ORF307 – Optimization

13. Duality
Ed forum
Ed forum : The auxiliary LP problem

<table>
<thead>
<tr>
<th>Initial problem</th>
<th>Auxiliary problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>minimize $c^T x$</td>
<td>minimize $1^T y$</td>
</tr>
<tr>
<td>subject to $Ax = b$</td>
<td>subject to $Ax + y = b$</td>
</tr>
<tr>
<td>$x \geq 0$</td>
<td>$x \geq 0, y \geq 0$</td>
</tr>
</tbody>
</table>
minimize $x_n$
subject to $\epsilon \leq x_1 \leq 1$
$\epsilon x_{i-1} \leq x_i \leq 1 - \epsilon x_{i-1}$, \quad $i = 2, \ldots, n$

Figure 1: Klee-Minty cube for $n = 3, \epsilon = 1/3$
(from Henk and Ziegler, Combinatorica, 1998)
Today’s agenda

- Obtaining lower bounds
- The dual problem
- Weak and strong duality

Readings

- Chapter 4, Bertsimas & Tsitsiklis
- Chapter 5, Vanderbei
Linear programs (standard forms)

**Standard form LP**
- minimize \( c^T x \)
- subject to \( Ax = b \)
- \( x \geq 0 \)

**Inequality form LP**
- minimize \( c^T x \)
- subject to \( Ax \leq b \)
Obtaining lower bounds
Obtaining lower bounds

A simple example

minimize \( x_1 + 3x_2 \)
subject to \( x_1 + 3x_2 \geq 2 \)

What is a lower bound on the optimal cost?
Obtaining lower bounds

Another example

minimize \( x_1 + 3x_2 \)

subject to \( x_1 + x_2 \geq 2 \)
\[ x_2 \geq 1 \]

What is a lower bound on the optimal cost?

Sum the constraints:

\[
1 \cdot (x_1 + x_2 \geq 2) \\
+ 2 \cdot (x_2 \geq 1) \\
= x_1 + 3x_2 \geq 4
\]

\[ \implies \quad \text{A lower bound is } 4 \]
Obtaining lower bounds
A more interesting example

minimize \( x_1 + 3x_2 \)
subject to \( x_1 + x_2 \geq 2 \)
\( x_2 \geq 1 \)
\( x_1 - x_2 \geq 3 \)

Add constraints
\[ y_1 \cdot (x_1 + x_2 \geq 2) \]
\[ + y_2 \cdot (x_2 \geq 1) \]
\[ + y_3 \cdot (x_1 - x_2 \geq 3) \]
\[ = x_1 + 3x_2 \geq 2y_1 + y_2 + 3y_3 \]

Match the cost
\[ y_1 + y_3 = 1 \]
\[ y_1 + y_2 - y_3 = 3 \]
\[ y_1, y_2, y_3 \geq 0 \]

Bound

Many options
\( y = (1, 2, 0) \Rightarrow \text{Bound 4} \)
\( y = (0, 4, 1) \Rightarrow \text{Bound 7} \)
How can we get the best one?
Obtaining lower bounds

A more interesting example — Best lower bound

We can obtain the **best lower bound** by solving the following problem

\[
\begin{align*}
\text{maximize} & \quad 2y_1 + y_2 + 3y_3 \\
\text{subject to} & \quad y_1 + y_3 = 1 \\
& \quad y_1 + y_2 - y_3 = 3 \\
& \quad y_1, y_2, y_3 \geq 0
\end{align*}
\]

This linear optimization problem is called the **dual problem**
Dual linear programming
The Lagrangian function

A general linear program

\[
\min_{x \in \mathcal{X}} \quad c^T x \\
\text{subject to} \quad a_i^T x - b_i \leq 0 \quad i = 1 \ldots m \\
d_i^T x - f_i = 0 \quad i = 1 \ldots p
\]

Lagrangian Function: \( L : \mathcal{X} \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R} \)

\[
L(x, y, z) = c^T x + \sum_{i=1}^{m} y_i (a_i^T x - b_i) + \sum_{i=1}^{p} z_i (d_i^T x - f_i)
\]

- \( y_i \): inequality Lagrange multiplier.
- \( z_i \): equality Lagrange multiplier.
The Lagrangian function

A general linear program

\[
\begin{align*}
\min_{x \in \mathcal{X}} & \quad c^T x \\
\text{subject to} & \quad Ax - b \leq 0 \\
& \quad Dx - f = 0
\end{align*}
\]

Lagrangian Function: \( L : \mathcal{X} \times \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R} \)

\[
L(x, y, z) = c^T x + y^T (Ax - b) + z^T (Dx - f)
\]

- \( y \): inequality Lagrange multiplier (same size as \( b \)).
- \( z \): equality Lagrange multiplier (same size as \( f \)).
Lagrange dual function

The dual function $g : \mathbb{R}^m \times \mathbb{R}^p$ is

$$g(y, z) = \inf_{x \in \mathcal{X}} L(x, y, z)$$

$$= \inf_{x \in \mathbb{R}^n} [c^T x + y^T (Ax - b) + z^T (Dx - f)]$$
Lagrange dual function

The dual function $g : \mathbb{R}^m \times \mathbb{R}^p$ is

$$g(y, z) = \inf [c^T x + y^T (Ax - b) + z^T (Dx - f)]$$

The dual function $g(y, z)$ is always a **concave** function.
Lagrange dual function

The dual function $g : \mathbb{R}^m \times \mathbb{R}^p$ is

$$g(y, z) = \inf [c^T x + y^T (Ax - b) + z^T (Dx - f)]$$

The dual function $g(y, z)$ generates lower bounds for $p^*$ when $y \geq 0$: 
Lagrange dual function

The dual function $g : \mathbb{R}^m \times \mathbb{R}^p$ is

$$g(y, z) = \inf [c^T x + y^T (Ax - b) + z^T (Dx - f)]$$

The dual function $g(y, z)$ might be $-\infty$ for some arguments:

$$\text{dom } g = \{ y, z \mid g(\lambda, \nu) > -\infty \}$$
Finding the dual function

The dual function $g : \mathbb{R}^m \times \mathbb{R}^p$ is

$$g(y, z) = \inf [c^T x + y^T (Ax - b) + z^T (Dx - f)]$$
Finding the dual function

The dual function $g : \mathbb{R}^m \times \mathbb{R}^p$ is

$$g(y, z) = \inf \left[ c^T x + y^T (Ax - b) + z^T (Dx - f) \right]$$

$$= \inf \left[ (c + A^T y + D^T z)x - b^T y - f^T z \right]$$

$$g(y, z) = \begin{cases} -b^T y - f^T z & \text{if } c + A^T y + D^T z = 0 \\ -\infty & \text{otherwise} \end{cases}$$

Lower bound property: $-b^T y - f^T z \leq p^*$ whenever $c + A^T y + D^T z = 0$ and $y \geq 0.$
The dual problem

Every $y \geq 0$ and $z$ produces a lower bound for $p^*$ using the dual function.

Which is the best?

$$\max_{y, z} \ g(y, z)$$

subject to $y \geq 0$

• Problem $(\mathcal{D})$ is **convex**, even if $(\mathcal{P})$ is not.

• Problem $(\mathcal{D})$ has optimal value $d^* \leq p^*$. 
The dual problem

Every \( y \geq 0 \) and \( z \) produces a lower bound for \( p^* \) using the dual function.

Which is the best?

\[
\max_{y,z} \quad g(y, z) \quad \quad \quad : (\mathcal{D}) \\
\text{subject to} \quad y \geq 0
\]

• The point \((y, z)\) is \textbf{dual feasible} if \( y \geq 0 \) and \((y, z) \in \text{dom } g\).

• Can often impose the constraint \((y, z) \in \text{dom } g\) explicitly in \((\mathcal{D})\).
The dual of a linear program

Our \textbf{primal} problem:

\[
\begin{aligned}
\text{min} & \quad c^T x \\
\text{subject to} & \quad Ax \leq b \\
& \quad Dx = f
\end{aligned} : (P)
\]

Its \textbf{dual} problem:

\[
\begin{aligned}
\text{max} & \quad -b^T y - f^T z \\
\text{subject to} & \quad A^T y + D^T z = -c \\
& \quad y \geq 0
\end{aligned} : (D)
\]

The dual is also a linear program!
Another example

Our **primal** problem:

\[
\begin{align*}
\min & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\]

: (\(\mathcal{P}\))

The Lagrangian:
Another example

Our **primal** problem:

\[
\begin{align*}
\min & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\] : (\(\mathcal{P}\))

The dual function:
Another example

Our primal problem:

\[
\begin{align*}
\min & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\] : (P)

The dual problem:
General forms

**Standard form LP**

minimize $c^T x$
subject to $Ax = b$
$x \geq 0$

maximize $-b^T y$
subject to $A^T y + c \geq 0$

**LP with inequalities and equalities**

minimize $c^T x$
subject to $Ax \leq b$
$Dx = f$

maximize $-b^T y - f^T z$
subject to $A^T y + D^T z + c = 0$
$y \geq 0$

**Inequality form LP**

minimize $c^T x$
subject to $Ax \leq b$

maximize $-b^T y$
subject to $A^T y + c = 0$
$y \geq 0$
Weak and strong duality
## Optimal objective values

### Primal

<table>
<thead>
<tr>
<th>Objective</th>
<th>Constraint</th>
</tr>
</thead>
<tbody>
<tr>
<td>minimize $c^T x$</td>
<td>$Ax \leq b$</td>
</tr>
</tbody>
</table>

$p^*$ is the primal optimal value

- Primal infeasible: $p^* = +\infty$
- Primal unbounded: $p^* = -\infty$

### Dual

<table>
<thead>
<tr>
<th>Objective</th>
<th>Constraint</th>
</tr>
</thead>
<tbody>
<tr>
<td>maximize $-b^T y$</td>
<td>$A^T y + c = 0$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$y \geq 0$</th>
</tr>
</thead>
</table>

$d^*$ is the dual optimal value

- Dual infeasible: $d^* = -\infty$
- Dual unbounded: $d^* = +\infty$
## Unbounded problems

<table>
<thead>
<tr>
<th>Primal</th>
<th>Dual</th>
</tr>
</thead>
<tbody>
<tr>
<td>minimize $c^T x$</td>
<td>maximize $-b^T y$</td>
</tr>
<tr>
<td>subject to $Ax \leq b$</td>
<td>subject to $A^T y + c = 0$, $y \geq 0$</td>
</tr>
</tbody>
</table>

Primal unbounded: $p^* = -\infty$

Dual unbounded: $d^* = +\infty$
Weak duality

Theorem

If:

- $x$ is a primal feasible point

- $y$ is a dual feasible point

Then:

$$-b^T y \leq c^T x$$

Proof
Weak duality

Theorem

If:
- $x$ is a primal feasible point
- $y$ is a dual feasible point

$$-b^T y \leq c^T x$$

Important!:
- Any dual feasible $y$ gives a **lower bound** on $p^*$
- Any primal feasible $x$ gives an **upper bound** on $d^*$
- $c^T x + b^T y$ is the **duality gap**
- $d^* \leq p^*$ always
Weak duality corollaries

Unboundedness vs feasibility

- Primal unbounded \((p^* = -\infty)\) \(\Rightarrow\) dual infeasible \((d^* = -\infty)\)
- Dual unbounded \((d^* = +\infty)\) \(\Rightarrow\) primal infeasible \((p^* = +\infty)\)

Optimality condition

If:
- \(x\) is a primal feasible point
- \(y\) is a dual feasible point
- The duality gap \(b^T y + c^T x = 0\)

Then \(x\) and \(y\) are optimal solutions to the primal and dual problems
Strong duality

Theorem

If a linear optimization problem has an optimal solution, then

• so does its dual

• the optimal values of the primal and dual are equal
Exception to strong duality

Primal
minimize $x$
subject to $0 \cdot x \leq -1$

Optimal value is $p^* = +\infty$

Dual
maximize $y$
subject to $0 \cdot y + 1 = 0$
$y \geq 0$

Optimal value is $d^* = -\infty$

NB: primal and dual are both infeasible
## Relationship between primal and dual

<table>
<thead>
<tr>
<th></th>
<th>$p^* = +\infty$</th>
<th>$p^* \text{ finite}$</th>
<th>$p^* = -\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d^* = +\infty$</td>
<td>primal inf.</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>dual unb.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$d^* \text{ finite}$</td>
<td>optimal values equal</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$d^* = -\infty$</td>
<td>exception</td>
<td></td>
<td>primal unb. dual inf</td>
</tr>
</tbody>
</table>
Example from earlier

minimize \[ x_1 + 3x_2 \]
subject to \[ x_1 + x_2 \geq 2 \]
\[ x_2 \geq 1 \]
\[ x_1 - x_2 \geq 3 \]

Primal
minimize \[ c^T x \]
subject to \[ Ax \leq b \]

Dual
maximize \[ -b^T y \]
subject to \[ A^T y + c = 0 \]
\[ y \geq 0 \]
Example from the beginning

minimize \( x_1 + 3x_2 \)
subject to
\[
\begin{align*}
  x_1 + x_2 & \geq 2 \\
  x_2 & \geq 1 \\
  x_1 - x_2 & \geq 3
\end{align*}
\]

Dual
maximize \(-b^T y\)
subject to
\[
\begin{align*}
  A^T y + c &= 0 \\
  y &\geq 0
\end{align*}
\]

Inequality form LP
minimize \( c^T x \)
subject to
\[
\begin{align*}
  A x &\leq b
\end{align*}
\]
\[
\begin{pmatrix}
  c \\
  A
\end{pmatrix} = \begin{pmatrix}
  (1, 3) \\
  \begin{pmatrix}
  -1 & -1 \\
  0 & -1 \\
  -1 & 1
\end{pmatrix}
\end{pmatrix}
\]
\[
b = (-2, -1, -3)
\]

maximize \( 2y_1 + y_2 + 3y_3 \)
subject to
\[
\begin{align*}
  -y_1 - y_3 &= -1 \\
  -y_1 - y_2 + y_3 &= -3 \\
  (y_1, y_2, y_3) &\geq 0
\end{align*}
\]
Next time

- KKT Conditions
- Interpretation of dual problems