

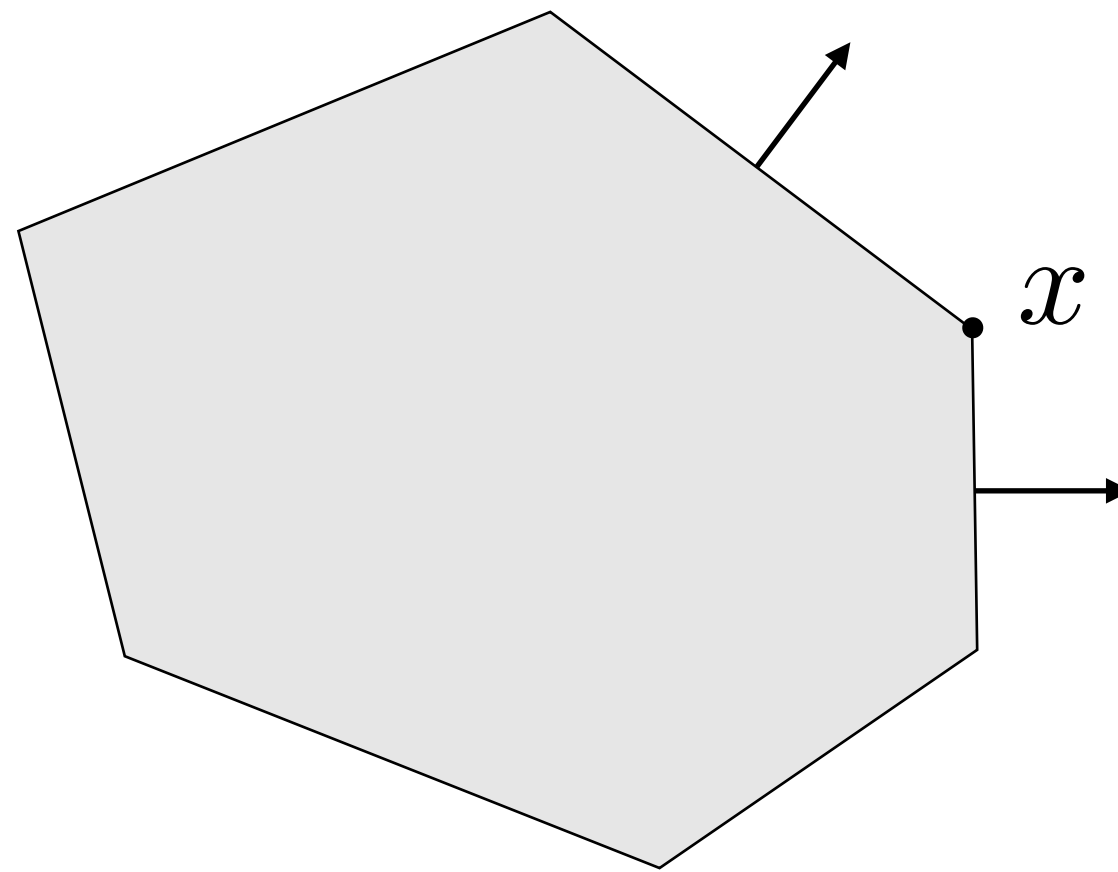
ORF522 – Linear and Nonlinear Optimization

4. The simplex method

Recap

Equivalence Theorem

Given a nonempty polyhedron $P = \{x \mid Ax \leq b\}$



Let $x \in P$

x is a **vertex** $\iff x$ is an **extreme point** $\iff x$ is a **basic feasible solution**

Basic feasible solution

$$P = \{x \mid a_i^T x \leq b_i, \quad i = 1, \dots, m\}$$

Active constraints at \bar{x}

$$\mathcal{I}(\bar{x}) = \{i \in \{1, \dots, m\} \mid a_i^T \bar{x} = b_i\}$$

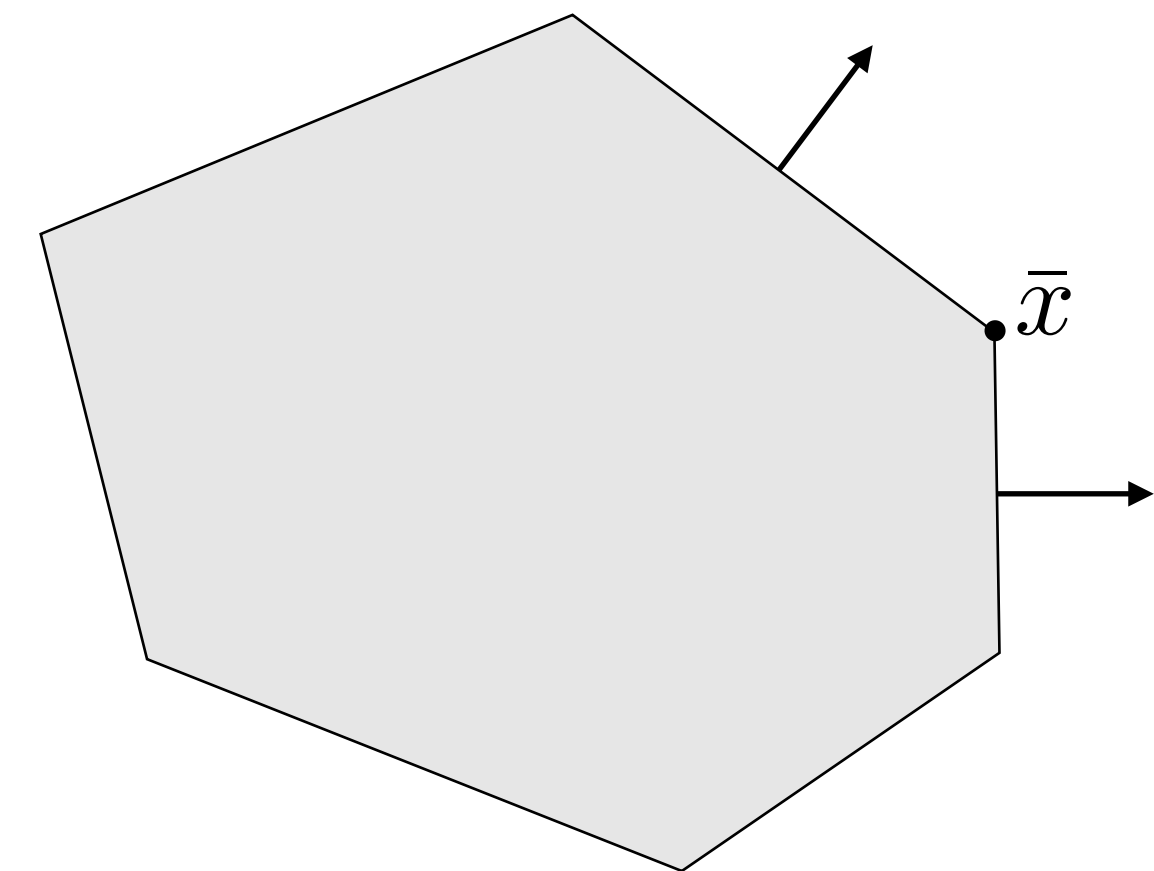
Index of all the constraints
satisfied as **equality**

Basic solution \bar{x}

- $\{a_i \mid i \in \mathcal{I}(\bar{x})\}$ has n linearly independent vectors

Basic feasible solution \bar{x}

- $\bar{x} \in P$
- $\{a_i \mid i \in \mathcal{I}(\bar{x})\}$ has n linearly independent vectors



Basic solutions

Standard form polyhedra

$$P = \{x \mid Ax = b, x \geq 0\} \quad \text{with} \quad A \in \mathbf{R}^{m \times n} \text{ has full row rank } m \leq n$$

Theorem

x is a **basic solution** if and only if

- $Ax = b$
- There exist indices $B(1), \dots, B(m)$ such that
 - columns $A_{B(1)}, \dots, A_{B(m)}$ are linearly independent
 - $x_i = 0$ for $i \neq B(1), \dots, B(m)$

x is a **basic feasible solution** if x is a **basic solution** and $x \geq 0$

Intuition: from geometry to standard form

$$\begin{array}{l}
 \text{minimize } c^T x \\
 \text{subject to } Ax \leq b
 \end{array}
 \longrightarrow
 \begin{array}{l}
 \text{minimize } c^T (x^+ - x^-) \\
 \text{subject to } \begin{bmatrix} A & -A & I \end{bmatrix} \begin{bmatrix} x^+ \\ x^- \\ s \end{bmatrix} = b \\
 (x^+, x^-, s) \geq 0
 \end{array}
 \longrightarrow
 \begin{array}{l}
 \text{minimize } \tilde{c}^T \tilde{x} \\
 \text{subject to } \tilde{A} \tilde{x} = b \\
 \tilde{x} \geq 0
 \end{array}$$

Variables: $\tilde{n} = 2n + m$

(Equality) constraints: $\tilde{m} = m \implies$ **active**

For a **basic solution** \longrightarrow We need $\tilde{n} - \tilde{m} = 2n$
 active inequalities $\implies \tilde{x}_i = 0$ (non basic)

Which corresponds to m inequalities inactive $\implies \tilde{x}_i > 0$ (basic)

Constructing basic solution

1. Choose any m independent columns of A : $A_{B(1)}, \dots, A_{B(m)}$
2. Let $x_i = 0$ for all $i \neq B(1), \dots, B(m)$
3. Solve $Ax = b$ for the remaining $x_{B(1)}, \dots, x_{B(m)}$

Basis
matrix

Basis columns

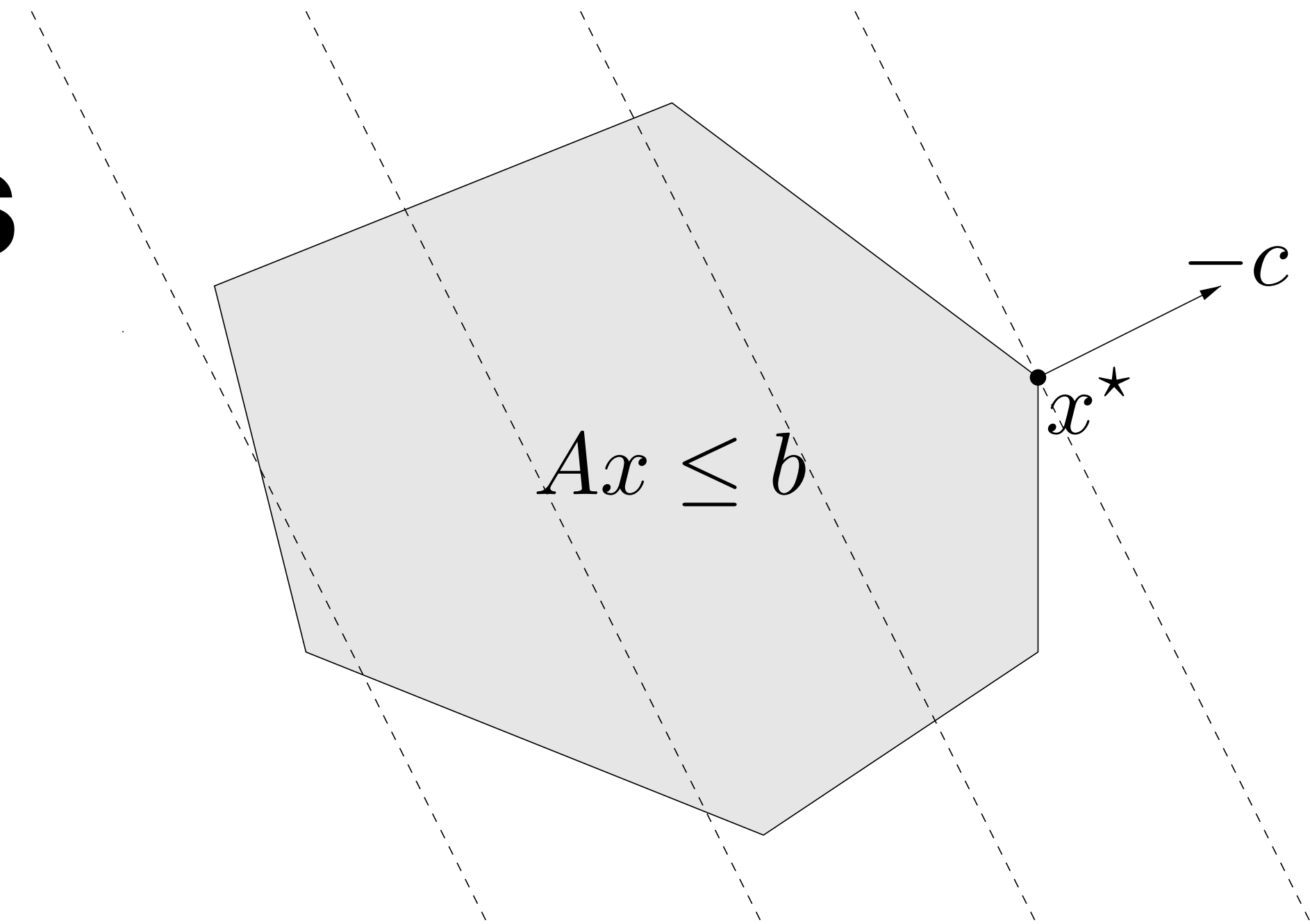
Basic variables

$$A_B = \begin{bmatrix} | & | & & | \\ A_{B(1)} & A_{B(2)} & \dots & A_{B(m)} \\ | & | & & | \end{bmatrix}, \quad x_B = \begin{bmatrix} x_{B(1)} \\ \vdots \\ x_{B(m)} \end{bmatrix} \longrightarrow \text{Solve } A_B x_B = b$$

If $x_B \geq 0$, then x is a **basic feasible solution**

Optimality of extreme points

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$



Theorem

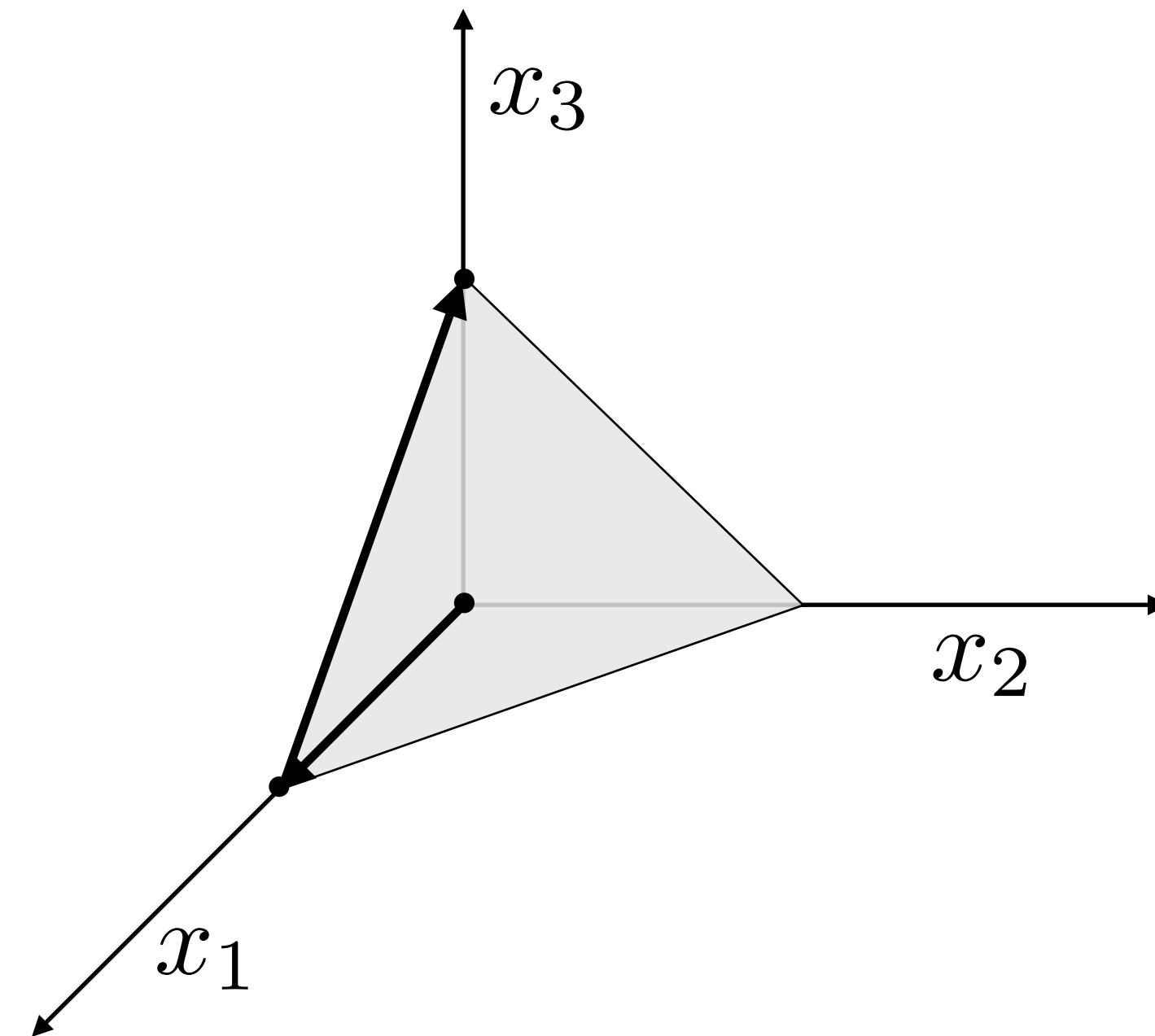
- If
- P has at least one extreme point
 - There exists an optimal solution x^*

Then, there exists an optimal solution which is an **extreme point** of P

We only need to search between **extreme points**

Conceptual algorithm

- Start at corner
- Visit neighboring corner that improves the objective



Today's agenda

Readings: [Chapter 3, LO]

Simplex method

- Iterate between neighboring basic solutions
- Optimality conditions
- Simplex iterations

The simplex method

Top 10 algorithms of the 20th century

1946: Metropolis algorithm

1947: Simplex method 

1950: Krylov subspace method

1951: The decompositional approach to matrix computations

1957: The Fortran optimizing compiler

1959: QR algorithm

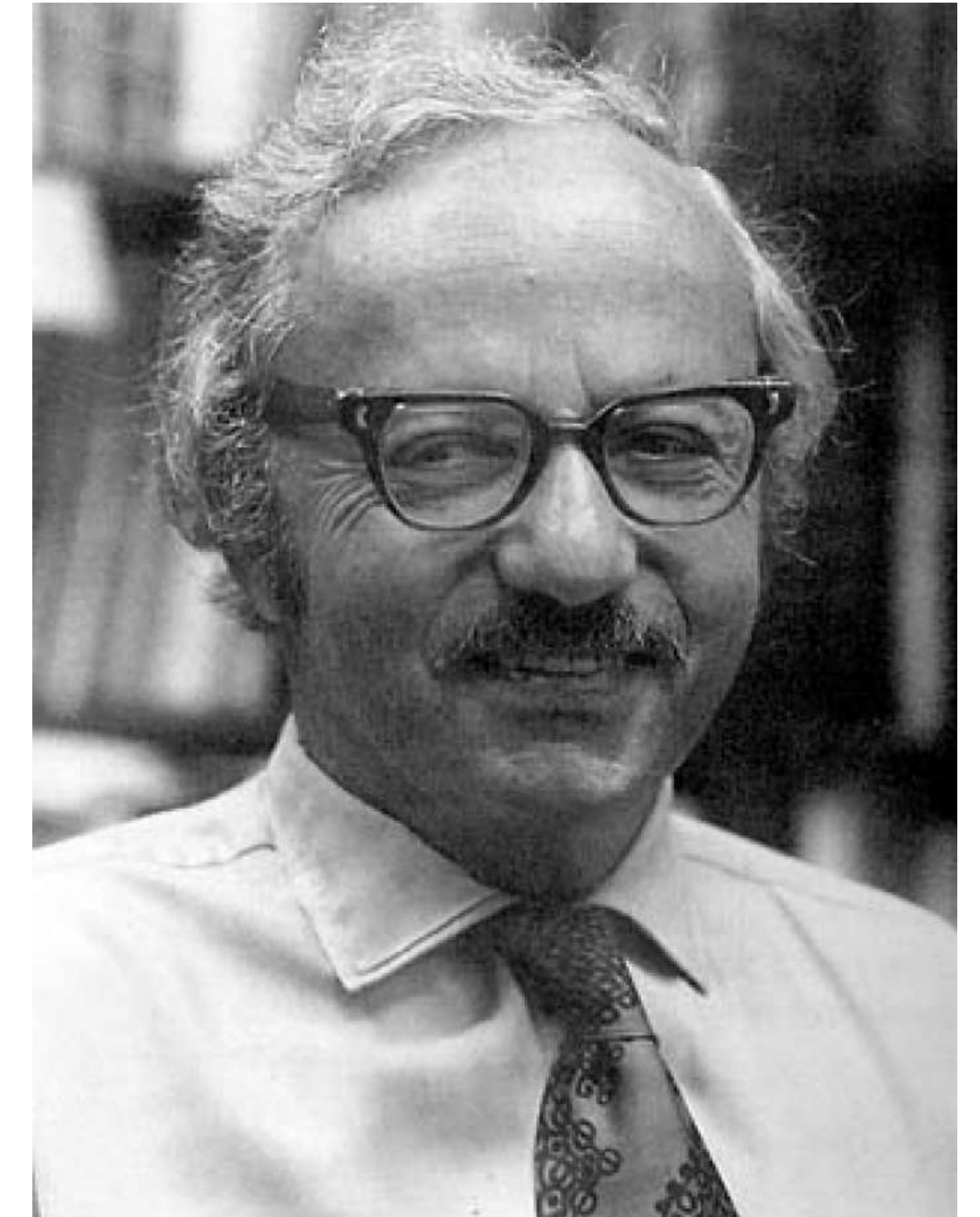
1962: Quicksort

1965: Fast Fourier transform

1977: Integer relation detection

1987: Fast multipole method

George Dantzig



Neighboring basic solutions

Feasible directions

Conditions

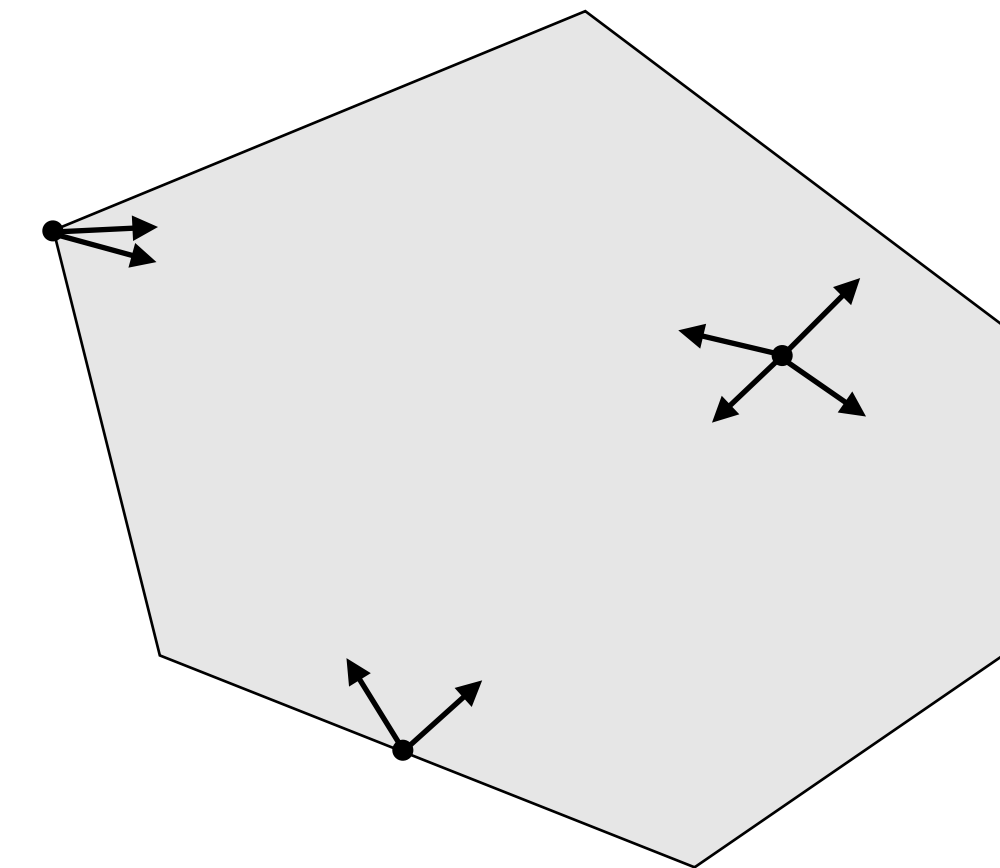
$$P = \{x \mid Ax = b, x \geq 0\}$$

Given a basis matrix $A_B = \begin{bmatrix} A_{B(1)} & \dots & A_{B(m)} \end{bmatrix}$

we have basic feasible solution x :

- x_B solves $A_B x_B = b$
- $x_i = 0, \forall i \neq B(1), \dots, B(m)$

Let $x \in P$, a vector d is a **feasible direction** at x if $\exists \theta > 0$ for which $x + \theta d \in P$



Feasible direction d

- $A(x + \theta d) = b \implies Ad = 0$
- $x + \theta d \geq 0$

Feasible directions

Computation

Feasible direction d

- $A(x + \theta d) = b \implies Ad = 0$
- $x + \theta d \geq 0$

Nonbasic indices

- $d_j = 1 \longrightarrow$ **Basic direction**
- $d_k = 0, \forall k \notin \{j, B(1), \dots, B(m)\}$

Basic indices

$$Ad = 0 = \sum_{i=1}^n A_i d_i = A_B d_B + A_j = 0 \implies d_B = -A_B^{-1} A_j$$

Non-negativity (non-degenerate assumption)

- Non-basic variables: $x_i = 0$. Nonnegative direction $d_i \geq 0$
- Basic variables: $x_B > 0$. Therefore $\exists \theta > 0$ such that $x_B + \theta d_B \geq 0$

Feasible directions

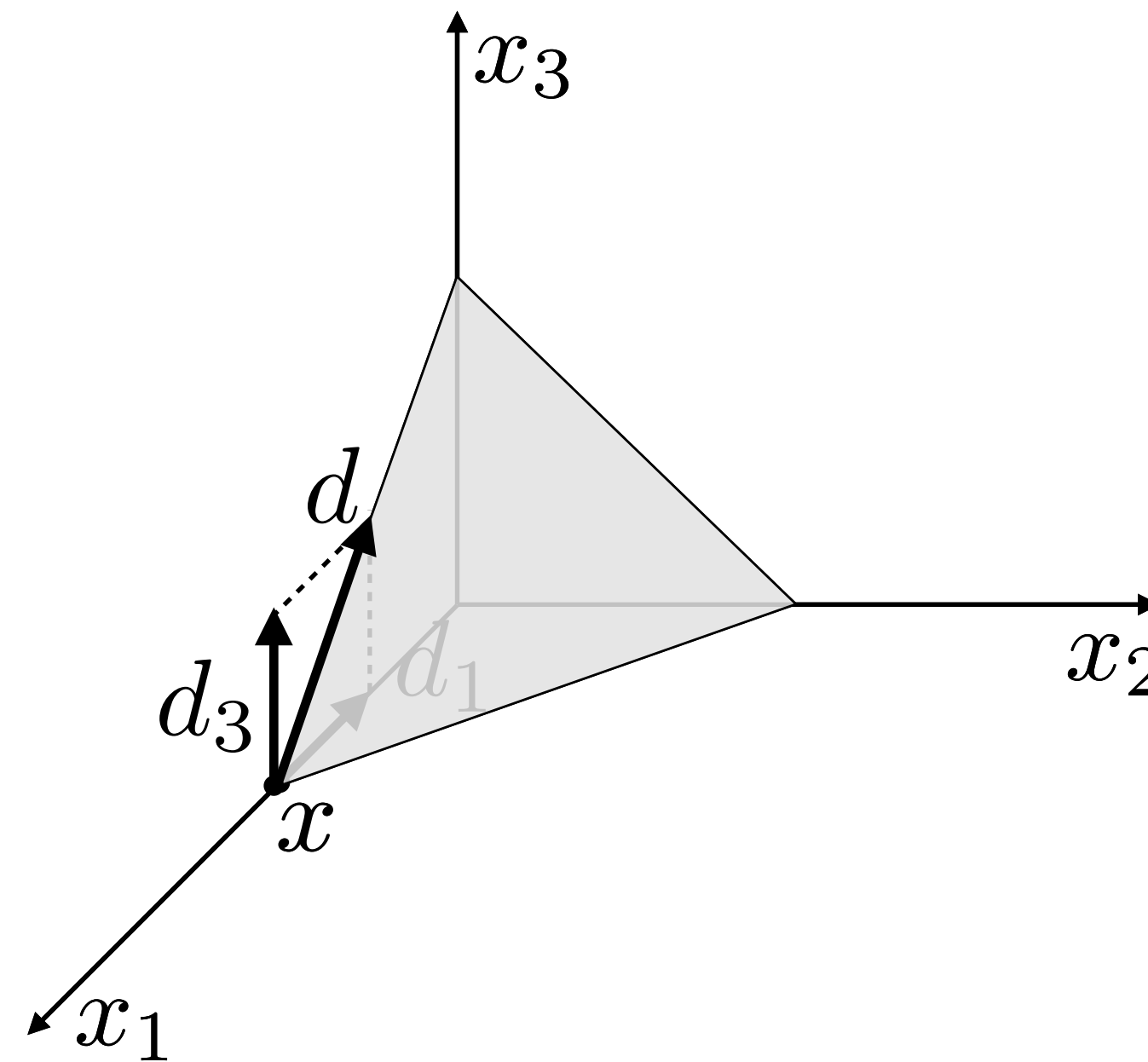
Example

$$P = \{x \mid x_1 + x_2 + x_3 = 2, \quad x \geq 0\}$$

$$x = (2, 0, 0) \quad B = \{1\}$$

Nonbasic index $j = 3 \longrightarrow d = (-1, 0, 1)$

$$A_B d_B = -A_j \quad \Rightarrow \quad d_j = 1$$
$$A_B d_B = -A_j \quad \Rightarrow \quad d_B = -1$$



How does the cost change?

Cost improvement

$$c^T(x + \theta d) - c^T x = \theta c^T d$$

New cost

Old cost

We call \bar{c}_j the **reduced cost** of (introducing) variable x_j in the basis

$$\bar{c}_j = c^T d = \sum_{i=1}^n c_i d_i = c_j + c_B^T d_B = c_j - c_B^T A_B^{-1} A_j$$

Reduced costs

Interpretation

Change in objective/marginal cost of adding x_j to the basis

$$\bar{c}_j = c_j - c_B^T A_B^{-1} A_j$$

Cost per-unit increase
of variable x_j

Cost to change other variables
compensating for x_j
to enforce $Ax = b$

- $\bar{c}_j > 0$: adding x_j will increase the objective (bad)
- $\bar{c}_j < 0$: adding x_j will decrease the objective (good)

Reduced costs for basic variables is 0

$$\begin{aligned}\bar{c}_{B(i)} &= c_{B(i)} - c_B^T A_B^{-1} A_{B(i)} = c_{B(i)} - c_B^T (A_B^{-1} A_B) e_i \\ &= c_{B(i)} - c_B^T e_i = c_{B(i)} - c_{B(i)} = 0\end{aligned}$$

Vector of reduced costs

Reduced costs

$$\bar{c}_j = c_j - c_B^T A_B^{-1} A_j$$

Full vector in one shot?

$$\bar{c} = (\bar{c}_1, \dots, \bar{c}_n)$$

Isolate basis B -related components p
(they are the same across j)

$$\bar{c}_j = c_j - A_j^T (A_B^{-1})^T c_B = c_j - A_j^T p$$

Obtain p by solving linear system

$$p = (A_B^{-1})^T c_B \quad \Rightarrow \quad A_B^T p = c_B$$

Note: $(M^{-1})^T = (M^T)^{-1}$
for any square invertible M

Computing reduced cost vector

1. Solve $A_B^T p = c_B$
2. $\bar{c} = c - A^T p$

Optimality conditions

Optimality conditions

Theorem

Let x be a basic feasible solution associated with basis matrix A_B
Let \bar{c} be the vector of reduced costs.

If $\bar{c} \geq 0$, then x is **optimal**

Remark

This is a **stopping criterion** for the simplex algorithm.

If the **neighboring solutions** do not improve the cost, we are done (because of convexity).

Optimality conditions

Proof

For a basic feasible solution x with basis B the reduced costs are $\bar{c} \geq 0$.

Consider any feasible solution y and define $d = y - x$

Since x and y are feasible, then $Ax = Ay = b$ and $Ad = 0$

$$Ad = A_B d_B + \sum_{i \in N} A_i d_i = 0 \quad \Rightarrow \quad d_B = - \sum_{i \in N} A_B^{-1} A_i d_i$$

N are the
nonbasic indices

The change in objective is

$$c^T d = c_B^T d_B + \sum_{i \in N} c_i d_i = \sum_{i \in N} (c_i - c_B^T A_B^{-1} A_i) d_i = \sum_{i \in N} \bar{c}_i d_i$$

Since $y \geq 0$ and $x_i = 0, i \in N$, then $d_i = y_i - x_i \geq 0, i \in N$

$$c^T d = c^T (y - x) \geq 0 \quad \Rightarrow \quad c^T y \geq c^T x.$$



Simplex iterations

Stepsize

What happens if some $\bar{c}_j < 0$?

We can decrease the cost by bringing x_j into the basis

How far can we go?

$$\theta^* = \max\{\theta \mid \theta \geq 0 \text{ and } x + \theta d \geq 0\} \quad d \text{ is the } j\text{-th basic direction}$$

Unbounded

If $d \geq 0$, then $\theta^* = \infty$. The LP is unbounded.

Bounded

If $d_i < 0$ for some i , then

$$\theta^* = \min_{\{i \mid d_i < 0\}} \left(-\frac{x_i}{d_i} \right) = \min_{\{i \in B \mid d_i < 0\}} \left(-\frac{x_i}{d_i} \right)$$

(Since $d_i \geq 0$, $i \notin B$)

Moving to a new basis

Next feasible solution

$$x + \theta^* d$$

Let $B(\ell) \in \{B(1), \dots, B(m)\}$ be the index such that $\theta^* = -\frac{x_{B(\ell)}}{d_{B(\ell)}}$. Then,

$$x_{B(\ell)} + \theta^* d_{B(\ell)} = 0$$

New solution

- $x_{B(\ell)}$ becomes 0 (exits)
- x_j becomes θ^* (enters)

New basis

$$A_{\bar{B}} = \left[A_{B(1)} \quad \dots \quad A_{B(\ell-1)} \quad A_j \quad A_{B(\ell+1)} \quad \dots \quad A_{B(m)} \right]$$

An iteration of the simplex method

Initialization

- a basic feasible solution x
- a basis matrix $A_B = \left[A_{B(1)} \quad \dots, A_{B(m)} \right]$

Iteration steps

1. Compute the reduced costs \bar{c}
 - Solve $A_B^T p = c_B$
 - $\bar{c} = c - A^T p$
2. If $\bar{c} \geq 0$, x **optimal. break**
3. Choose j such that $\bar{c}_j < 0$
4. Compute search direction d with $d_j = 1$ and $A_B d_B = -A_j$
5. If $d_B \geq 0$, the problem is **unbounded** and the optimal value is $-\infty$. **break**
6. Compute step length $\theta^* = \min_{\{i \in B \mid d_i < 0\}} \left(-\frac{x_i}{d_i} \right)$
7. Define y such that $y = x + \theta^* d$
8. Get new basis \bar{B} (i exits and j enters)

Example

$$P = \{x \mid x_1 + x_2 + x_3 = 2, \quad x \geq 0\}$$

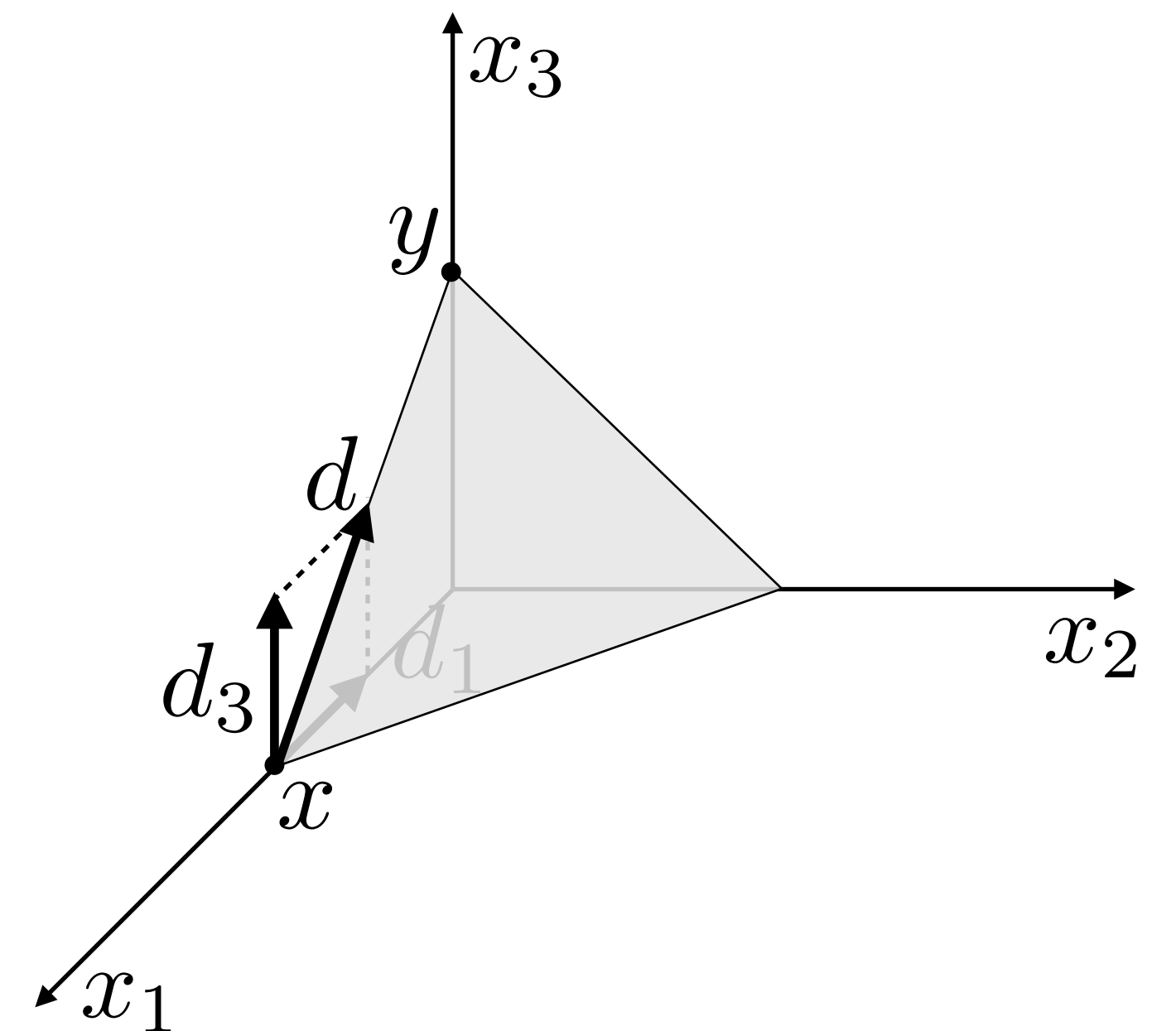
$$x = (2, 0, 0) \quad B = \{1\}$$

$$\text{Basic index } j = 3 \longrightarrow d = (-1, 0, 1) \\ d_j = 1$$

$$A_B d_B = -A_j \quad \Rightarrow \quad d_B = -1$$

$$\text{Stepsize } \theta^* = -\frac{x_1}{d_1} = 2$$

$$\text{New solution } y = x + \theta^* d = (0, 0, 2) \quad \bar{B} = \{3\}$$



Finite convergence

Theorem

- If**
- $P = \{x \mid Ax = b, x \geq 0\}$ not empty
 - Every basic feasible solution **non degenerate**

Then

- The simplex method **terminates after a finite number of iterations**
- At termination we either have one of the following
 - an **optimal basis** B
 - a **direction** d such that $Ad = 0, d \geq 0, c^T d < 0$ (with optimal cost $-\infty$)

Finite convergence

Proof sketch

At each iteration the algorithm improves

- by a **positive** amount θ^*
- along the **direction** d such that $c^T d < 0$

Therefore

- The cost strictly decreases
- No basic feasible solution can be visited twice

Since there is a **finite number of basic feasible solutions**

The algorithm **must eventually terminate**



The simplex method

Today, we learned to:

- **Iterate** between basic feasible solutions
- **Verify** optimality and unboundedness conditions
- **Apply** a single iteration of the simplex method
- **Prove** finite convergence of the simplex method in the non-degenerate case

Next lecture

- Finding initial basic feasible solution
- Degeneracy
- Complexity