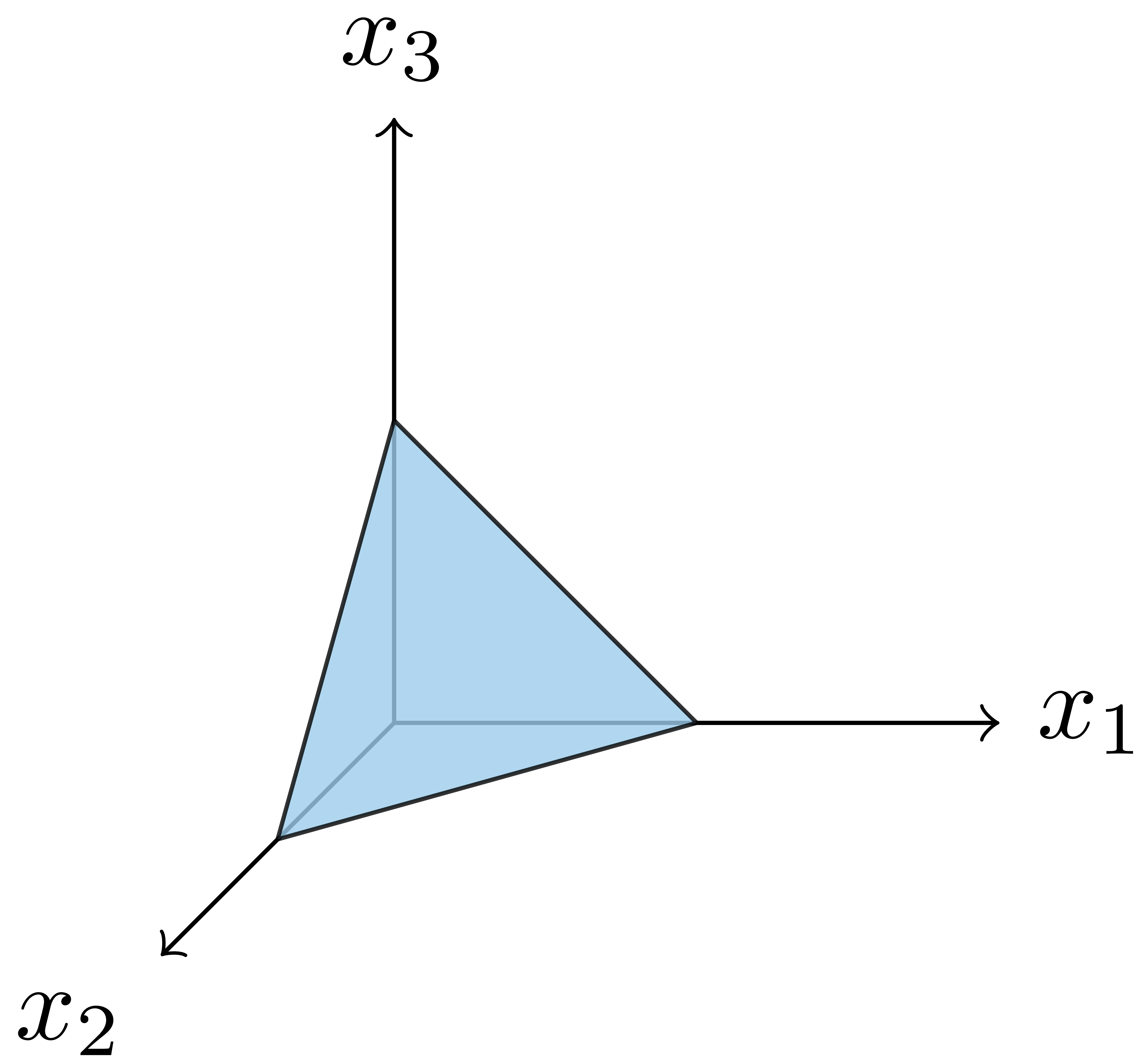
Define $c = \sum_{i \in \mathcal{I}(x)} a_i$

Constructing basic solutions

3D example

One equality $(m = 1, n = 3)$

$$
\begin{aligned}
\text{minimize} \quad & c^T x \\
\text{subject to} \quad & x_1 + x_2 + x_3 = 1 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$



Basic feasible solution $\bar{x}$ has n linearly independent active constraints.

↓

$n - m = 2$ inequalities have to be tight: $x_i = 0$

3D example

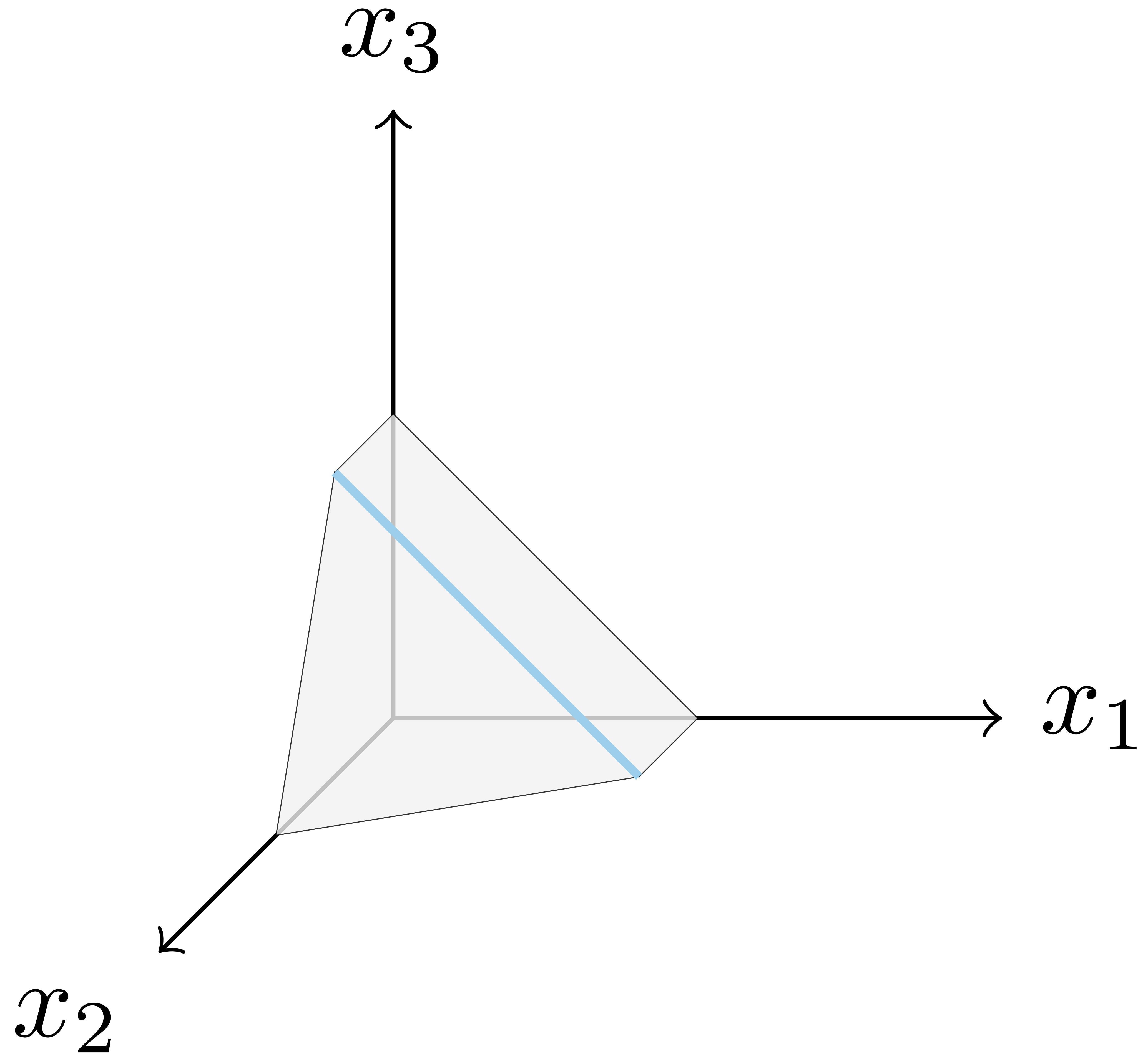
Two equalities $(m = 2, n = 3)$

minimize $c^T x$

subject to $x_1 + x_3 = 1$

$(1/2)x_1 + x_2 + (1/2)x_3 = 1$

$x_1, x_2, x_3 \geq 0$



Basic feasible solution $\bar{x}$ has n linearly independent active constraints.

↓

$n - m = 1$ inequalities have to be tight: $x_i = 0$

3D example

Three equalities $(m = 3, n = 3)$

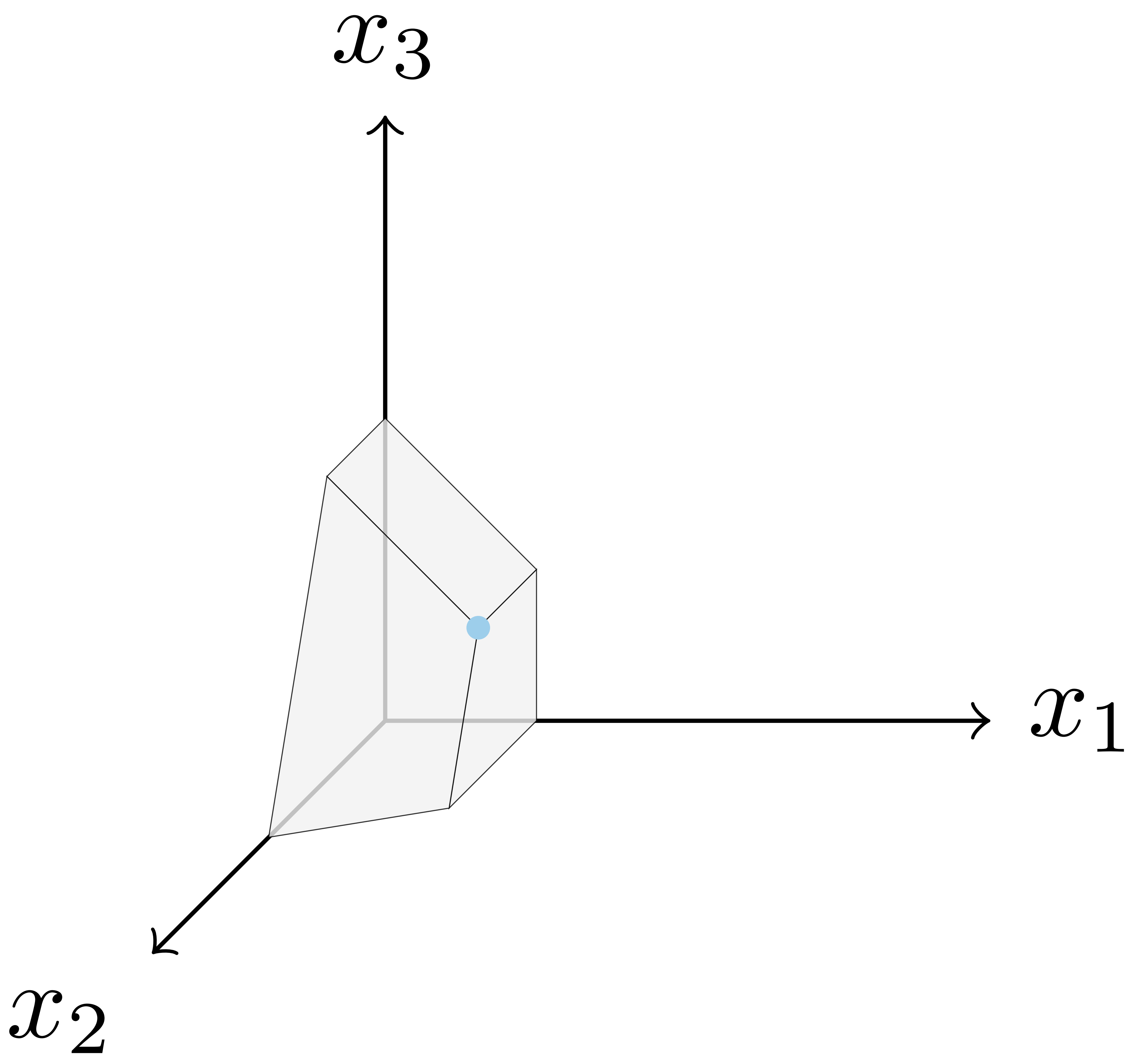
minimize $\quad c^T x$

subject to $\quad x_1 + x_3 = 1$

$(1/2)x_1 + x_2 + (1/2)x_3 = 1$

$2x_1 = 1$

$x_1, x_2, x_3 \geq 0$



Basic feasible solution $\bar{x}$ has n linearly independent active constraints.

↓

$n - m = 0$ inequalities have to be tight: $x_i = 0$

Standard form polyhedra

Standard form LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Assumption

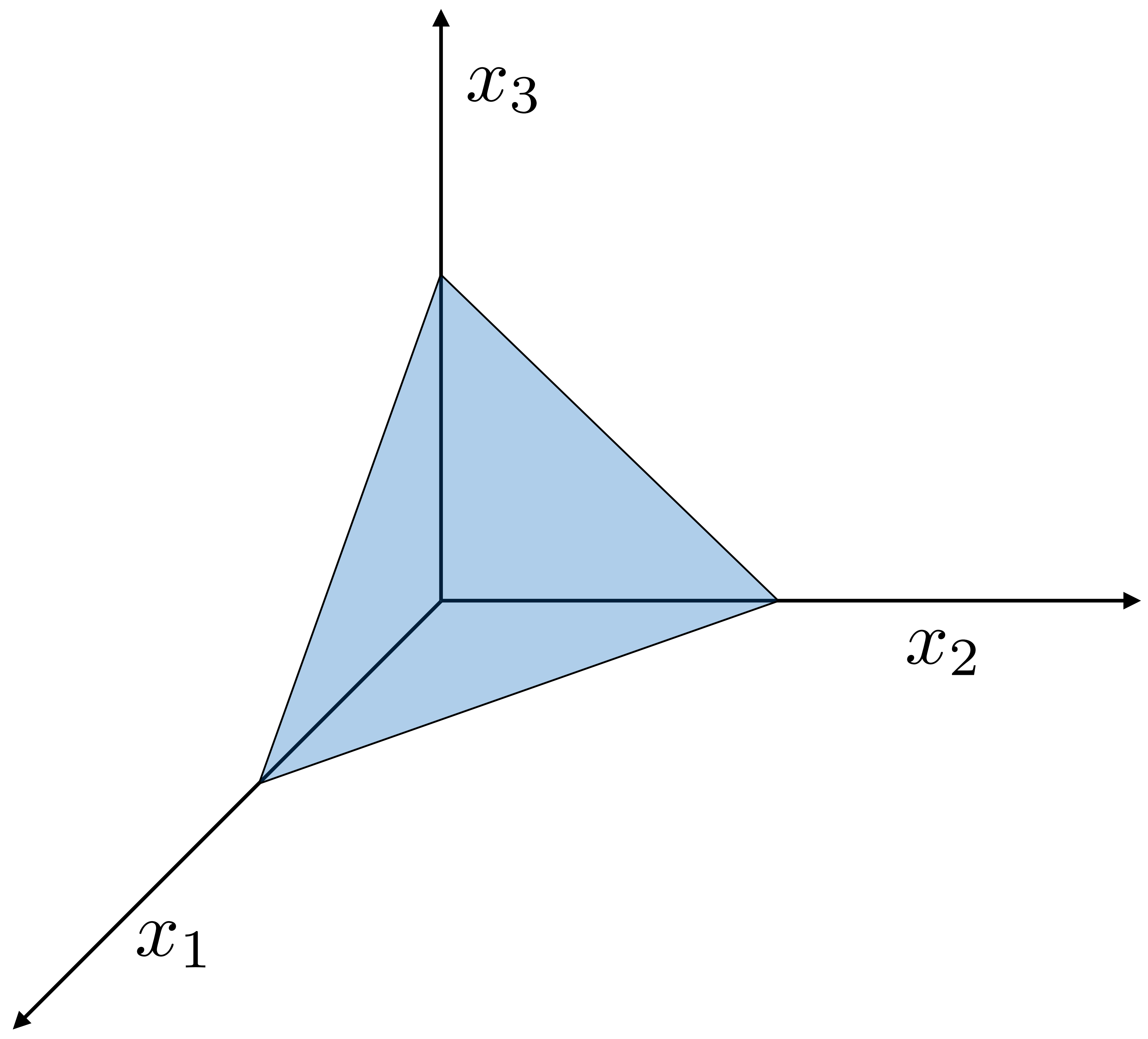
$A \in \mathbf{R}^{m \times n}$ has full row rank $m \leq n$

Interpretation

P is an $(n-m)$-dimensional surface

Standard form polyhedron

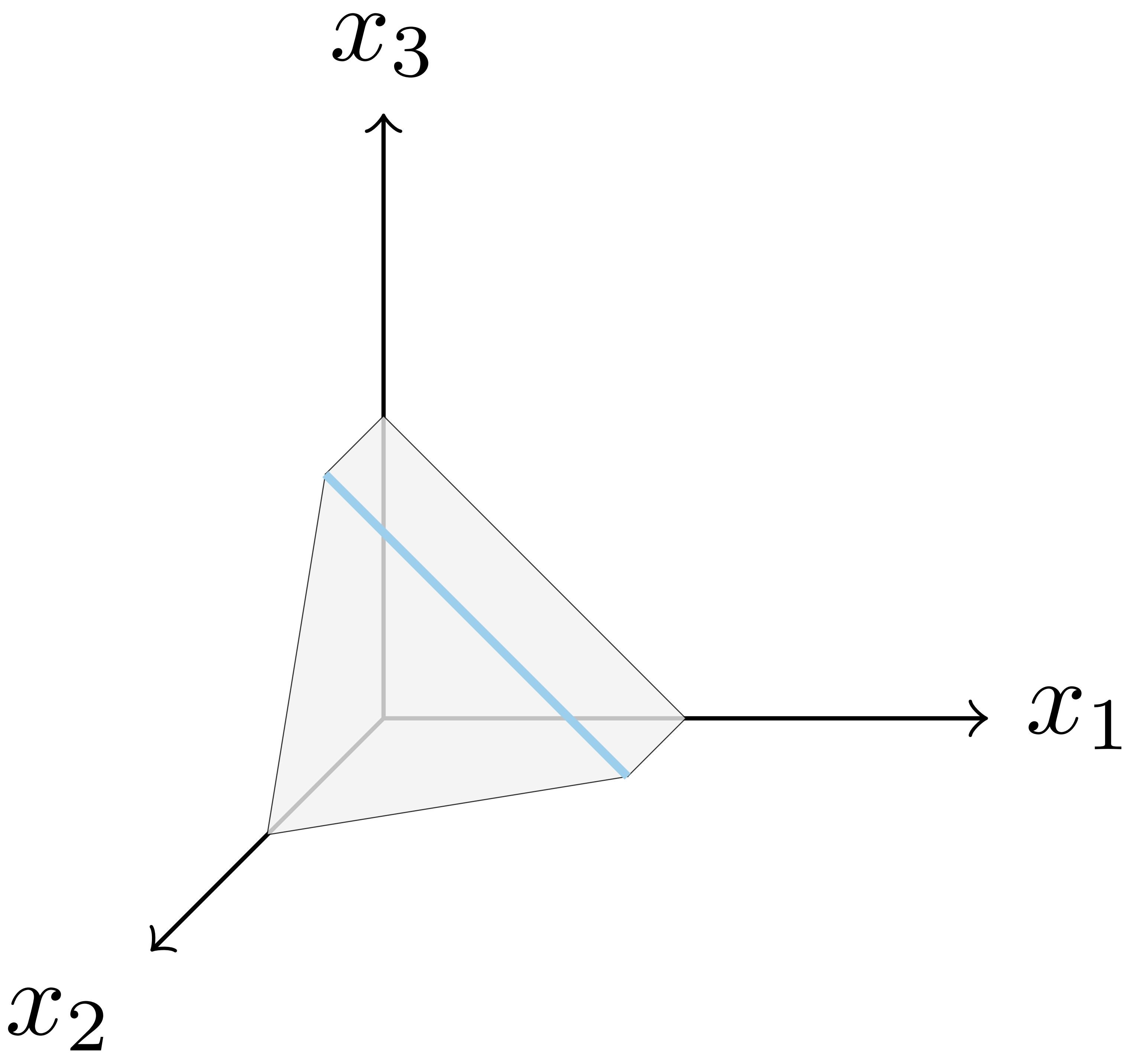
$$P = \{x \mid Ax = b,\ x \geq 0\}$$



$n = 3, m = 1$

Constructing a basic solution

Two equalities $(m = 2, n = 3)$



$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & x_1 + x_3 = 1 \\ & (1/2)x_1 + x_2 + (1/2)x_3 = 1 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

$n - m = 1$ inequalities have to be tight: $x_i = 0$

Set $x_1 = 0$ and solve

$$\begin{bmatrix} 1 & 0 & 1 \\ 1/2 & 1 & 1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 \\ 1 & 1/2 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \longrightarrow (x_2, x_3) = (0.5, 1)$$

Basic solutions

Standard form polyhedra

$P = \{x \mid Ax = b,\ x \geq 0\}$ with $A \in \mathbf{R}^{m \times n}$ has full row rank $m \leq n$

x is a **basic solution** if and only if

- $Ax = b$
- There exist indices $B(1), \dots, B(m)$ such that
  - columns $A_{B(1)}, \dots, A_{B(m)}$ are linearly independent
  - $x_i = 0$ for $i \neq B(1), \dots, B(m)$

x is a **basic feasible solution** if x is a **basic solution** and $x \geq 0$

Constructing basic solution

1. Choose any m independent columns of A: $A_{B(1)}, \ldots, A_{B(m)}$
2. Let $x_i = 0$ for all $i \neq B(1), \ldots, B(m)$
3. Solve $Ax = b$ for the remaining $x_{B(1)}, \ldots, x_{B(m)}$

Basis matrix

Basis columns

Basic variables

$$A_B = \begin{bmatrix} | & | & & | \\ A_{B(1)} & A_{B(2)} & \ldots & A_{B(m)} \\ | & | & & | \end{bmatrix}, \quad x_B = \begin{bmatrix} x_{B(1)} \\ \vdots \\ x_{B(m)} \end{bmatrix}$$

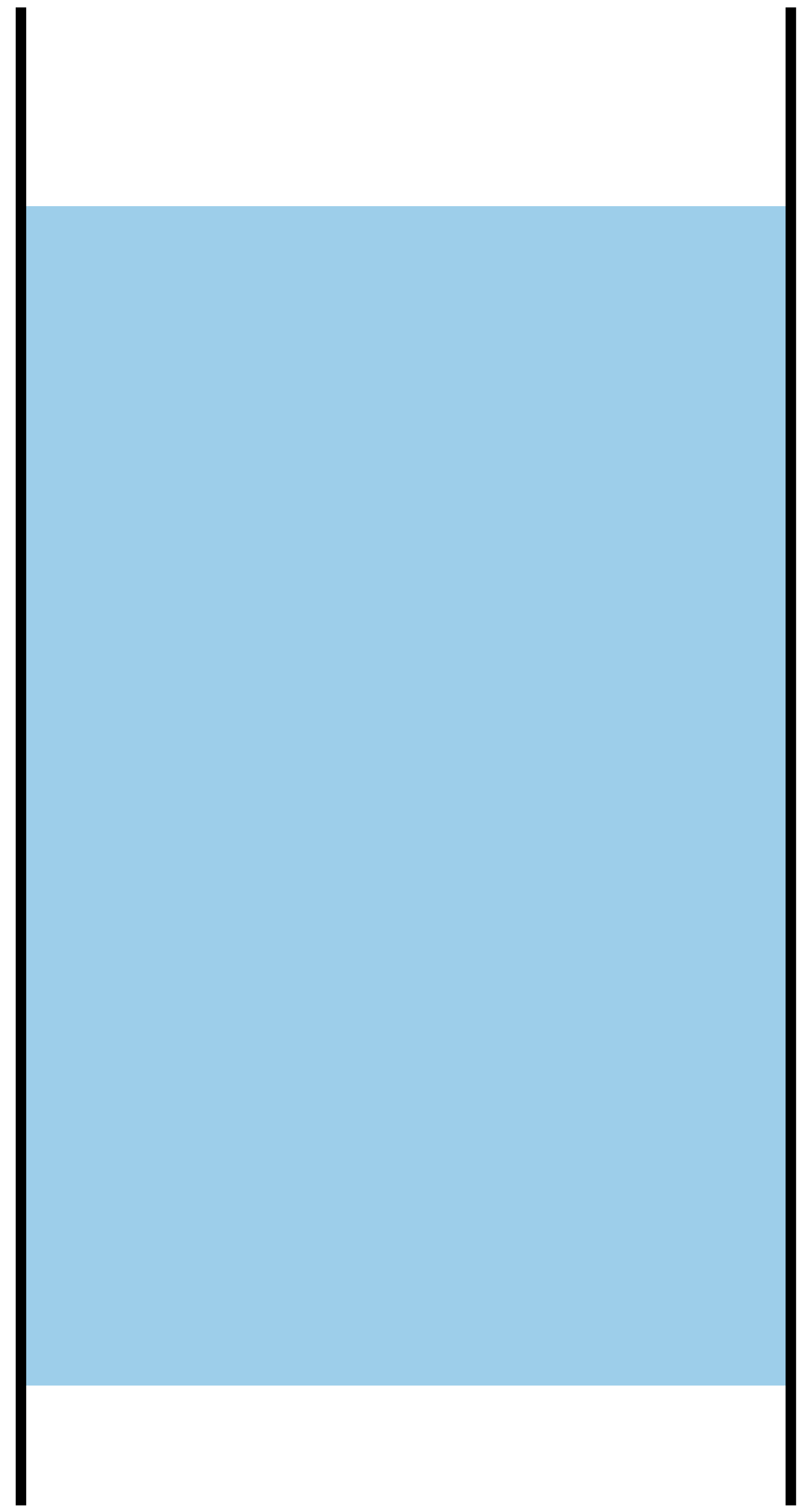
$\longrightarrow$ Solve $A_B x_B = b$

If $x_B \geq 0$, then x is a **basic feasible solution**

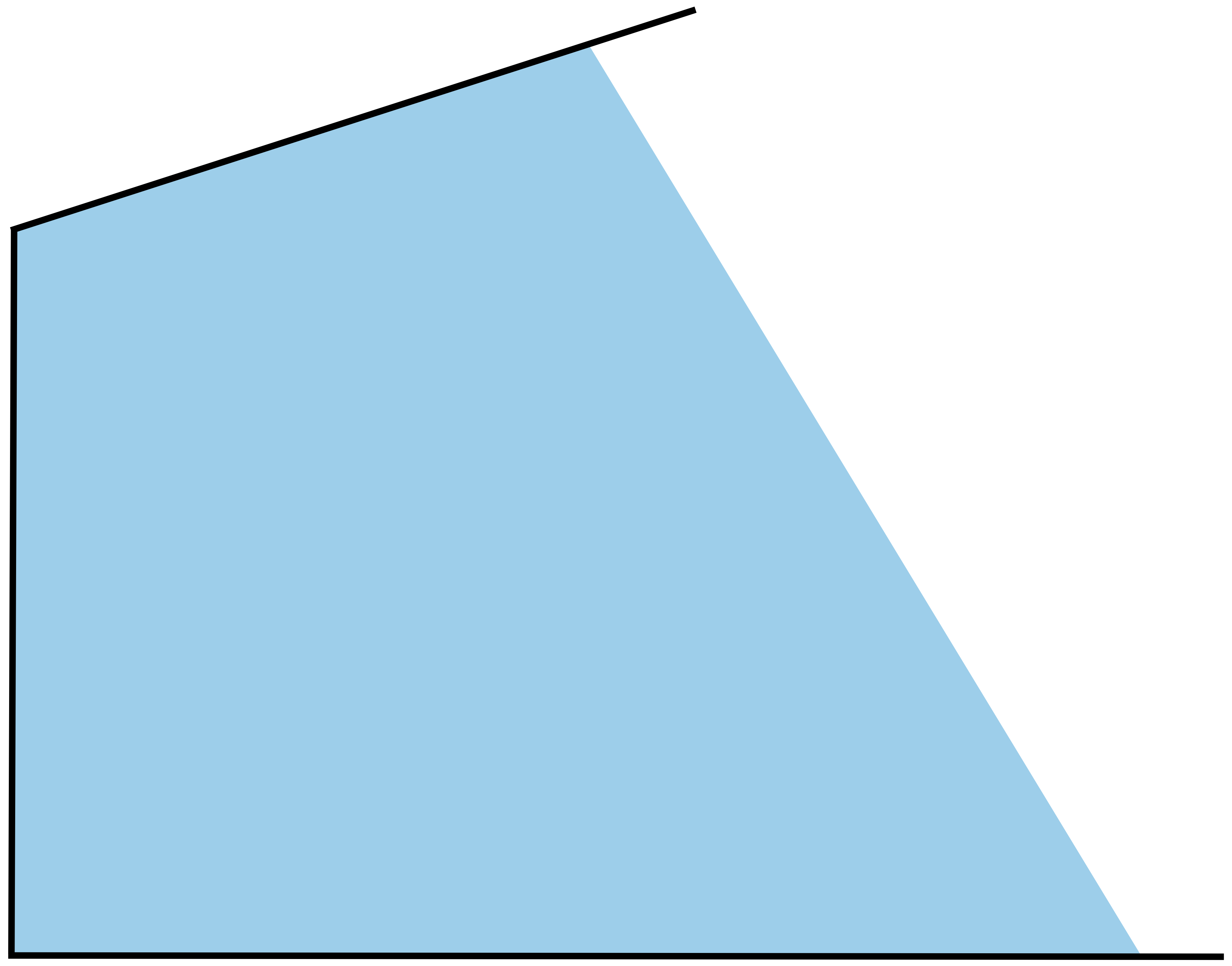
Existence and optimality of extreme points

Existence of extreme points

Example



No extreme points



Extreme points

Existence of extreme points

Characterization

A polyhedron P **contains a line** if

$\exists x \in P$ and a nonzero vector d such that $x + \lambda d \in P, \forall \lambda \in \mathbf{R}$.

Given a polyhedron $P = \{x \mid a_i^T x \leq b_i, \quad i = 1, \ldots, m\}$, the following are **equivalent**

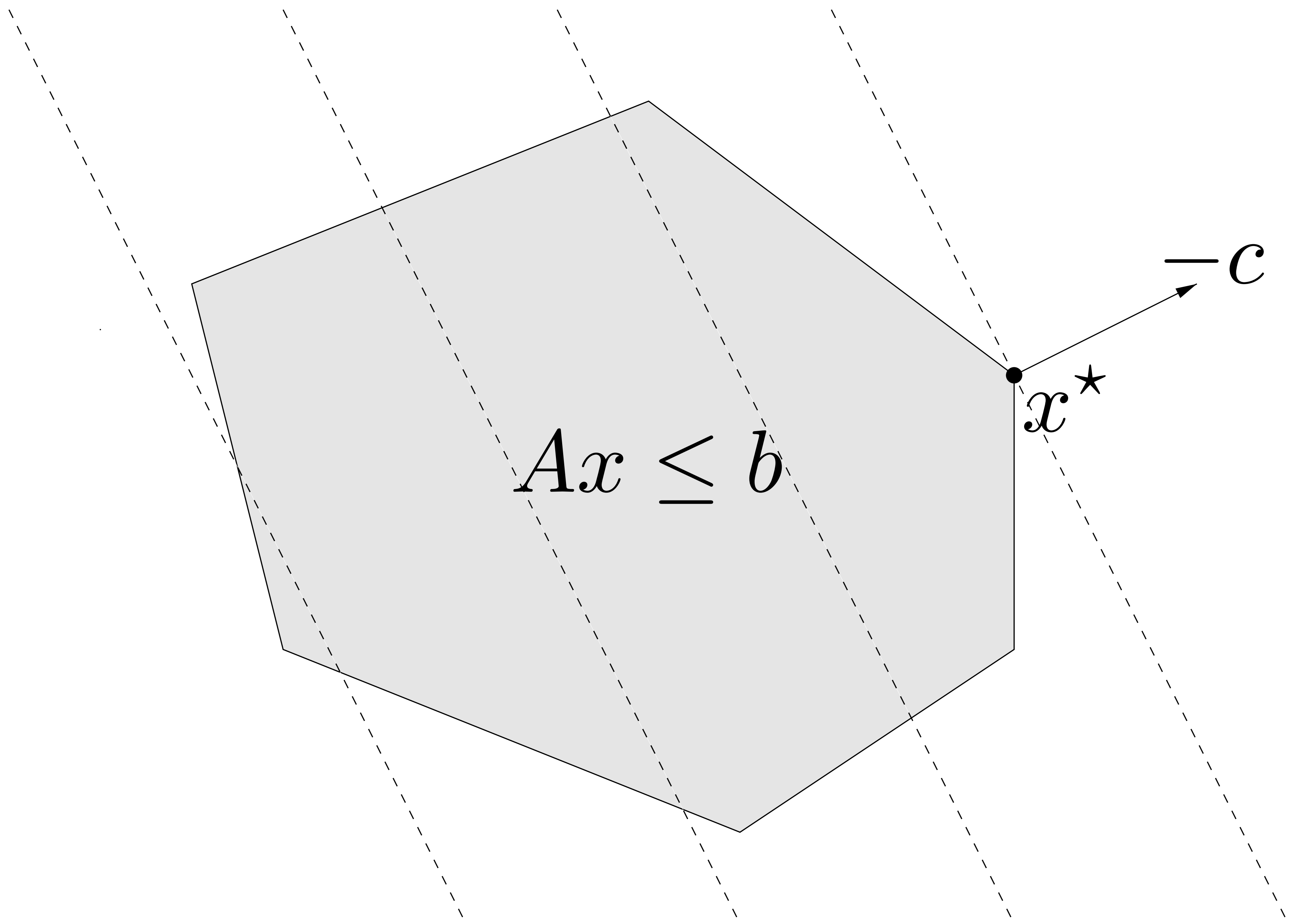
- P does not contain a line
- P has at least one extreme point
- n of the a_i vectors are linearly independent

Corollary

Every nonempty **bounded polyhedron** has **at least one basic feasible solution**

Optimality of extreme points

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \le b \end{array}$$



If

- P has at least one extreme point
- There exists an optimal solution $x^\star$

Then, there exists an optimal solution that is an **extreme point** of P.

Solution method: restrict search to **extreme points.**

How to search among basic feasible solutions?

Idea

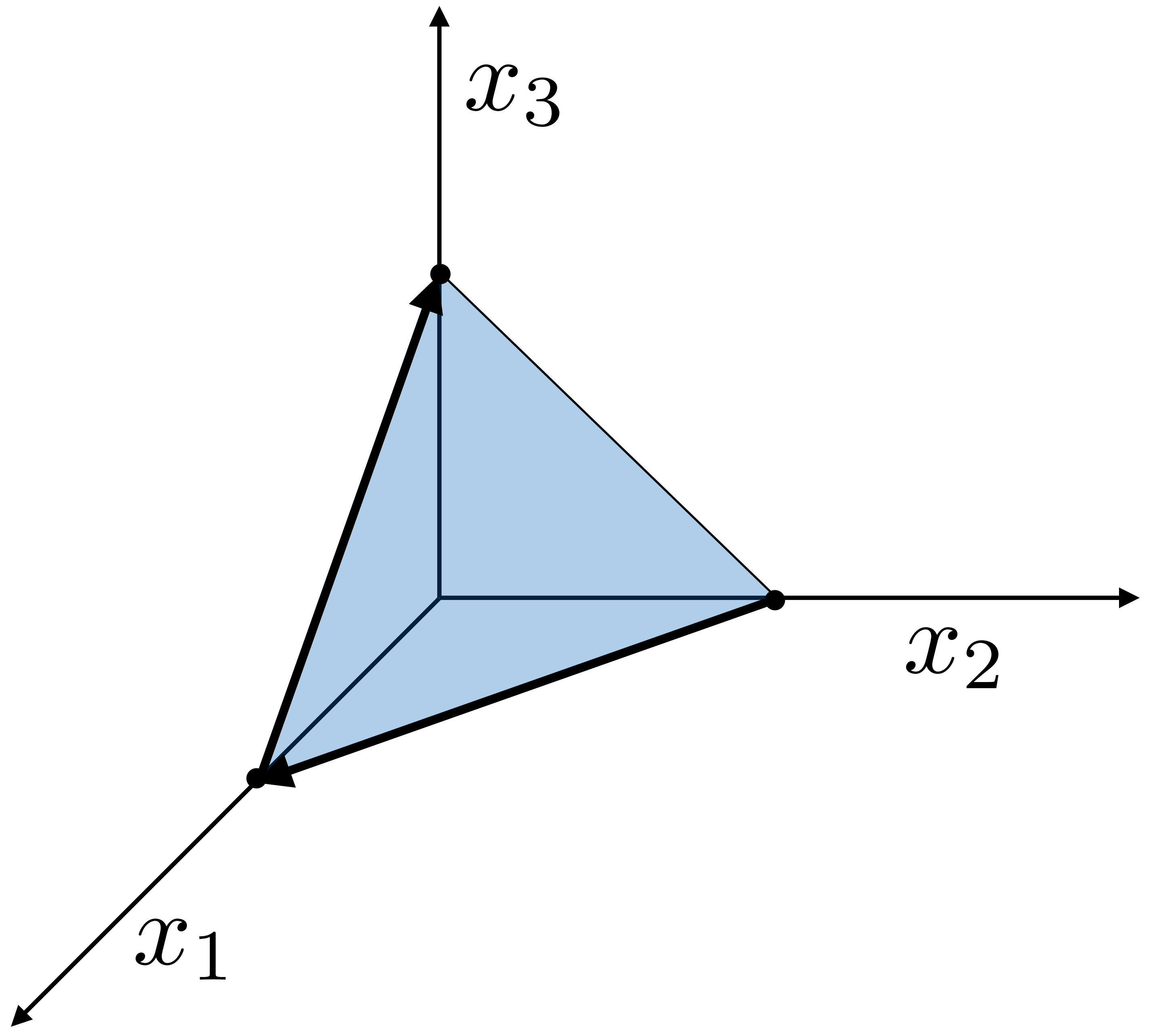
List all the basic feasible solutions, compare objective values and pick the best one.

Intractable!

If $n = 1000$ and $m = 100$, we have 10^{143} combinations!

Conceptual algorithm

- Start at corner
- Visit neighboring corner that improves the objective



Geometry of linear optimization

Today, we learned to:

- **Apply geometric and algebraic properties** of polyhedra to characterize the "corners" of the feasible region.
- **Construct basic feasible solutions** by solving a linear system.
- **Recognize existence and optimality** of extreme points.

References

- Bertsimas and Tsitsiklis: Introduction to Linear Programming
  - Chapter 2.1—2.6 : geometry of linear programming

Next topics

More applications

The simplex method