ORF307 – Optimization

8. Piecewise linear optimization

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Ed Forum

- What exactly does a closed form solution mean?
- What exactly is the purpose of a slack variable? Is it just so that we can write equalities instead of inequalities? When eliminating inequality constraints by using slack variables, how do we choose or obtain the exact value of the slack variables?
- Is it only a problem for the question to be unbounded below or is it also an issue to be unbounded above (given that we are solving a minimization problem)?
- What exactly is the difference between the 1-norm and the 2-norm.
  -> This lecture!

Today's lecture

Piecewise linear optimization

- Vector norms
- Piecewise linear optimization
- Turning vector norm problems as LPs
- Support vector machines

Vector norms

Vector norms

Euclidean norm

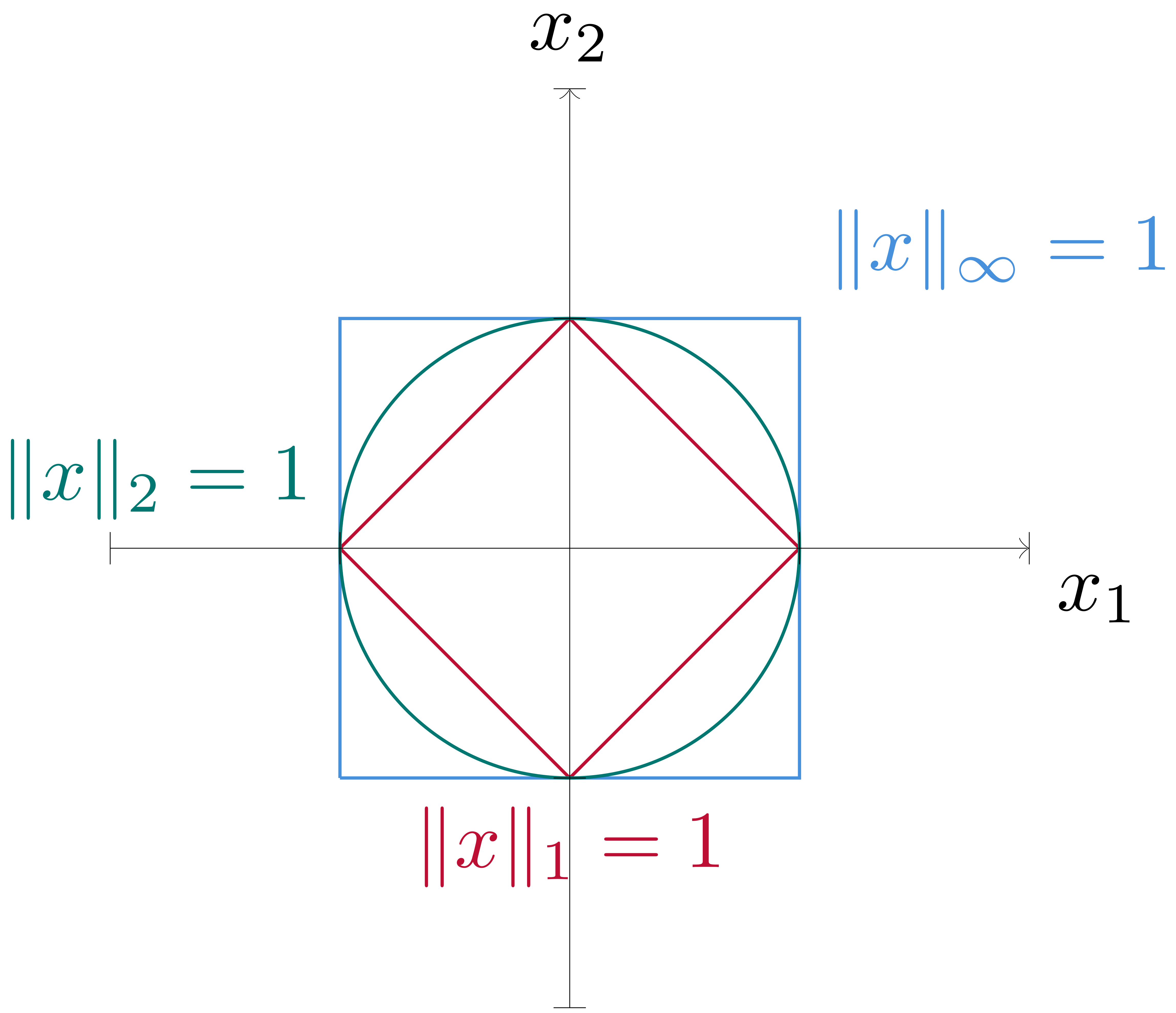
$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$$

1-norm (Manhattan norm)

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

∞-norm (max-norm)

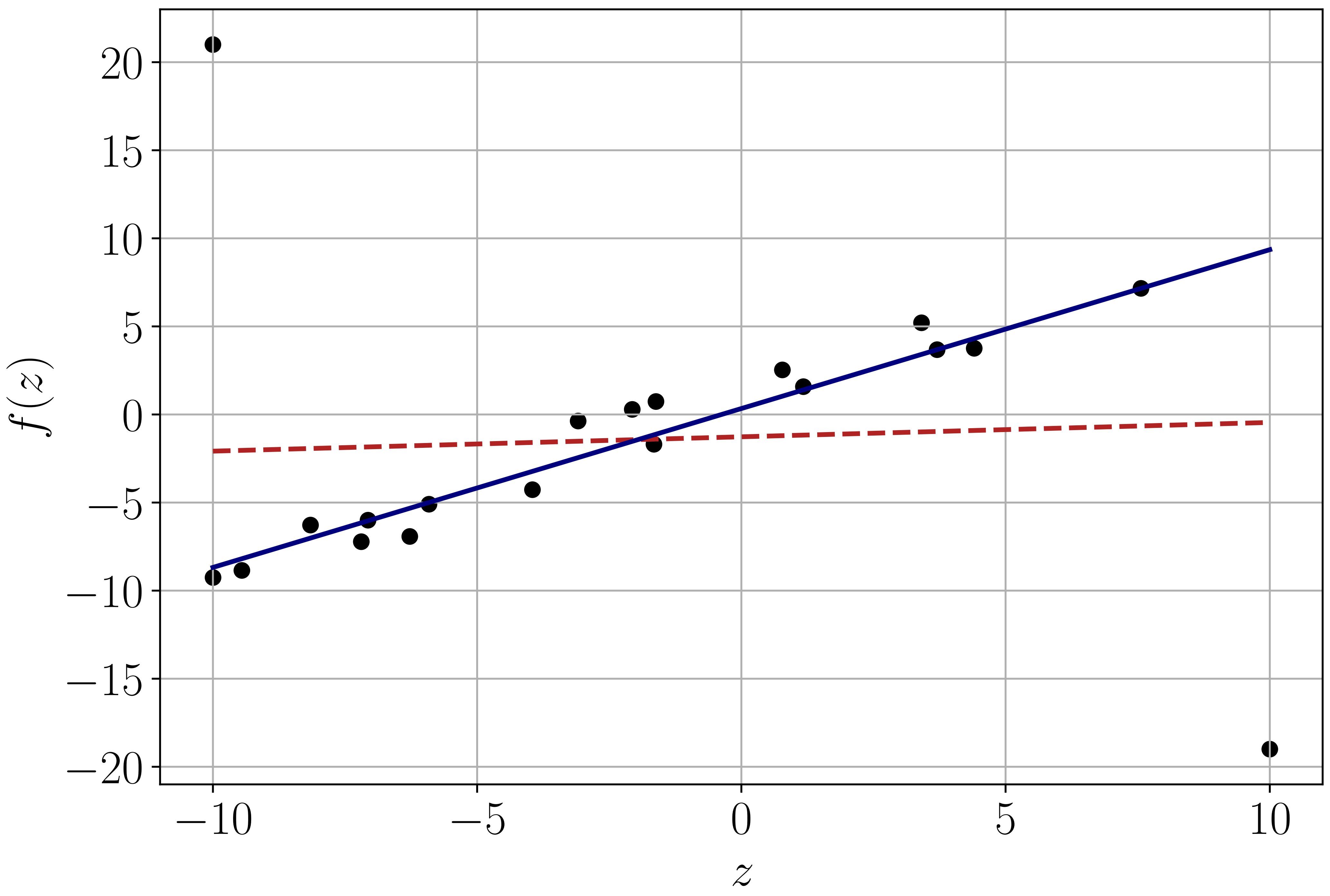
$$\|x\|_\infty = \max_i |x_i|$$



Data-fitting example

Fit a linear function $f(z) = a + bz$ to m data points (z_i, f_i):

Approximation problem $Ax \approx b$ where $\underbrace{\begin{bmatrix} 1 & z_1 \\ \vdots & \vdots \\ 1 & z_m \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{x} \approx \underbrace{\begin{bmatrix} f_1 \\ \vdots \\ f_m \end{bmatrix}}_{b}$



Recall our regression problem:

minimize $\sum_{i=1}^{m} |Ax - b|_i = \|Ax - b\|_1$

Why is it a linear program?

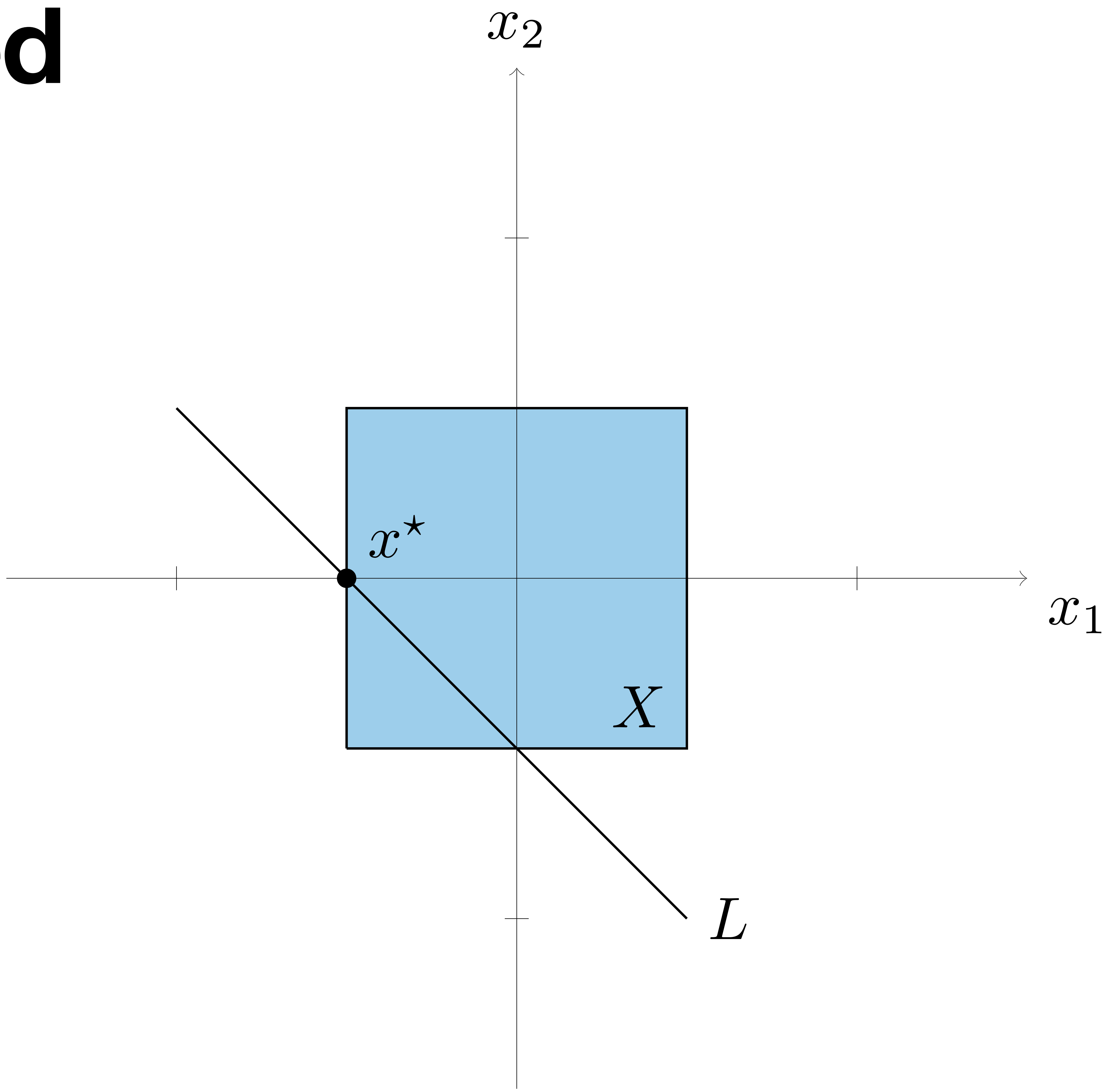
Simple example revisited

Goal find point as far left as possible,
in the unit box X,
and restricted to the line L

$$\begin{array}{ll} \text{minimize} & x_1 \\ \text{subject to} & \|x\|_\infty \leq 1 \\ & x_1 + x_2 = -1 \end{array}$$

The (nonlinear) norm function
appears in the constraints

Why is it a linear progam?

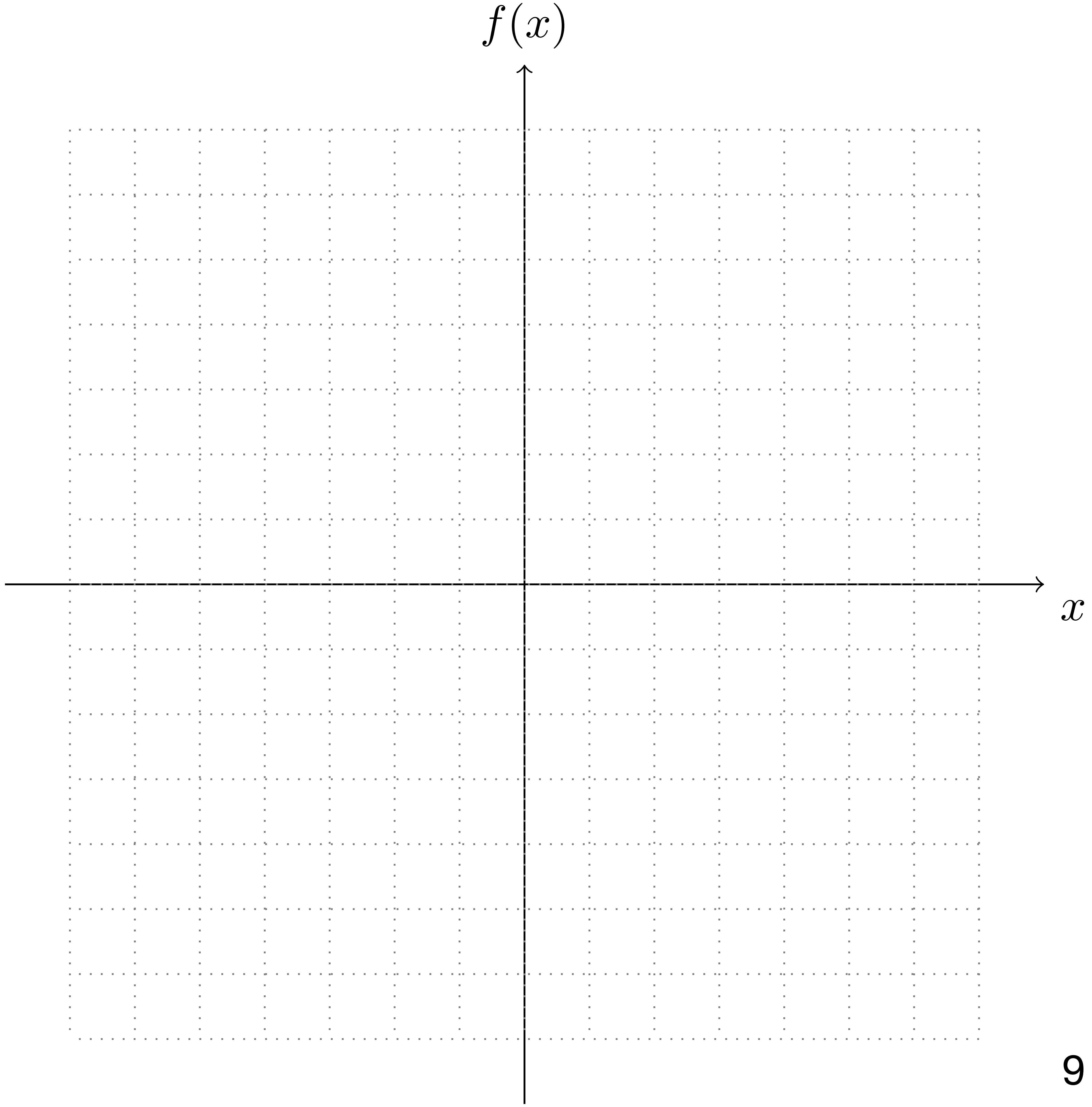


Piecewise linear optimization

Linear, affine and convex functions

Linear function: $f(x) = a^T x$

Affine function: $f(x) = a^T x + b$

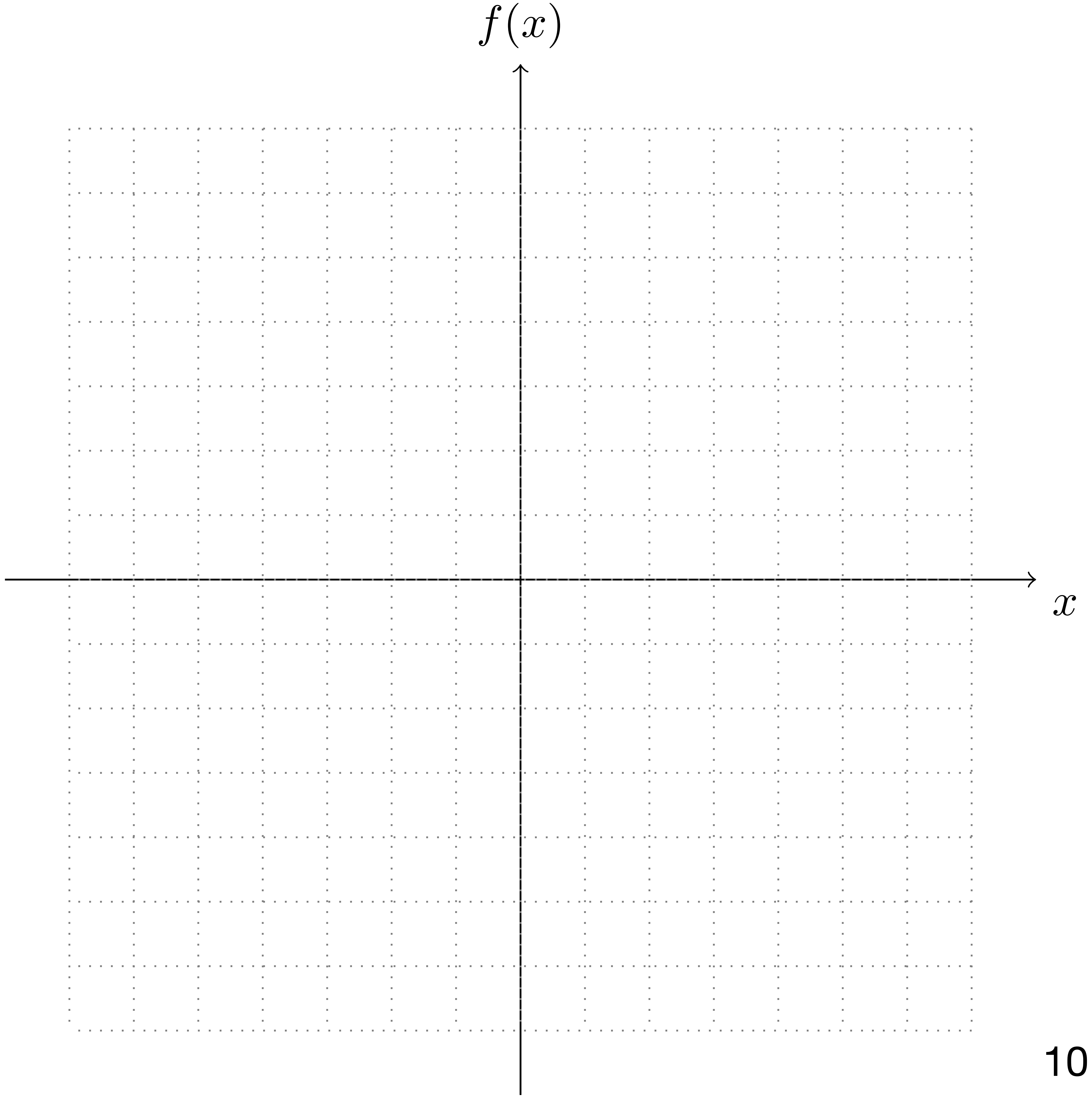


Linear, affine and convex functions

Convex function:

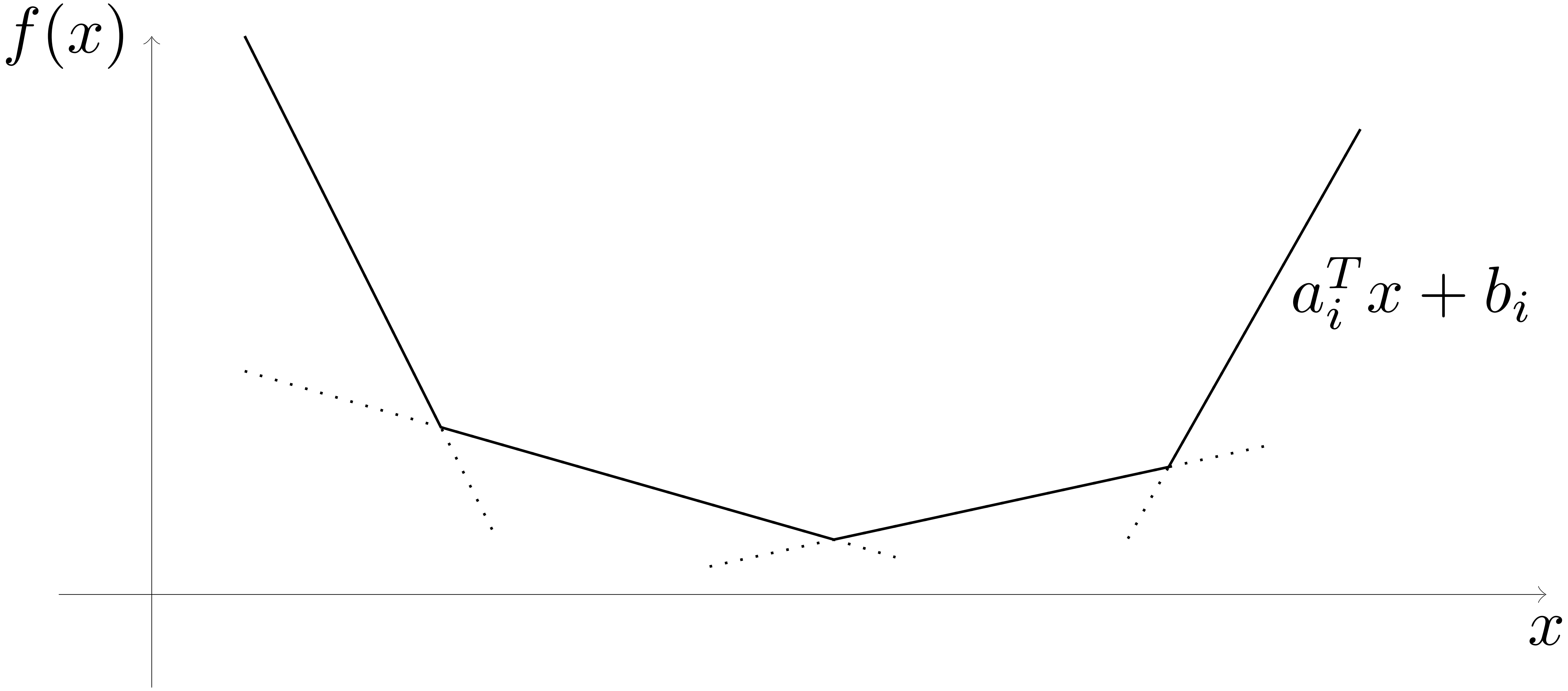
$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y),$$

$$\forall x, y \in \mathbf{R}^n,\ \alpha \in [0,1]$$



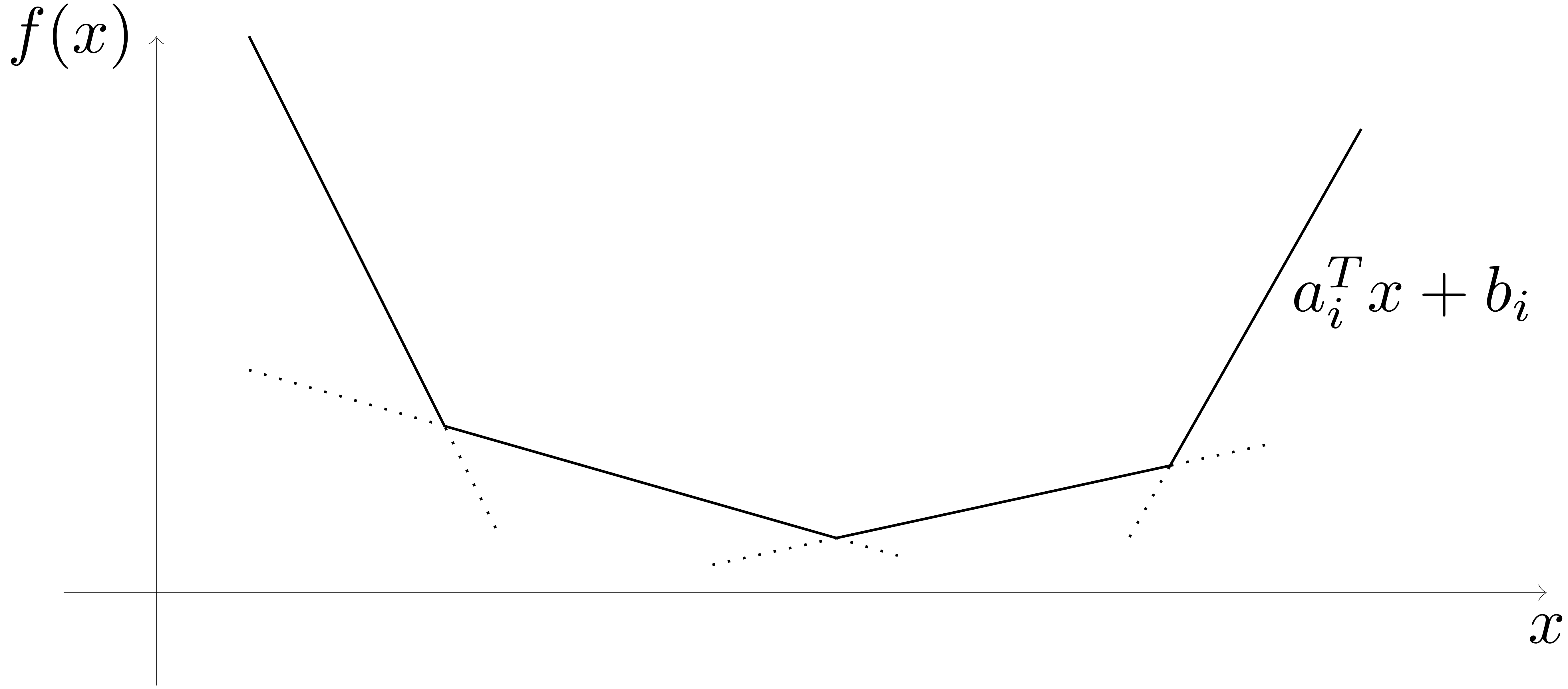
Convex piecewise-linear functions

$$f(x) = \max_{i=1,\ldots,m} (a_i^T x + b_i)$$



Convex piecewise-linear minimization

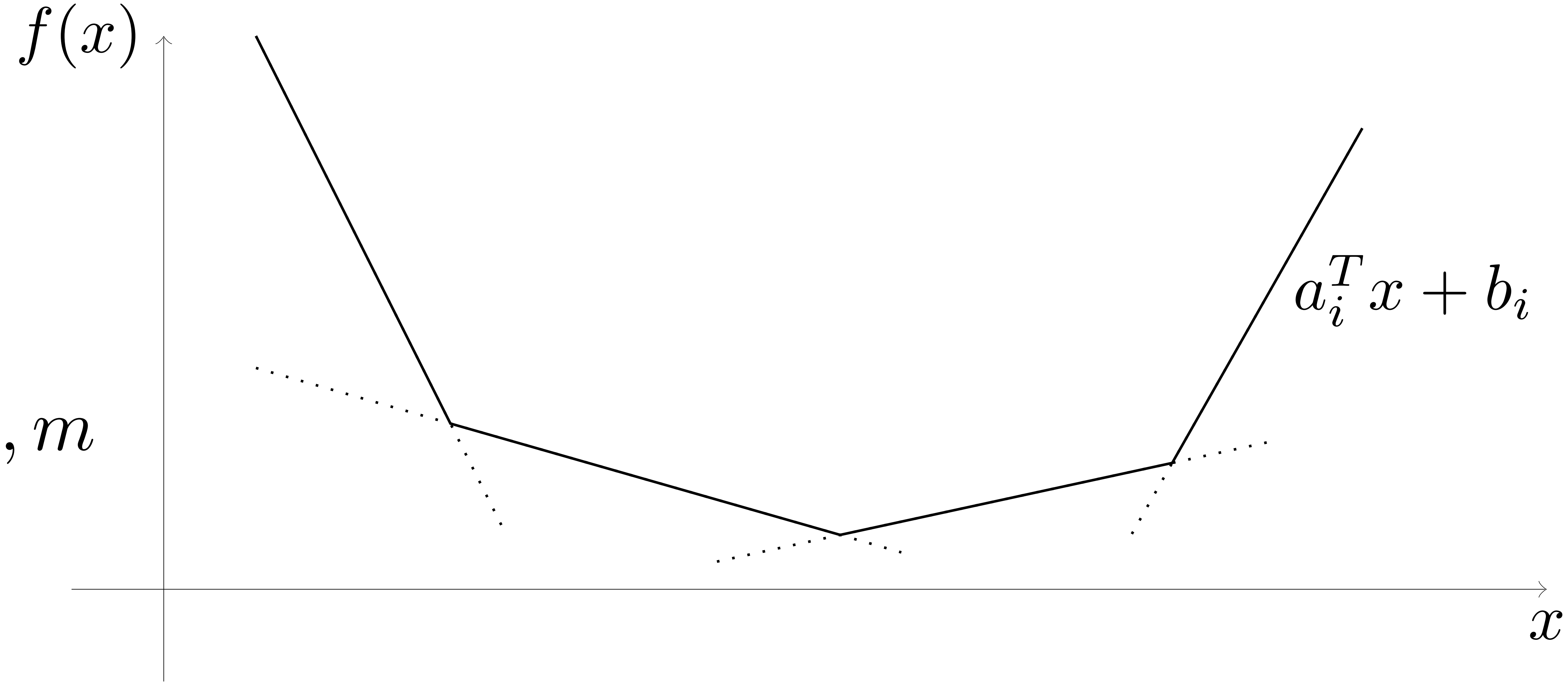
minimize $\max_{i=1,\ldots,m} (a_i^T x + b_i)$



Equivalent linear optimization

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & a_i^T x + b_i \leq t, \quad i = 1, \ldots, m \end{array}$$

Convex piecewise-linear minimization



Equivalent linear optimization

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & a_i^T x + b_i \leq t, \quad i = 1, \ldots, m \end{array}$$

Matrix notation

$$\begin{array}{ll} \text{minimize} & \tilde{c}^T \tilde{x} \\ \text{subject to} & \tilde{A}\tilde{x} \leq \tilde{b} \end{array}$$

$$\tilde{x} = \begin{bmatrix} x \\ t \end{bmatrix}, \quad \tilde{c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} a_1^T & -1 \\ \vdots & \vdots \\ a_m^T & -1 \end{bmatrix}, \quad \tilde{b} = \begin{bmatrix} -b_1 \\ \vdots \\ -b_m \end{bmatrix}$$

Vector norm problems as linear optimization

∞-norm regression

$$\text{minimize} \quad \|Ax - b\|_\infty$$

The ∞**-norm** of m-vector y is

$$\|y\|_\infty = \max_{i=1,\ldots,m} |y_i| = \max_{i=1,\ldots,m} \max\{y_i, -y_i\}$$

Equivalent problem

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & (Ax - b)_i \leq t, \quad i = 1, \ldots, m \\ & -(Ax - b)_i \leq t, \quad i = 1, \ldots, m \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & t \\ \text{subject to} & Ax - b \leq t\mathbf{1} \\ & -(Ax - b) \leq t\mathbf{1} \end{array}$$

∞-norm regression

$$\text{minimize} \quad \|Ax - b\|_\infty$$

The ∞**-norm** of m-vector y is

$$\|y\|_\infty = \max_{i=1,\ldots,m} |y_i| = \max_{i=1,\ldots,m} \max\{y_i, -y_i\}$$

Equivalent problem

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & Ax - b \leq t\mathbf{1} \\ & -(Ax - b) \leq t\mathbf{1} \end{array}$$

Matrix notation

$$\begin{array}{ll} \text{minimize} & \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} x \\ t \end{bmatrix} \\ \text{subject to} & \begin{bmatrix} A & -\mathbf{1} \\ -A & -\mathbf{1} \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \leq \begin{bmatrix} b \\ -b \end{bmatrix} \end{array}$$

Sum of piecewise-linear functions

minimize $\quad f(x) + g(x) = \max_{i=1,\dots,m} (a_i^T x + b_i) + \max_{i=1,\dots,p} (c_i^T x + d_i)$

Equivalent linear optimization

$$
\begin{array}{ll}
\text{minimize} & t_1 + t_2 \\
\text{subject to} & a_i^T x + b_i \leq t_1, \quad i = 1, \dots, m \\
& c_i^T x + d_i \leq t_2, \quad i = 1, \dots, p
\end{array}
$$

1-norm regression

minimize $\quad \|Ax - b\|_1$

The 1-**norm** of m-vector y is

$$\|y\|_1 = \sum_{i=1}^{m} |y_i| = \sum_{i=1}^{m} \max\{y_i, -y_i\}$$

Equivalent problem

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{m} u_i \\ \text{subject to} & (Ax - b)_i \leq u_i, \quad i = 1, \dots, m \\ & -(Ax - b)_i \leq u_i, \quad i = 1, \dots, m \end{array} \longrightarrow \begin{array}{ll} \text{minimize} & \mathbf{1}^T u \\ \text{subject to} & Ax - b \leq u \\ & -(Ax - b) \leq u \end{array}$$

1-norm regression

$$\text{minimize} \quad \|Ax - b\|_1$$

The 1-**norm** of m-vector y is

$$\|y\|_1 = \sum_{i=1}^{m} |y_i| = \sum_{i=1}^{m} \max\{y_i, -y_i\}$$

Equivalent problem

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T u \\ \text{subject to} & Ax - b \leq u \\ & -(Ax - b) \leq u \end{array}$$

Matrix notation

$$\begin{array}{ll} \text{minimize} & \begin{bmatrix} 0 \\ \mathbf{1} \end{bmatrix}^T \begin{bmatrix} x \\ u \end{bmatrix} \\ \text{subject to} & \begin{bmatrix} A & -I \\ -A & -I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \leq \begin{bmatrix} b \\ -b \end{bmatrix} \end{array}$$

Summary: 1 and ∞-norm regression

∞-norm

minimize $\|Ax - b\|_\infty$

Equivalent to

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & Ax - b \leq t\mathbf{1} \\ & -(Ax - b) \leq t\mathbf{1} \end{array}$$

Absolute value of every element $(Ax - b)_i$ is bounded by the same **scalar** t

1-norm

minimize $\|Ax - b\|_1$

Equivalent to

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T u \\ \text{subject to} & Ax - b \leq u \\ & -(Ax - b) \leq u \end{array}$$

Absolute value of every element $(Ax - b)_i$ is bounded by a component of the **vector** u

Example : converting to an LP

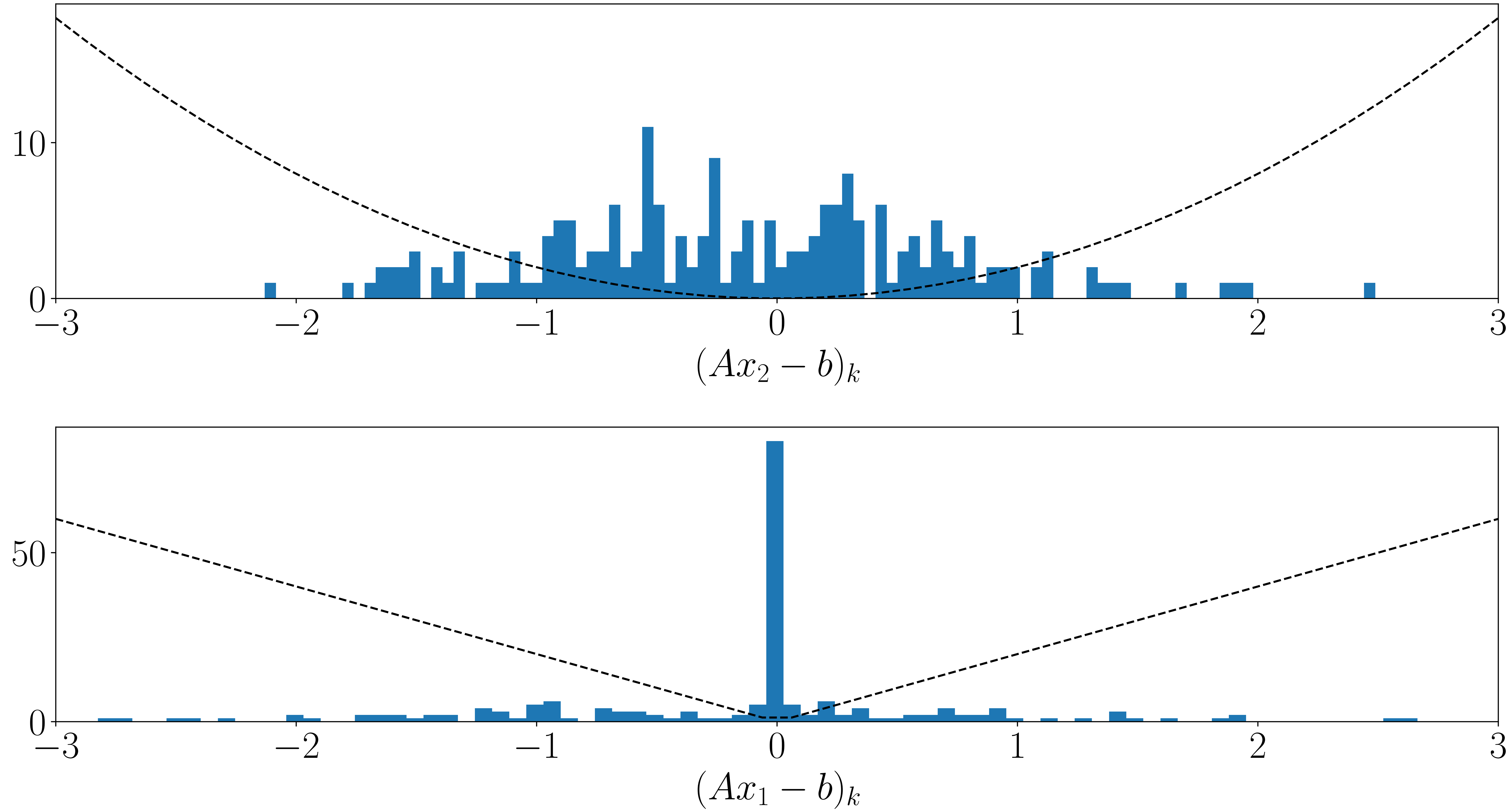
minimize $\|Ax - b\|_\infty$

subject to $\|x\|_1 \leq k$

Comparison with least-squares

Histogram of residuals $Ax - b$ with randomly generated $A \in \mathbf{R}^{200 \times 80}$

$$x_2 = \text{argmin } \|Ax - b\|_2^2, \qquad x_1 = \text{argmin } \|Ax - b\|_1$$



1-norm distribution is **wider** with a **high peak at zero**

Modeling software does most of this for you

∞-norm

minimize $\|Ax - b\|_\infty$

```python
import numpy as np
import cvxpy as cp

m = 200; n = 80

A = np.random.randn(200, 80)
b = np.random.randn(200)
x = cp.Variable(80)

objective = cp.norm(A @ x - b, np.inf)
problem = cp.Problem(cp.Minimize(objective))
problem.solve()
```

1-norm

minimize $\|Ax - b\|_1$

```python
import numpy as np
import cvxpy as cp

m = 200; n = 80

A = np.random.randn(200, 80)
b = np.random.randn(200)
x = cp.Variable(80)

objective = cp.norm(A @ x - b, 1)
problem = cp.Problem(cp.Minimize(objective))
problem.solve()
```

Sparse signal recovery

Sparse signal recovery via 1–norm minimization

$\hat{x} \in \mathbf{R}^n$ is unknown signal, known to be sparse

We make linear measurements $y = A\hat{x}$ with $A \in \mathbf{R}^{m \times n}, m < n$

Estimate signal with smallest ℓ_1-norm, consistent with measurements

$$
\begin{array}{ll}
\text{minimize} & \|x\|_1 \\
\text{subject to} & Ax = y
\end{array}
$$

Equivalent linear optimization

$$
\begin{array}{ll}
\text{minimize} & \mathbf{1}^T u \\
\text{subject to} & -u \leq x \leq u \\
& Ax = y
\end{array}
$$

Sparse signal recovery via 1–norm minimization

Example

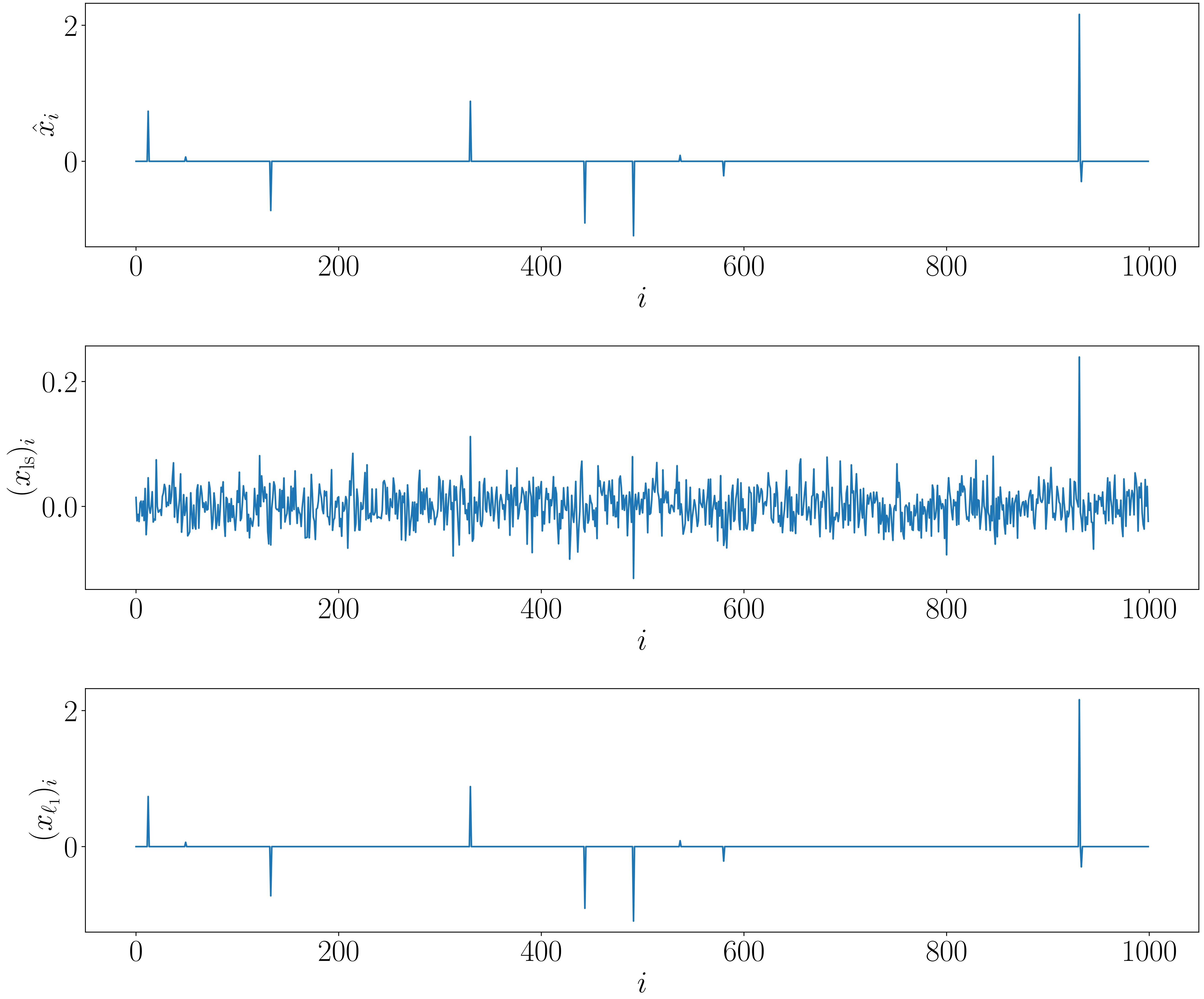
Exact signal $\hat{x} \in \mathbf{R}^{1000}$

10 nonzero components

Random $A \in \mathbf{R}^{100 \times 1000}$

The least squares estimate cannot recover the sparse signal

The 1-norm estimate is **exact**



Support vector machines

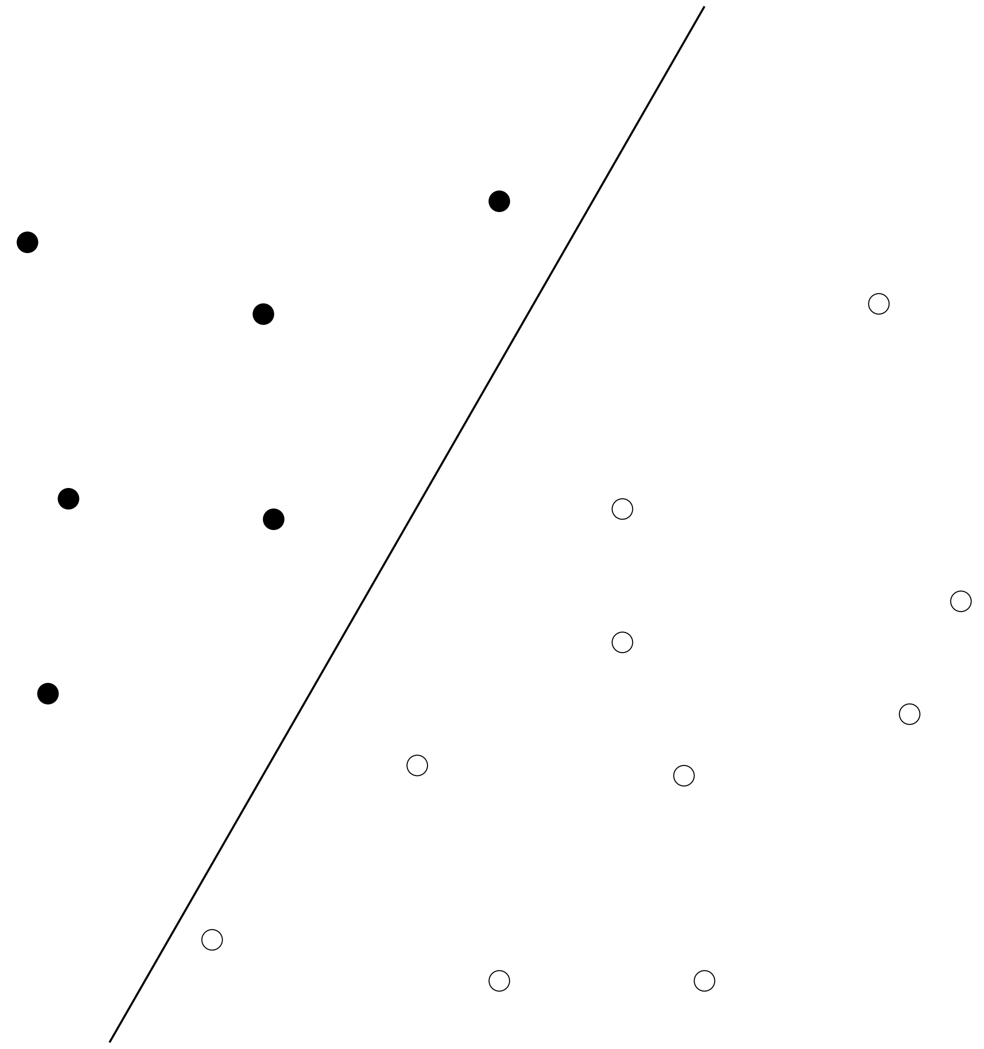
Linear classification

Support vector machine (linear separation)

Given a set of points $\{v_1, \ldots, v_N\}$ with binary labels $s_i \in \{-1, 1\}$

Find hyperplane that strictly separates the tho classes

$$a^T v_i + b > 0 \quad \text{if} \quad s_i = 1$$
$$a^T v_i + b < 0 \quad \text{if} \quad s_i = -1$$



Homogeneous in (a, b), hence equivalent to the linear inequalities (in a, b)

$$s_i(a^T v_i + b) \geq 1$$

Linear classification

Separable case

Feasibility problem

$$\begin{array}{ll} \text{find} & a, b \\ \text{subject to} & s_i(a^T v_i + b) \geq 1, \quad i = 1, \dots, N \end{array}$$

Which can be seen as a special case of LP with

$$\begin{array}{ll} \text{minimize} & 0 \\ \text{subject to} & s_i(a^T v_i + b) \geq 1, \quad i = 1, \dots, N \end{array}$$

$p^\star = 0$ if problem feasible (points separable)

$p^\star = \infty$ if problem infeasible (points not separable) ⟶ **What then?**

Linear classification

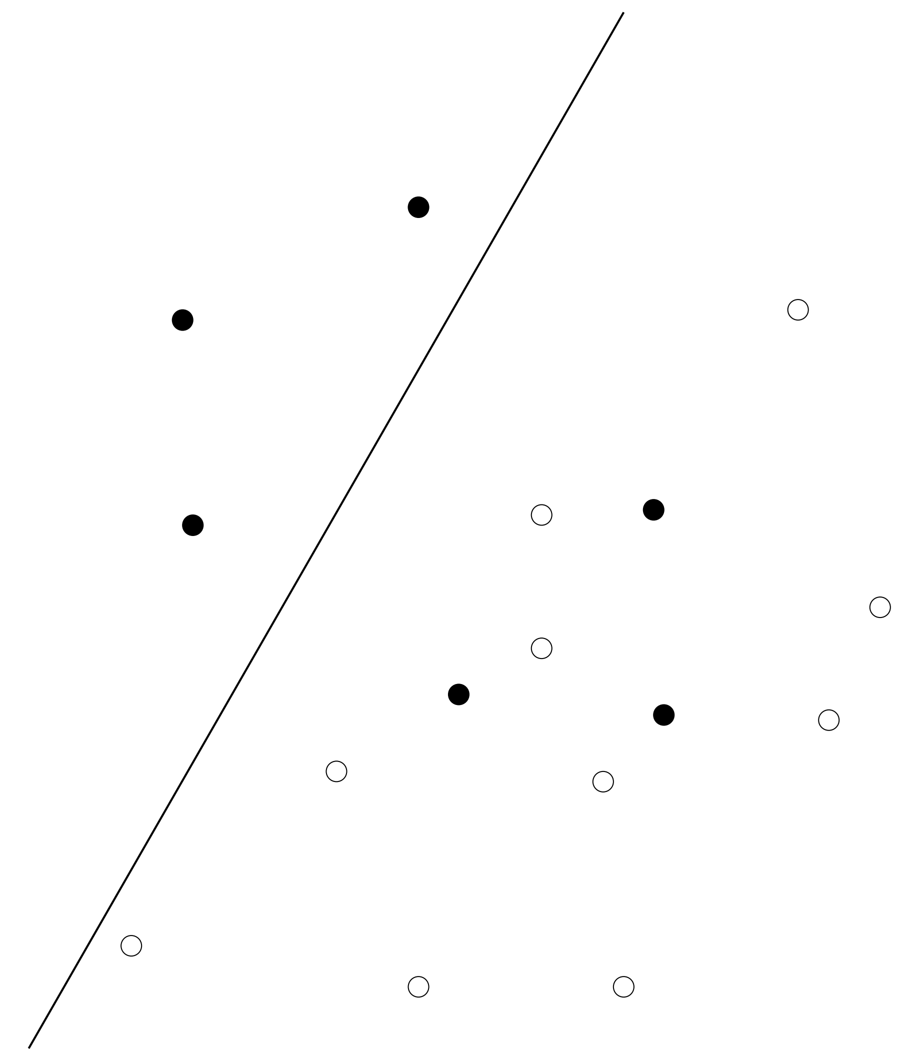
Approximate linear separation of non-separable points

Each of our constraints is

$$s_i(a^T v_i + b) \geq 1 \longrightarrow$$

Violation

$$\max\{0, 1 - s_i(a^T v_i + b)\}$$



Goal

Minimize sum of the violations

minimize $\sum_{i=1}^{N} \max\{0, 1 - s_i(a^T v_i + b)\}$

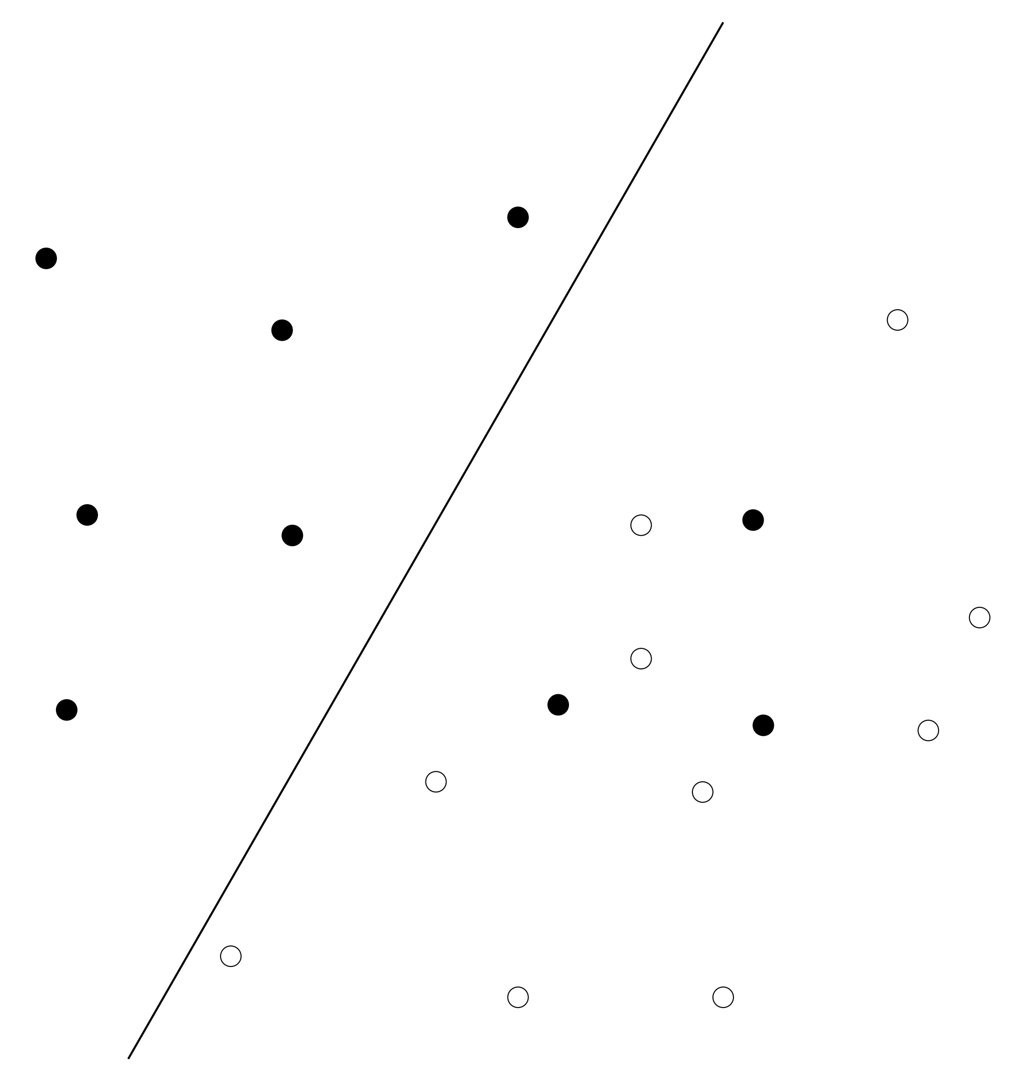
Piecewise-linear minimization problem with variables a, b

Linear classification

Approximate linear separation of non-separable points

minimize $\sum_{i=1}^{N} \max\{0, 1 - s_i(a^T v_i + b)\}$

As a linear optimization problem



Piecewise-linear optimization

Today, we learned to:

- **Understand** the differences between vector norms
- **Reformulate** convex piecewise linear minimization as linear optimization
- **Apply** these techniques to sparse signal recovery and classification problems

References

- Bertsimas, Tsitsiklis: Introduction to Linear Optimization
  - Chapter 1.3: piecewise linear optimization
- R. Vanderbei: Linear Programming — Foundations and Extensions
  - Chapter 12.4,12.7: 1-norm regression and SVMs

Next time

- Linear optimization geometry
- Optimality conditions