ORF522 – Linear and Nonlinear Optimization

7. Linear optimization duality
Recap
Linear optimization formulations

**Standard form LP**

minimize \( c^T x \)

subject to \( Ax = b \)

\( x \geq 0 \)

**Inequality form LP**

minimize \( c^T x \)

subject to \( Ax \leq b \)
Today’s agenda
[Chapter 4, LO][Chapter 5, LP]

• Obtaining lower bounds
• The dual problem
• Weak and strong duality
Obtaining lower bounds
Obtaining lower bounds

A simple example

minimize \( x_1 + 3x_2 \)
subject to \( x_1 + 3x_2 \geq 2 \)

What is a lower bound on the optimal cost?
Obtaining lower bounds
A simple example

\[
\begin{align*}
\text{minimize} & \quad x_1 + 3x_2 \\
\text{subject to} & \quad x_1 + 3x_2 \geq 2
\end{align*}
\]

What is a lower bound on the optimal cost?

A lower bound is 2 because \( x_1 + 3x_2 \geq 2 \)
Obtaining lower bounds

Another example

minimize \( x_1 + 3x_2 \)
subject to \( x_1 + x_2 \geq 2 \)
\( x_2 \geq 1 \)

What is a lower bound on the optimal cost?
Obtaining lower bounds

Another example

minimize \[ x_1 + 3x_2 \]
subject to \[ x_1 + x_2 \geq 2 \]
\[ x_2 \geq 1 \]

What is a lower bound on the optimal cost?

Let’s sum the constraints

\[ 1 \cdot (x_1 + x_2 \geq 2) \]
\[ + 2 \cdot (x_2 \geq 1) \]
\[ = x_1 + 3x_2 \geq 4 \]
Obtaining lower bounds
Another example

minimize \( x_1 + 3x_2 \)
subject to \( x_1 + x_2 \geq 2 \)
\( x_2 \geq 1 \)

What is a lower bound on the optimal cost?

Let’s sum the constraints
\[
1 \cdot (x_1 + x_2 \geq 2) \\
+ 2 \cdot (x_2 \geq 1) \\
= x_1 + 3x_2 \geq 4
\]

A lower bound is 4
Obtaining lower bounds

A more interesting example

minimize \[ x_1 + 3x_2 \]

subject to

\[ x_1 + x_2 \geq 2 \]
\[ x_2 \geq 1 \]
\[ x_1 - x_2 \geq 3 \]

How can we obtain a lower bound?
Obtaining lower bounds

A more interesting example

minimize \( x_1 + 3x_2 \)
subject to \( x_1 + x_2 \geq 2 \)
\( x_2 \geq 1 \)
\( x_1 - x_2 \geq 3 \)

How can we obtain a lower bound?

Add constraints

\[ y_1 \cdot (x_1 + x_2 \geq 2) \]
\[ + y_2 \cdot (x_2 \geq 1) \]
\[ + y_3 \cdot (x_1 - x_2 \geq 3) \]
\[ = x_1 + 3x_2 \geq 2y_1 + y_2 + 3y_3 \]
 Obtaining lower bounds

A more interesting example

minimize \( x_1 + 3x_2 \)
subject to \( x_1 + x_2 \geq 2 \)
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How can we obtain a lower bound?

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\(+ y_3 \cdot (x_1 - x_2 \geq 3) \)
\( = x_1 + 3x_2 \geq 2y_1 + y_2 + 3y_3 \)

Bound
Obtaining lower bounds
A more interesting example

minimize \( x_1 + 3x_2 \)
subject to \( x_1 + x_2 \geq 2 \)
\( x_2 \geq 1 \)
\( x_1 - x_2 \geq 3 \)

How can we obtain a lower bound?

Add constraints
\begin{align*}
y_1 \cdot (x_1 + x_2 & \geq 2) \\
y_2 \cdot (x_2 & \geq 1) \\
y_3 \cdot (x_1 - x_2 & \geq 3)
\end{align*}
\[= x_1 + 3x_2 \geq 2y_1 + y_2 + 3y_3\]

Match cost coefficients
\begin{align*}
y_1 + y_3 &= 1 \\
y_1 + y_2 - y_3 &= 3 \\
y_1, y_2, y_3 &\geq 0
\end{align*}
# Obtaining lower bounds

## A more interesting example

Minimize: $x_1 + 3x_2$

Subject to:

- $x_1 + x_2 \geq 2$
- $x_2 \geq 1$
- $x_1 - x_2 \geq 3$

How can we obtain a lower bound?

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<th>Add constraints</th>
<th>Match cost coefficients</th>
<th>Many options</th>
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<td>$y_1 \cdot (x_1 + x_2 \geq 2)$</td>
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**Bound**
Obtaining lower bounds

A more interesting example

minimize \[ x_1 + 3x_2 \]
subject to \[ x_1 + x_2 \geq 2 \]
\[ x_2 \geq 1 \]
\[ x_1 - x_2 \geq 3 \]

How can we obtain a lower bound?

Add constraints
\[ y_1 \cdot (x_1 + x_2 \geq 2) \]
\[ + y_2 \cdot (x_2 \geq 1) \]
\[ + y_3 \cdot (x_1 - x_2 \geq 3) \]
\[ = x_1 + 3x_2 \geq \boxed{2y_1 + y_2 + 3y_3} \]

Match cost coefficients
\[ y_1 + y_3 = 1 \]
\[ y_1 + y_2 - y_3 = 3 \]
\[ y_1, y_2, y_3 \geq 0 \]

Many options
\[ y = (1, 2, 0) \Rightarrow \text{Bound 4} \]
\[ y = (0, 4, 1) \Rightarrow \text{Bound 7} \]
Obtaining lower bounds

A more interesting example

minimize \[ x_1 + 3x_2 \]
subject to \[ x_1 + x_2 \geq 2 \]
\[ x_2 \geq 1 \]
\[ x_1 - x_2 \geq 3 \]

How can we obtain a lower bound?

Add constraints
\[ y_1 \cdot (x_1 + x_2 \geq 2) \]
\[ + y_2 \cdot (x_2 \geq 1) \]
\[ + y_3 \cdot (x_1 - x_2 \geq 3) \]
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Match cost coefficients
\[ y_1 + y_3 = 1 \]
\[ y_1 + y_2 - y_3 = 3 \]
\[ y_1, y_2, y_3 \geq 0 \]

Many options
\[ y = (1, 2, 0) \Rightarrow \text{Bound 4} \]
\[ y = (0, 4, 1) \Rightarrow \text{Bound 7} \]

Many options

How can we get the best one?
We can obtain the best lower bound by solving the following problem

maximize \[ 2y_1 + y_2 + 3y_3 \]

subject to \[ y_1 + y_3 = 1 \]
\[ y_1 + y_2 - y_3 = 3 \]
\[ y_1, y_2, y_3 \geq 0 \]
Obtaining lower bounds

A more interesting example — Best lower bound

We can obtain the best lower bound by solving the following problem

maximize \[ 2y_1 + y_2 + 3y_3 \]
subject to \[ y_1 + y_3 = 1 \]
\[ y_1 + y_2 - y_3 = 3 \]
\[ y_1, y_2, y_3 \geq 0 \]

This linear optimization problem is called the dual problem
The dual problem
Lagrange multipliers

Consider the LP in standard form

minimize \( c^T x \)

subject to \( Ax = b \)
\( x \geq 0 \)
Lagrange multipliers

Consider the LP in standard form

minimize \( c^T x \)
subject to \( Ax = b \)
\( x \geq 0 \)

Relax the constraint

\[ g(y) = \minimize_x c^T x + y^T (Ax - b) \]
subject to \( x \geq 0 \)
Lagrange multipliers

Consider the LP in standard form

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\]

Relax the constraint

\[
\begin{align*}
g(y) &= \min_{x} \quad c^T x + y^T (Ax - b) \\
& \quad x \geq 0
\end{align*}
\]

Lower bound

\[
g(y) \leq c^T x^* + y^T (Ax^* - b) = c^T x^*
\]
Lagrange multipliers

Consider the LP in standard form

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\text{minimize} & \quad c^T x \\
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g(y) = \min_{x} \quad c^T x + y^T (Ax - b)
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subject to \( x \geq 0 \)

Lower bound

\[
g(y) \leq c^T x^* + y^T (Ax^* - b) = c^T x^*
\]

Best lower bound

\[
\max_y g(y)
\]
The dual

Dual function

\[ g(y) = \min_{x \geq 0} \left( c^T x + y^T (Ax - b) \right) \]

\[ = -b^T y + \min_{x \geq 0} \left( c + A^T y \right)^T x \]
The dual

Dual function

\[ g(y) = \min_{x \geq 0} \left( c^T x + y^T (Ax - b) \right) \]

\[ = -b^T y + \min_{x \geq 0} \left( c + A^T y \right)^T x \]

\[ g(y) = \begin{cases} 
- b^T y & \text{if } c + A^T y \geq 0 \\
-\infty & \text{otherwise} 
\end{cases} \]
The dual

**Dual function**

\[ g(y) = \min_{x \geq 0} (c^T x + y^T (Ax - b)) \]

\[ = -b^T y + \min_{x \geq 0} (c + A^T y)^T x \]

\[ g(y) = \begin{cases} 
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-\infty & \text{otherwise} 
\end{cases} \]

**Dual problem** (find the best bound)

\[ \max_y g(y) = \max \quad -b^T y \]

subject to \[ A^T y + c \geq 0 \]
Primal and dual problems

Primal problem

minimize \( c^T x \)
subject to \( Ax = b \)
\( x \geq 0 \)

Primal variable \( x \in \mathbb{R}^n \)

Dual problem

maximize \( -b^T y \)
subject to \( A^T y + c \geq 0 \)

Dual variable \( y \in \mathbb{R}^m \)
The dual problem carries **useful information** for the primal problem.
Primal and dual problems

Primal problem

minimize \( c^T x \)
subject to \( Ax = b \)
\( x \geq 0 \)

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Primal variable \( x \in \mathbb{R}^n \)

Dual variable \( y \in \mathbb{R}^m \)

The dual problem carries **useful information** for the primal problem

Duality is useful also to **solve** optimization problems
Dual of inequality form LP

What if you find an LP with inequalities?

minimize \( c^T x \)

subject to \( Ax \leq b \)
Dual of inequality form LP

What if you find an LP with inequalities?

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax \leq b
\end{align*}
\]

1. We could first transform it to standard form
Dual of inequality form LP

What if you find an LP with inequalities?

minimize \( c^T x \)
subject to \( Ax \leq b \)

1. We could first transform it to standard form
2. We can compute the dual function (same procedure as before)

Relax the constraint

\[
g(y) = \minimize_x c^T x + y^T (Ax - b)
\]
Dual of inequality form LP

What if you find an LP with inequalities?

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax \leq b
\end{align*}
\]

1. We could first transform it to standard form
2. We can compute the dual function (same procedure as before)

Relax the constraint

\[
g(y) = \min_{x} c^T x + y^T (Ax - b)
\]

Lower bound

\[
g(y) \leq c^T x^* + y^T (Ax^* - b) \leq c^T x^*
\]

we must have \( y \geq 0 \)
Dual of LP with inequalities

Derivation

Dual function

\[
g(y) = \min_x \left( c^T x + y^T (Ax - b) \right)
\]

\[
= -b^T y + \min_x \left( c + A^T y \right)^T x
\]
Dual of LP with inequalities

Derivation

**Dual function**

\[ g(y) = \min_{x} \left( c^T x + y^T (A x - b) \right) \]

\[ = -b^T y + \min_{x} \left( c + A^T y \right)^T x \]

\[ g(y) = \begin{cases} 
-b^T y & \text{if } c + A^T y = 0 \quad \text{(and } y \geq 0) \\
-\infty & \text{otherwise}
\end{cases} \]
Dual of LP with inequalities

Derivation

**Dual function**

\[
g(y) = \min_x \left( c^T x + y^T (Ax - b) \right)
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\[
= -b^T y + \min_x (c + A^T y)^T x
\]

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-\infty & \text{otherwise}
\end{cases}
\]

**Dual problem** (find the best bound)

\[
\max_y g(y) = \max_x -b^T y
\]

subject to \( A^T y + c = 0 \)

\( y \geq 0 \)
## General forms

### Primal

<table>
<thead>
<tr>
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</tr>
<tr>
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**Inequality form LP**

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**General forms**

**Primal**
- **Standard form LP**
  - Minimize: $c^T x$
  - Subject to: $Ax = b$
  - $x \geq 0$

**Dual**
- Maximize: $-b^T y$
- Subject to: $A^T y + c \geq 0$

**Inequality form LP**
- Minimize: $c^T x$
- Subject to: $Ax \leq b$

**Dual**
- Maximize: $-b^T y$
- Subject to: $A^T y + c = 0$
- $y \geq 0$

**LP with inequalities and equalities**
- Minimize: $c^T x$
- Subject to: $Ax \leq b$
- $Cx = d$

**Dual**
- Maximize: $-b^T y - d^T z$
- Subject to: $A^T y + C^T z + c = 0$
- $y \geq 0$
Example from before

minimize $x_1 + 3x_2$
subject to $x_1 + x_2 \geq 2$
$x_2 \geq 1$
$x_1 - x_2 \geq 3$
Example from before

minimize \( x_1 + 3x_2 \)
subject to \( x_1 + x_2 \geq 2 \)
\( x_2 \geq 1 \)
\( x_1 - x_2 \geq 3 \)

Inequality form LP
minimize \( c^T x \)
subject to \( Ax \leq b \)
\( c = (1, 3) \)
\( A = \begin{bmatrix} -1 & -1 \\ 0 & -1 \\ -1 & 1 \end{bmatrix} \)
\( b = (-2, -1, -3) \)
Example from before

minimize \( x_1 + 3x_2 \)
subject to \( x_1 + x_2 \geq 2 \)
\( x_2 \geq 1 \)
\( x_1 - x_2 \geq 3 \)

Inequality form LP
minimize \( c^T x \)
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Dual
maximize \( -b^T y \)
subject to \( A^T y + c = 0 \)
\( y \geq 0 \)
Example from before

minimize \[ x_1 + 3x_2 \]
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Dual

maximize \[ -b^T y \]
subject to \[ A^T y + c = 0 \]
\[ y \geq 0 \]

maximize \[ 2y_1 + y_2 + 3y_3 \]
subject to \[ -y_1 - y_3 = -1 \]
\[ -y_1 - y_2 + y_3 = -3 \]
\[ y_1, y_2, y_3 \geq 0 \]
To memorize

Ways to get the dual

• Derive dual function directly
• Transform the problem in inequality form LP and dualize

Sanity-checks and signs convention

• Consider constraints as $g(x) \leq 0$ or $g(x) = 0$
• Each dual variable is associated to a primal constraint
• $y$ free for primal equalities and $y \geq 0$ for primal inequalities
Dual of the dual

Theorem
If we transform the primal into its dual and then transform the dual to its dual, we obtain a problem equivalent to the original problem. In other words, the dual of the dual is the primal.
Dual of the dual

Theorem
If we transform the primal into its dual and then transform the dual to its dual, we obtain a problem equivalent to the original problem. In other words, the dual of the dual is the primal.

Exercise
Derive dual and dualize again

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Dual of the dual

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Theorem
If we transform a linear optimization problem to another form (inequality form, standard form, inequality and equality form), the dual of the two problems will be equivalent.
Weak and strong duality
Optimal objective values

**Primal**

minimize \( c^T x \)

subject to \( Ax \leq b \)

**Dual**

maximize \( -b^T y \)

subject to \( A^T y + c = 0 \)
\( y \geq 0 \)

\( p^* \) is the primal optimal value
\( d^* \) is the dual optimal value

Primal infeasible: \( p^* = +\infty \)
Primal unbounded: \( p^* = -\infty \)

Dual infeasible: \( d^* = -\infty \)
Dual unbounded: \( d^* = +\infty \)
Weak duality

Theorem
If $x, y$ satisfy:

- $x$ is a feasible solution to the primal problem
- $y$ is a feasible solution to the dual problem

$$-b^T y \leq c^T x$$
Weak duality

Theorem
If $x, y$ satisfy:

- $x$ is a feasible solution to the primal problem
- $y$ is a feasible solution to the dual problem

Then

$$-b^T y \leq c^T x$$

Proof
We know that $Ax \leq b, A^T y + c = 0$ and $y \geq 0$. Therefore,

$$0 \leq y^T (b - Ax) = b^T y - y^T Ax = c^T x + b^T y$$
Weak duality

Theorem
If $x, y$ satisfy:
- $x$ is a feasible solution to the primal problem
- $y$ is a feasible solution to the dual problem

\[ -b^T y \leq c^T x \]

Proof
We know that $Ax \leq b$, $A^T y + c = 0$ and $y \geq 0$. Therefore,

\[
0 \leq y^T (b - Ax) = b^T y - y^T Ax = c^T x + b^T y
\]

Remark
- Any dual feasible $y$ gives a lower bound on the primal optimal value
- Any primal feasible $x$ gives an upper bound on the dual optimal value
- $c^T x + b^T y$ is the duality gap
Weak duality

Corollaries

Unboundedness vs feasibility

- Primal unbounded \((p^* = -\infty)\) \(\Rightarrow\) dual infeasible \((d^* = -\infty)\)
- Dual unbounded \((d^* = +\infty)\) \(\Rightarrow\) primal infeasible \((p^* = +\infty)\)
Weak duality
Corollaries

Unboundedness vs feasibility
- Primal unbounded \( (p^* = -\infty) \) \( \Rightarrow \) dual infeasible \( (d^* = -\infty) \)
- Dual unbounded \( (d^* = +\infty) \) \( \Rightarrow \) primal infeasible \( (p^* = +\infty) \)

Optimality condition
If \( x, y \) satisfy:
  - \( x \) is a feasible solution to the primal problem
  - \( y \) is a feasible solution to the dual problem
  - The duality gap is zero, i.e., \( c^T x + b^T y = 0 \)

Then \( x \) and \( y \) are **optimal solutions** to the primal and dual problem respectively
Strong duality

Theorem
If a linear optimization problem has an optimal solution, so does its dual, and the optimal value of primal and dual are equal

\[ d^* = p^* \]
Strong duality

Constructive proof

Given a primal optimal solution $x^*$ we will construct a dual optimal solution $y^*$
Strong duality

Constructive proof

Given a primal optimal solution \( x^* \) we will construct a dual optimal solution \( y^* \)

Apply simplex to problem in **standard form**

- minimize \( c^T x \)
- subject to \( Ax = b \)
- \( x \geq 0 \)
Strong duality
Constructive proof

Given a primal optimal solution $x^*$ we will construct a dual optimal solution $y^*$

Apply simplex to problem in standard form

minimize $c^T x$
subject to $Ax = b$
$x \geq 0$

• optimal basis $B$
• optimal solution $x^*$ with $A_B x^*_B = b$
• reduced costs $\bar{c} = c - A^T A_B^{-T} c_B \geq 0$
Strong duality

Constructive proof

Given a primal optimal solution $x^*$ we will construct a dual optimal solution $y^*$

Apply simplex to problem in **standard form**

\[
\begin{align*}
\text{minimize} & \quad c^T x & \quad \text{optimal basis } B \\
\text{subject to} & \quad Ax = b & \quad \text{optimal solution } x^* \text{ with } A_Bx_B^* = b \\
& \quad x \geq 0 & \quad \text{reduced costs } \bar{c} = c - A^T A_B^{-T} c_B \geq 0
\end{align*}
\]

Define $y^*$ such that $y^* = -A_B^{-T} c_B$. Therefore, $A^T y^* + c \geq 0 (y^* \text{ dual feasible})$. 
Strong duality
Constructive proof

Given a primal optimal solution \( x^* \) we will construct a dual optimal solution \( y^* \).

Apply simplex to problem in **standard form**

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
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\end{align*}
\]

- optimal basis \( B \)
- optimal solution \( x^* \) with \( A_B x_B^* = b \)
- reduced costs \( \bar{c} = c - A^T A_B^{-T} c_B \geq 0 \)

Define \( y^* \) such that \( y^* = -A_B^{-T} c_B \). Therefore, \( A^T y^* + c \geq 0 \) (\( y^* \) dual feasible).

\[
-b^T y^* = -b^T (-A_B^{-T} c_B) = c_B^T (A_B^{-1} b) = c_B^T x_B^* = c^T x^*
\]
Strong duality
Constructive proof

Given a primal optimal solution \( x^* \) we will construct a dual optimal solution \( y^* \).

Apply simplex to problem in \textbf{standard form}

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\]

\( \cdot \) optimal basis \( B \)

\( \cdot \) optimal solution \( x^* \) with \( A_B x_B^* = b \)

\( \cdot \) reduced costs \( \bar{c} = c - A^T A_B^{-T} c_B \geq 0 \)

Define \( y^* \) such that \( y^* = -A_B^{-T} c_B \). Therefore, \( A^T y^* + c \geq 0 \) (\( y^* \) dual feasible).

\[
-b^T y^* = -b^T (-A_B^{-T} c_B) = c_B^T (A_B^{-1} b) = c_B^T x_B^* = c^T x^*
\]

By weak duality theorem corollary, \( y^* \) is an optimal solution of the dual. Therefore, \( d^* = p^* \).
Exception to strong duality

**Primal**

minimize \( x \)

subject to \( 0 \cdot x \leq -1 \)

Optimal value is \( p^* = +\infty \)

**Dual**

maximize \( y \)

subject to \( 0 \cdot y + 1 = 0 \)
\( y \geq 0 \)

Optimal value is \( d^* = -\infty \)
Exception to strong duality

Primal

minimize $x$
subject to $0 \cdot x \leq -1$

Optimal value is $p^* = +\infty$

Dual

maximize $y$
subject to $0 \cdot y + 1 = 0$
$y \geq 0$

Optimal value is $d^* = -\infty$

Both primal and dual infeasible
**Relationship between primal and dual**

<table>
<thead>
<tr>
<th></th>
<th>$p^* = +\infty$</th>
<th>$p^*$ finite</th>
<th>$p^* = -\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d^* = +\infty$</td>
<td>primal inf. dual unb.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$d^*$ finite</td>
<td></td>
<td>optimal values equal</td>
<td></td>
</tr>
<tr>
<td>$d^* = -\infty$</td>
<td>exception</td>
<td></td>
<td>primal unb. dual inf</td>
</tr>
</tbody>
</table>

- Upper-right excluded by **weak duality**
- $(1, 1)$ and $(3, 3)$ proven by **weak duality**
- $(3, 1)$ and $(2, 2)$ proven by **strong duality**
Example
Production problem

maximize \quad x_1 + 2x_2

subject to \quad x_1 \leq 100
\quad 2x_2 \leq 200
\quad x_1 + x_2 \leq 150
\quad x_1, x_2 \geq 0
Production problem

maximize \( x_1 + 2x_2 \)  \[\rightarrow\] Profits

subject to

\[ x_1 \leq 100 \]
\[ 2x_2 \leq 200 \]
\[ x_1 + x_2 \leq 150 \]
\[ x_1, x_2 \geq 0 \]
Production problem

maximize  \[ x_1 + 2x_2 \]  \[ \text{Profits} \]

subject to

\[ x_1 \leq 100 \]  \[ \text{Resources} \]

\[ 2x_2 \leq 200 \]

\[ x_1 + x_2 \leq 150 \]

\[ x_1, x_2 \geq 0 \]
Production problem

maximize \( x_1 + 2x_2 \)  
subject to
\[
\begin{align*}
x_1 & \leq 100 \\
2x_2 & \leq 200 \\
x_1 + x_2 & \leq 150 \\
x_1, x_2 & \geq 0
\end{align*}
\]

Dualize

1. Transform in inequality form

minimize \( c^T x \)
subject to \( Ax \leq b \)
\[
\begin{align*}
c &= (-1, -2) \\
A &= \begin{bmatrix} 1 & 0 \\
0 & 2 \\
1 & 1 \\
-1 & 0 \\
0 & -1 \end{bmatrix} \\
b &= (100, 200, 150, 0, 0)
\end{align*}
\]
Production problem

maximize \( x_1 + 2x_2 \) \hspace{1cm} \text{Profits}
subject to
\[
\begin{align*}
  x_1 &\leq 100 \\
  2x_2 &\leq 200 \\
  x_1 + x_2 &\leq 150 \\
  x_1, x_2 &\geq 0
\end{align*}
\]

Dualize

1. Transform in inequality form

minimize \( c^T x \) \hspace{1cm} \text{subject to} \hspace{1cm} Ax \leq b
\[
A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}
\]
\[
b = (100, 200, 150, 0, 0)
\]

2. Derive dual

maximize \( -b^T y \) \hspace{1cm} \text{subject to} \hspace{1cm} A^T y + c = 0 \\
\text{and} \hspace{1cm} y \geq 0
Production problem

The dual

minimize \[ 100y_1 + 200y_2 + 150y_3 \]
subject to
\[ y_1 + y_3 \geq 1 \]
\[ 2y_2 + y_3 \geq 2 \]
\[ y_1, y_2, y_3 \geq 0 \]
Production problem

The dual

minimize \[ 100y_1 + 200y_2 + 150y_3 \]
subject to \[ y_1 + y_3 \geq 1 \]
\[ 2y_2 + y_3 \geq 2 \]
\[ y_1, y_2, y_3 \geq 0 \]

Interpretation

- **Sell all your resources** at a fair (minimum) price
- Selling must be more convenient than producing:
  - Product 1 (price 1, needs 1× resource 1 and 3): \[ y_1 + y_3 \geq 1 \]
  - Product 2 (price 2, needs 2× resource 2 and 1× resource 3): \[ 2y_2 + y_3 \geq 2 \]
Linear optimization duality

Today, we learned to:

• **Dualize** linear optimization problems
• **Prove** weak and strong duality conditions
• **Interpret** simple dual optimization problems
Next lecture

More on duality:

• Game theoretic interpretation
• Complementary slackness
• Alternative systems