ORF522 – Linear and Nonlinear Optimization

5. The simplex method
In the worst case scenario I will need to go through all the vertex, which would be very costly. Is there a way to guarantee that in most cases this will not be the outcome? Is there a way to choose the starting point of the algorithm so we avoid the worst case?

How can we tell every basic feasible solution is non-degenerate from the problem settings? Do we need to compute all extreme points?
Recap
Standard form polyhedra

Definition

Standard form LP

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\]

Assumption

\( A \in \mathbb{R}^{m \times n} \) has full row rank \( m \leq n \)

Interpretation

\( P \) lives in \( (n - m) \)-dimensional subspace
Standard form polyhedra

Visualization

\[ P = \{ x \mid Ax = b, \ x \geq 0 \}, \quad n - m = 2 \]

Three dimensions

Higher dimensions
Neighboring solutions

Two basic solutions are neighboring if their basic indices differ by exactly one variable.

Example

\[
A = \begin{bmatrix}
1 & -1 & 0 & 3 & -2 \\
2 & 0 & -1 & -1 & 0 \\
0 & 2 & 4 & -1 & 4 \\
\end{bmatrix}, \quad b = \begin{bmatrix}
-5 \\
-1 \\
14 \\
\end{bmatrix}
\]

\[
A_B x_B = b \quad \Rightarrow \quad x_B = \begin{bmatrix}
x_1 \\
x_3 \\
x_5 \\
\end{bmatrix} = \begin{bmatrix}
0 \\
1 \\
2.5 \\
\end{bmatrix}
\]

\[
B = \{1, 3, 5\} \quad x_2 = x_4 = 0
\]

\[
\bar{B} = \{1, 3, 4\} \quad y_2 = y_5 = 0
\]

\[
A_{\bar{B}} y_{\bar{B}} = b \quad \Rightarrow \quad y_{\bar{B}} = \begin{bmatrix}
y_1 \\
y_3 \\
y_4 \\
\end{bmatrix} = \begin{bmatrix}
0.1 \\
3.0 \\
-1.7 \\
\end{bmatrix}
\]

\[
B = \{1, 3, 5\} \quad x_2 = x_4 = 0
\]

\[
\bar{B} = \{1, 3, 4\} \quad y_2 = y_5 = 0
\]
Feasible directions

Conditions

\[ P = \{ x \mid Ax = b, \ x \geq 0 \} \]

Given a basis matrix \( A_B = \begin{bmatrix} A_{B(1)} & \cdots & A_{B(m)} \end{bmatrix} \)

we have basic feasible solution \( x \):

- \( x_B \) solves \( A_B x_B = b \)
- \( x_i = 0, \ \forall i \neq B(1), \ldots, B(m) \)

Let \( x \in P \), a vector \( d \) is a feasible direction at \( x \)
if \( \exists \theta > 0 \) for which \( x + \theta d \in P \)

Feasible direction \( d \)

- \( A(x + \theta d) = b \implies Ad = 0 \)
- \( x + \theta d \geq 0 \)
Feasible directions

Computation

Nonbasic indices

- \( d_j = 1 \)
- \( d_k = 0, \forall k \notin \{j, B(1), \ldots, B(m)\} \)

Basic indices

\[
Ad = 0 = \sum_{i=1}^{n} A_i d_i = A_B d_B + A_j = 0 \implies d_B = -A_B^{-1} A_j
\]

Non-negativity (non-degenerate assumption)

- Non-basic variables: \( x_i = 0 \). Nonnegative direction \( d_i \geq 0 \)
- Basic variables: \( x_B > 0 \). Therefore \( \exists \theta > 0 \) such that \( x_B + \theta d_B \geq 0 \)
Stepsize

What happens if some \( \bar{c}_j < 0 \)?
We can decrease the cost by bringing \( x_j \) into the basis

How far can we go?

\[
\theta^* = \max\{\theta \mid \theta \geq 0 \text{ and } x + \theta d \geq 0\}
\]

\( d \) is the \( j \)-th basic direction

Unbounded
If \( d \geq 0 \), then \( \theta^* = \infty \). The LP is unbounded.

Bounded
If \( d_i < 0 \) for some \( i \), then

\[
\theta^* = \min_{\{i \mid d_i < 0\}} \left( \frac{-x_i}{d_i} \right) = \min_{\{i \in B \mid d_i < 0\}} \left( \frac{-x_i}{d_i} \right)
\]

(Since \( d_i \geq 0 \), \( i \notin B \))
Moving to a new basis

Next feasible solution

\[ x + \theta^* d \]

Let \( B(\ell) \in \{B(1), \ldots, B(m)\} \) be the index such that \( \theta^* = -\frac{x_{B(\ell)}}{d_{B(\ell)}} \). Then,

\[ x_{B(\ell)} + \theta^* d_{B(\ell)} = 0 \]

New solution

- \( x_{B(\ell)} \) becomes 0 (exits)
- \( x_j \) becomes \( \theta^* \) (enters)

New basis

\[ A_{\bar{B}} = \begin{bmatrix} A_{B(1)} & \ldots & A_{B(\ell-1)} & A_j & A_{B(\ell+1)} & \ldots & A_{B(m)} \end{bmatrix} \]
An iteration of the simplex method

Initialization
- a basic feasible solution \( x \)
- a basis matrix \( A_B = \begin{bmatrix} A_B(1) & \ldots & A_B(m) \end{bmatrix} \)

Iteration steps

<table>
<thead>
<tr>
<th>Step</th>
<th>Description</th>
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</thead>
<tbody>
<tr>
<td>1.</td>
<td>Compute the reduced costs ( \bar{c} )</td>
</tr>
<tr>
<td></td>
<td>- Solve ( A_B^T p = c_B )</td>
</tr>
<tr>
<td></td>
<td>- ( \bar{c} = c - A^T p )</td>
</tr>
<tr>
<td>2.</td>
<td>If ( \bar{c} \geq 0 ), ( x ) optimal. break</td>
</tr>
<tr>
<td>3.</td>
<td>Choose ( j ) such that ( \bar{c}_j &lt; 0 )</td>
</tr>
<tr>
<td>4.</td>
<td>Compute search direction ( d ) with</td>
</tr>
<tr>
<td></td>
<td>( d_j = 1 ) and ( A_B d_B = -A_j )</td>
</tr>
<tr>
<td>5.</td>
<td>If ( d_B \geq 0 ), the problem is unbounded</td>
</tr>
<tr>
<td></td>
<td>and the optimal value is (-\infty). break</td>
</tr>
<tr>
<td>6.</td>
<td>Compute step length ( \theta^* = \min_{{i \in B</td>
</tr>
<tr>
<td>7.</td>
<td>Define ( y ) such that ( y = x + \theta^* d )</td>
</tr>
<tr>
<td>8.</td>
<td>Get new basis ( \tilde{B} ) (( i ) exits and ( j ) enters)</td>
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Today’s agenda
[Chapter 3, LO]

• Find initial feasible solution
• Degeneracy
• Complexity
Find an initial point in simplex method
Initial basic feasible solution

minimize $c^T x$
subject to $Ax = b$
$x \geq 0$

How do we get an initial basic feasible solution $x$ and a basis $B$?

Does it exist?
Finding an initial basic feasible solution

**Auxiliary problem**

Minimize \( c^T x \)

subject to

\[
Ax = b \\
x \geq 0
\]

Minimize \( \mathbf{1}^T y \)

subject to

\[
Ax + y = b \\
x \geq 0, y \geq 0
\]

**Assumption** \( b \geq 0 \) w.l.o.g. (if not multiply constraint by \(-1\))

**Trivial** basic feasible solution: \( x = 0, y = b \)

**Possible outcomes**

- **Feasible problem** (cost = 0): \( y^* = 0 \) and \( x^* \) is a basic feasible solution
- **Infeasible problem** (cost > 0): \( y^* > 0 \) are the violations
Two-phase simplex method

Phase I
1. Construct auxiliary problem such that $b \geq 0$
2. Solve auxiliary problem using simplex method starting from $(x, y) = (0, b)$
3. If the optimal value is greater than 0, problem infeasible. break.

Phase II
1. Recover original problem (drop variables $y$ and restore original cost)
2. Solve original problem starting from the solution $x$ and its basis $B$. 
Big-M method

minimize \( c^T x + M 1^T y \)
subject to \( Ax + y = b \)
\( x \geq 0, y \geq 0 \)

**Incorporate penalty in the cost**

- We can still use \( y = b \geq 0 \) as initial basic feasible solution
- If the problem is **feasible**, \( y \) will not be in the basis.

**Remarks**

- **Pro:** need to solve only one LP
- **Con:** it is not easy to pick \( M \) and it makes the problem badly scaled
Degeneracy
Degenerate basic feasible solutions

Inequality form polyhedron

A solution \( y \) is degenerate if \( |\mathcal{I}(\bar{x})| > n \)

\[
P = \{ x \mid Ax \leq b \}
\]
Degenerate basic feasible solutions

Standard form polyhedron

Given a basis matrix $A_B = \begin{bmatrix} A_B(1) & \ldots & A_B(m) \end{bmatrix}$

we have basic feasible solution $x$:

- $A_B x_B = b$
- $x_i = 0$, $\forall i \neq B(1), \ldots, B(m)$

If some of the $x_B = 0$, then it is a degenerate solution

$$P = \{ x \mid Ax = b, \ x \geq 0 \}$$
Degenerate basic feasible solutions

Example

\[
x_1 + x_2 + x_3 = 1 \\
-x_1 + x_2 - x_3 = 1 \\
x_1, x_2, x_3 \geq 0
\]

Degenerate solutions

Basis \( B = \{1, 2\} \) \[\rightarrow\] \( x = (0, 1, 0) \)

Basis \( B = \{2, 3\} \) \[\rightarrow\] \( y = (0, 1, 0) \)
Cycling

Stepsize

6. Compute step length \( \theta^* = \min_{\{i \in B | d_i < 0\}} \left( -\frac{x_i}{d_i} \right) \)

If \( i \in B \), \( d_i < 0 \) and \( x_i = 0 \) (degenerate)

\[ \theta^* = 0 \]

Therefore \( y = x + \theta^*x = x \) and \( B = \bar{B} \)

Same solution and cost

Different basis

Finite termination no longer guaranteed!

How can we fix it?

Pivoting rules
Pivoting rules
Choose the index entering the basis

Simplex iterations
3. Choose \( j \) such that \( \bar{c}_j < 0 \)  \hspace{1cm} \text{Which } j?\]

Possible rules
- **Smallest subscript**: smallest \( j \) such that \( \bar{c}_j < 0 \)
- **Most negative**: choose \( j \) with the most negative \( \bar{c}_j \)
- **Largest cost decrement**: choose \( j \) with the largest \( \theta^*|\bar{c}_j| \)
Pivoting rules
Choose index exiting the basis

Simplex iterations

6. Compute step length \( \theta^* = \min_{\{i \in B | d_i < 0\}} \left( -\frac{x_i}{d_i} \right) \)

We can have more than one \( i \) for which \( x_i = 0 \)
(next solution is degenerate)

Which \( i \)?

Smallest index rule

Smallest \( i \) such that \( \theta^* = -\frac{x_i}{d_i} \)
Bland’s rule to avoid cycles

Theorem
If we use the **smallest index rule** for choosing both the \( j \) entering the basis and the \( i \) leaving the basis, then no cycling will occur.

Proof idea [Ch 3, Sec 4, LP][Sec 3.4, LO]
- Assume Bland’s rule is applied and there exists a cycle with different bases.
- Obtain contradiction.
Perturbation approach to avoid cycles

\[ Ax = b + \epsilon \]

\[ x_1 = 0 \]
\[ x_2 = 0 \]
\[ x_3 = 0 \]
\[ x_4 = 0 \]
\[ x_5 = 0 \]
\[ x_6 = 0 \]
Complexity
Complexity

**Basic operation:** one simplex iteration

**Estimate complexity of an algorithm**
- Write number of basic operations as a *function of problem dimensions*
- Simplify and keep only leading terms
Complexity

Notation
We write $g(x) \sim O(f(x))$ if and only if there exist $c > 0$ and an $x_0$ such that

$$|g(x)| \leq cf(x), \quad \forall x \geq x_0$$

Polynomial
Practical

Exponential
Impractical!
\( \mathcal{P} \) and \( \mathcal{NP} \)

**Complexity class \( \mathcal{P} \)**
There exists a polynomial time algorithms to solve it.

**Complexity class \( \mathcal{NP} \)**
Given a candidate solution, there exists a polynomial time algorithm to verify it.

**Complexity class \( \mathcal{NP} \)-hard**
At least as hard as the hardest problem in \( \mathcal{NP} \)

We don’t know any polynomial time algorithm.

**Million dollar problem**: \( \mathcal{P} = \mathcal{NP} \)?
- We know that \( \mathcal{P} \subset \mathcal{NP} \)
- Does it exist a polynomial time algorithm for \( \mathcal{NP} \)-hard problems?
Complexity of the simplex method

Example of worst-case behavior

Innocent-looking problem

minimize \(-x_n\)
subject to \(0 \leq x \leq 1\)

\[
\begin{align*}
\text{2}^n & \text{ vertices} \\
\text{2}^n/2 & \text{ vertices: cost = 1} \\
\text{2}^n/2 & \text{ vertices: cost = 0}
\end{align*}
\]

Perturb unit cube

minimize \(-x_n\)
subject to \(\epsilon \leq x_1 \leq 1\)

\[
\epsilon x_{i-1} \leq x_i \leq 1 - \epsilon x_{i-1}, \quad i = 2, \ldots, n
\]
Complexity of the simplex method

Example of worst-case behavior

minimize \(-x_n\)
subject to \(\epsilon \leq x_1 \leq 1\)
\(\epsilon x_{i-1} \leq x_i \leq 1 - \epsilon x_{i-1}, \quad i = 2, \ldots, n\)

Theorem

- The vertices can be ordered so that each one is adjacent to and has a lower cost than the previous one
- There exists a pivoting rule under which the simplex method terminates after \(2^n - 1\) iterations

Remark

- A different pivot rule would have converged in one iteration.
- We have a bad example for every pivot rule.
Complexity of the simplex method

We do not know any polynomial version of the simplex method, no matter which pivoting rule we pick.

Still open research question!

Worst-case
There are problem instances where the simplex method will run an exponential number of iterations in terms of the dimensions $n$ and $m$: $O(2^n)$

Good news: average-case
Practical performance is very good. On average, it stops in $O(n)$ iterations.
The simplex method

Today, we learned to:

• **Formulate** auxiliary problem to find starting simplex solutions
• **Apply** pivoting rules to avoid cycling in degenerate linear programs
• **Analyze** complexity of the simplex method
Next lecture

- Numerical linear algebra
- "Realistic" simplex implementation
- Examples