ORF307 – Optimization

19. Linear optimization review
Ed Forum

• How do mathematicians know when to use "heuristics," or arbitrary measures to make the algorithm work better?

• Why is not being able to warm start a problem if we can start interior point methods with an infeasible solution?
Today’s lecture
Linear optimization review

- Formulations
- Piecewise linear optimization
- Duality
- Sensitivity analysis
- Simplex method
- Interior point methods
Formulations
Linear optimization

minimize \( c^T x \)
subject to \( Ax \leq b \)
\( Dx = f \)

- Minimization
- Less-than ineq. constraints
- Equality constraints

\( x \) is **feasible** if it satisfies the constraints \( Ax \leq b \) and \( Dx = f \)

The **feasible set** is the set of all feasible points

\( x^* \) is **optimal** if it is feasible and \( c^T x^* \leq c^T x \) for all feasible \( x \)

The **optimal value** is \( p^* = c^T x^* \)

**Unbounded problem:** \( c^T x \) is unbounded below on the feasible set \( (p^* = -\infty) \)

**Infeasible problem:** feasible set is empty \( (p^* = +\infty) \)
Feasibility problems

find \[ x \]
subject to \[ Ax \leq b \]
\[ Dx = f \]

minimize \[ 0 \]
subject to \[ Ax \leq b \]
\[ Dx = f \]

Possible results

\[ \begin{align*}
\text{\( p^* = 0 \)} & \quad \text{if constraints are feasible (consistent).} \\
& \quad \text{(Every feasible} \ x \ \text{is optimal)} \\
\text{\( p^* = \infty \)} & \quad \text{otherwise}
\end{align*} \]
Standard form

Definition

minimize $c^T x$
subject to $Ax = b$
$x \geq 0$

- Minimization
- Equality constraints
- Nonnegative variables

Useful to

- develop algorithms
- algebraic manipulations
Piecewise linear optimization
Piecewise-linear minimization

minimize \( f(x) = \max_{i=1,\ldots,m} (a_i^T x + b_i) \)

minimize \( t \)

subject to \( a_i^T x + b_i \leq t, \quad i = 1, \ldots, m \)
Piecewise-linear minimization

minimize \( f(x) = \max_{i=1,\ldots,m} (a_i^T x + b_i) \)

\[ \begin{align*} 
\text{minimize} & \quad t \\
\text{subject to} & \quad a_i^T x + b_i \leq t, \quad i = 1, \ldots, m 
\end{align*} \]

Matrix notation

minimize \( \tilde{c}^T \tilde{x} \)

subject to \( \tilde{A} \tilde{x} \leq \tilde{b} \)

\[ \begin{align*} 
\tilde{x} &= \begin{bmatrix} x \\ t \end{bmatrix}, \quad \tilde{c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} a_1^T & -1 \\ \vdots & \vdots \\ a_m^T & -1 \end{bmatrix}, \quad \tilde{b} = \begin{bmatrix} -b_1 \\ \vdots \\ -b_m \end{bmatrix} 
\end{align*} \]
1 and infinity norms reformulations

1-norm minimization:

\[ \text{minimize } \|Ax - b\|_1 = \sum_i |(Ax - b)_i| \]

Equivalent to:

\[ \text{minimize } \mathbf{1}^T u \]

subject to \(-u \leq Ax - b \leq u\)

\[ \text{Absolute value of every element } (Ax - b)_i \text{ is bounded by a component of the vector } u \]

\[ \text{min } |r_i| = > \text{min } 0_i \]

\[ \text{st.: } u \geq 0 \]

\[ \infty\text{-norm minimization:} \]

\[ \text{minimize } \|Ax - b\|_\infty = \max_i |(Ax - b)_i| \]

Equivalent to:

\[ \text{minimize } t \]

subject to \(-t\mathbf{1} \leq Ax - b \leq t\mathbf{1}\)

\[ \text{Absolute value of every element } (Ax - b)_i \text{ is bounded by the same scalar } t \]
Duality
Inequality form LP

minimize \[ c^T x \]
subject to \[ Ax \leq b \]
Inequality form LP

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax \leq b \\
\end{align*}
\]

Relax the constraint

\[
g(y) = \min_x \quad c^T x + y^T (Ax - b)
\]
Inequality form LP

\[
\text{minimize} \quad c^T x \\
\text{subject to} \quad Ax \leq b
\]

Relax the constraint

\[
g(y) = \min_x c^T x + y^T (Ax - b)
\]

Lower bound

\[
g(y) \leq c^T x^* + y^T (Ax^* - b) \leq c^T x^*
\]

we must have \( y \geq 0 \)
Inequality form LP

minimize \hspace{1cm} c^T x
subject to \hspace{1cm} Ax \leq b

Relax the constraint
\[ g(y) = \min_x c^T x + y^T (Ax - b) \]

Lower bound
\[ g(y) \leq c^T x^* + y^T (Ax^* - b) \leq c^T x^* \]
we must have \( y \geq 0 \)
Dual of LP with inequalities

Derivation

Dual function

\[ g(y) = \min_x \left( c^T x + y^T (Ax - b) \right) - b^T y + \min_x \left( c + A^T y \right)^T x \]
Dual of LP with inequalities

Derivation

Dual function

\[ g(y) = \min_x (c^T x + y^T (Ax - b)) \]

\[ -b^T y + \min_x (c + A^T y)^T x \]

\[ g(y) = \begin{cases} 
-b^T y & \text{if } c + A^T y = 0 \ (\text{and } y \geq 0) \\
-\infty & \text{otherwise}
\end{cases} \]
Dual of LP with inequalities

Derivation

Dual function

\[
g(y) = \min_x (c^T x + y^T (Ax - b))
- b^T y + \min_x (c + A^T y)^T x
\]

\[
g(y) = \begin{cases} 
-b^T y & \text{if } c + A^T y = 0 \quad \text{(and } y \geq 0) \\
-\infty & \text{otherwise}
\end{cases}
\]

Dual problem (find the best bound)

maximize \[ g(y) = \max_y -b^T y \]
subject to \[ A^T y + c = 0 \]
\[ y \geq 0 \]
General forms

Inequality form LP

minimize \( c^T x \)
subject to \( Ax \leq b \)
maximize \( -b^T y \)
subject to \( A^T y + c = 0 \)
\( y \geq 0 \)

Standard form LP

minimize \( c^T x \)
subject to \( Ax = b \)
\( x \geq 0 \)
maximize \( -b^T y \)
subject to \( A^T y + c \geq 0 \)

LP with inequalities and equalities

minimize \( c^T x \)
subject to \( Ax \leq b \)
\( Dx = f \)
maximize \( -b^T y - f^T z \)
subject to \( A^T y + D^T z + c = 0 \)
\( y \geq 0 \)
Weak duality

**Theorem**
If $x, y$ satisfy:

- $x$ is a feasible solution to the primal problem
- $y$ is a feasible solution to the dual problem

\[ -b^T y \leq c^T x \]
Weak duality

Theorem
If $x, y$ satisfy:

- $x$ is a feasible solution to the primal problem
- $y$ is a feasible solution to the dual problem

\[ -b^T y \leq c^T x \]

Proof
We know that $Ax \leq b$, $A^T y + c = 0$ and $y \geq 0$. Therefore,

\[ 0 \leq y^T (b - Ax) = b^T y - y^T Ax = c^T x + b^T y \]
Weak duality

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Remark
- Any dual feasible $y$ gives a **lower bound** on the primal optimal value
- Any primal feasible $x$ gives an **upper bound** on the dual optimal value
- $c^T x + b^T y$ is the **duality gap**
Weak duality
Corollaries

Unboundedness vs feasibility
• Primal unbounded \((p^* = -\infty)\) \(\Rightarrow\) dual infeasible \((d^* = -\infty)\)
• Dual unbounded \((d^* = +\infty)\) \(\Rightarrow\) primal infeasible \((p^* = +\infty)\)
Weak duality

Corollaries

Unboundedness vs feasibility

• Primal unbounded \( p^* = -\infty \) ⇒ dual infeasible \( d^* = -\infty \)
• Dual unbounded \( d^* = +\infty \) ⇒ primal infeasible \( p^* = +\infty \)

Optimality condition

If \( x, y \) satisfy:

• \( x \) is a feasible solution to the primal problem
• \( y \) is a feasible solution to the dual problem
• The duality gap is zero, i.e., \( c^T x + b^T y = 0 \)

Then \( x \) and \( y \) are optimal solutions to the primal and dual problem respectively
Strong duality

Primal
minimize \( c^T x \)
subject to \( Ax = b \)
\( x \geq 0 \)

Dual
maximize \( -b^T y \)
subject to \( A^T y + c \geq 0 \)

Theorem
If a linear optimization problem has an optimal solution, then

- so does its dual
- the optimal values of the primal and dual are equal
## Relationship between primal and dual

<table>
<thead>
<tr>
<th>$d^* = +\infty$</th>
<th>$p^* = +\infty$</th>
<th>$p^*$ finite</th>
<th>$p^* = -\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>primal inf.</td>
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Complementary slackness

\[ \begin{align*}
\text{Primal} & \quad \text{minimize} \quad c^T x \\
& \text{subject to} \quad Ax \leq b
\end{align*} \]

\[ \begin{align*}
\text{Dual} & \quad \text{maximize} \quad -b^T y \\
& \text{subject to} \quad A^T y + c = 0 \quad y \geq 0
\end{align*} \]

Theorem
Primal, dual feasible \( x, y \) are optimal if and only if

\[ y_i (b_i - a_i^T x) = 0, \quad i = 1, \ldots, m \]

i.e., at optimum, \( b - Ax \) and \( y \) have a \textbf{complementary sparsity} pattern:

\[ y_i > 0 \quad \Rightarrow \quad a_i^T x = b_i \]
\[ a_i^T x < b_i \quad \Rightarrow \quad y_i = 0 \]
Complementary slackness

**Primal**

minimize \( c^T x \)

subject to \( Ax \leq b \)

**Dual**

maximize \(-b^T y\)

subject to \( A^T y + c = 0 \)
\( y \geq 0 \)

Proof

The duality gap at primal feasible \( x \) and dual feasible \( y \) can be written as

\[
c^T x + b^T y = (-A^T y)^T x + b^T y = (b - Ax)^T y = \sum_{i=1}^{m} y_i (b_i - a_i^T x) = 0
\]
Complementary slackness

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| subject to $Ax \leq b$| subject to $A^T y + c = 0$
|                       | $y \geq 0$          |

Proof

The duality gap at primal feasible $x$ and dual feasible $y$ can be written as

$$c^T x + b^T y = (-A^T y)^T x + b^T y = (b - Ax)^T y = \sum_{i=1}^{m} y_i (b_i - a_i^T x) = 0$$

Since all the elements of the sum are nonnegative, they must all be 0 □
Complementary slackness

Primal
minimize \( c^T x \)
subject to \( Ax \leq b \)

Dual
maximize \(-b^T y\)
subject to \(A^T y + c = 0\)
\[ y \geq 0 \]

Proof
The duality gap at primal feasible \( x \) and dual feasible \( y \) can be written as
\[ c^T x + b^T y = (-A^T y)^T x + b^T y = (b - Ax)^T y = \sum_{i=1}^{m} y_i (b_i - a_i^T x) = 0 \]

Since all the elements of the sum are nonnegative, they must all be 0

For feasible \( x \) and \( y \) complementary slackness = zero duality gap
Farkas lemma

Theorem
Given $A$ and $b$, exactly one of the following statements is true:

1. There exists an $x$ with $Ax = b$, $x \geq 0$
2. There exists a $y$ with $A^Ty \geq 0$, $b^Ty < 0$
Farkas lemma

Proof

1 and 2 cannot be both true (easy)

\[ x \geq 0, \ Ax = b \text{ and } y^T A \geq 0 \quad \rightarrow \quad y^T b = y^T Ax \geq 0 \]
Farkas lemma

Proof

1 and 2 cannot be both true (easy)

\[ x \geq 0, \quad Ax = b \quad \text{and} \quad y^T A \geq 0 \quad \rightarrow \quad y^T b = y^T Ax \geq 0 \]

1 and 2 cannot be both false (duality)

<table>
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<tr>
<th>1) ( Ax=b, , x \geq 0 )</th>
<th>2) ( A^T y \geq 0, , B^T y &lt; 0 )</th>
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<td>minimize ( c^T x )</td>
<td>maximize ( -b^T y )</td>
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| subject to \( Ax = b \)  | subject to \( A^T y \geq -c \)  
| \( x \geq 0 \)          | \( c = 0 \geq \) \( A^T y \geq 0 \) |
Farkas lemma

Proof

1 and 2 cannot be both true (easy)

\[ x \geq 0, \ Ax = b \text{ and } y^T A \geq 0 \quad \rightarrow \quad y^T b = y^T Ax \geq 0 \]

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| \(x \geq 0\) | \(y = 0\) always feasible \(d^* \neq -\infty, \quad p^* = d^*\)
Farkas lemma

Proof

1 and 2 cannot be both true (easy)

\[ x \geq 0, \ Ax = b \text{ and } y^T A \geq 0 \quad \rightarrow \quad y^T b = y^T Ax \geq 0 \]

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Alternative 1: primal feasible \(p^* = d^* = 0\)

\(b^T y \geq 0\) for all \(y\) such that \(A^T y \geq 0\)
Farkas lemma

Proof

1 and 2 cannot be both true (easy)

\[ x \geq 0, \ Ax = b \text{ and } y^T A \geq 0 \quad \rightarrow \quad y^T b = y^T Ax \geq 0 \]

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**Alternative 1:** primal feasible \(p^* = d^* = 0\)

\(b^T y \geq 0\) for all \(y\) such that \(A^T y \geq 0\)

**Alternative 2:** primal infeasible \(p^* = d^* = +\infty\)

There exists \(y\) such that \(A^T y \geq 0\) and \(b^T y < 0\)
Farkas lemma

Proof

1 and 2 cannot be both true (easy)

\[ x \geq 0, \ Ax = b \text{ and } y^T A \geq 0 \quad \Rightarrow \quad y^T b = y^T Ax \geq 0 \]

1 and 2 cannot be both false (duality)

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Alternative 1: primal feasible, \(p^* = d^* = 0\)

\(b^T y \geq 0\) for all \(y\) such that \(A^T y \geq 0\)

Alternative 2: primal infeasible, \(p^* = d^* = +\infty\)

There exists \(y\) such that \(A^T y \geq 0\) and \(b^T y < 0\)

\(y\) is an infeasibility certificate
Sensitivity analysis
Changes in problem data

**Goal:** extract information from $x^*, y^*$ about their sensitivity with respect to changes in problem data

**Modified LP**

minimize $c^T x$

subject to $Ax = b + u$

$x \geq 0$

**Optimal value function**

$p^*(u) = \min \{ c^T x \mid Ax = b + u, \; x \geq 0 \}$
Changes in problem data

**Goal:** extract information from $x^*, y^*$ about their sensitivity with respect to changes in problem data

**Modified LP**

minimize \[ c^T x \]
subject to \[ Ax = b + u \]
\[ x \geq 0 \]

**Optimal value function**

\[ p^*(u) = \min\{c^T x \mid Ax = b + u, \ x \geq 0\} \]

**Assumption:** \( p^*(0) \) is finite

**Properties**

- \( p^*(u) > -\infty \) everywhere (from global lower bound)
- \( p^*(u) \) is piecewise-linear on its domain
Global sensitivity

Dual of modified LP

maximize \(- (b + u)^T y\)

subject to \(A^T y + c \geq 0\)
Global sensitivity

Dual of modified LP
maximize $-(b + u)^T y$
subject to $A^T y + c \geq 0$

Global lower bound
Given $y^*$ a dual optimal solution for $u = 0$, then
$$p^*(u) \geq -(b + u)^T y^*$$
$$= p^*(0) - u^T y^*$$
(from weak duality and dual feasibility)
Global sensitivity

**Dual of modified LP**

maximize \(- (b + u)^T y\)

subject to \(A^T y + c \geq 0\)

**Global lower bound**

Given \(y^*\) a dual optimal solution for \(u = 0\), then

\[ p^*(u) \geq - (b + u)^T y^* \]

\[ = p^*(0) - u^T y^* \]

(from weak duality and dual feasibility)

It holds for any \(u\)
Local sensitivity

\[ \text{in neighborhood of the origin} \]

Original LP  
\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\]

Optimal solution

Primal
\[
\begin{align*}
& \quad x_i = 0, \quad i \notin B \\
& \quad x_B^* = A_B^{-1} b
\end{align*}
\]

Dual
\[
\begin{align*}
& \quad y^* = -A_B^{-T} c_B
\end{align*}
\]
Local sensitivity

\( u \) in neighborhood of the origin

Original LP
\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\]

Optimal solution
\[
\begin{align*}
\text{Primal} & \quad x^*_i = 0, \quad i \notin B \\
& \quad x^*_B = A_B^{-1} b \\
\text{Dual} & \quad y^* = -A_B^{-T} c_B
\end{align*}
\]

Modified LP
\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax = b + u \\
& \quad x \geq 0
\end{align*}
\]

Modified dual
\[
\begin{align*}
\text{maximize} & \quad -(b + u)^T y \\
\text{subject to} & \quad A^T y + c \geq 0
\end{align*}
\]

Optimal basis does not change
Local sensitivity
$u$ in neighborhood of the origin

Original LP
\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\]

Optimal solution
\[
\begin{align*}
\text{Primal} & \quad x_i = 0, \quad i \notin B \\
& \quad x^*_B = A_B^{-1} b \\
\text{Dual} & \quad y^* = -A_B^{-T} c_B
\end{align*}
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Modified LP
\[
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\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax = b + u \\
& \quad x \geq 0
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Modified dual
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\begin{align*}
\text{maximize} & \quad -(b + u)^T y \\
\text{subject to} & \quad A^T y + c \geq 0
\end{align*}
\]

Modified optimal solution
\[
\begin{align*}
x^*_B(u) &= A_B^{-1}(b + u) = x^*_B + A_B^{-1} u \\
y^*(u) &= y^*
\end{align*}
\]
Derivative of the optimal value function

Modified optimal solution

\[ x_B^*(u) = A_B^{-1}(b + u) = x_B^* + A_B^{-1}u \]
\[ y^*(u) = y^* \]
Derivative of the optimal value function

Modified optimal solution
\[ x_B^*(u) = A_B^{-1}(b + u) = x_B^* + A_B^{-1}u \]
\[ y^*(u) = y^* \]

Optimal value function
\[ p^*(u) = c^T x^*(u) \]
\[ = c^T x^* + c_B^T A_B^{-1}u \]
\[ = p^*(0) - y^{*T}u \quad \text{(affine for small } u) \]
Derivative of the optimal value function

Modified optimal solution
\[ x^*_B(u) = A_B^{-1}(b + u) = x^*_B + A_B^{-1}u \]
\[ y^*(u) = y^* \]

Optimal value function
\[ p^*(u) = c^T x^*(u) \]
\[ = c^T x^* + c_B^T A_B^{-1}u \]
\[ = p^*(0) - y^*^T u \quad \text{(affine for small } u) \]

Local derivative
\[ \nabla p^*(u) = -y^* \quad (y^* \text{ are the shadow prices}) \]
Simplex method
Optimality of extreme points

minimize $c^T x$
subject to $Ax = b$
$x \geq 0$

If
- $P$ has at least one extreme point
- There exists an optimal solution $x^*$

Then, there exists an optimal solution which is an extreme point of $P$

We only need to search between extreme points
Equivalence

Theorem

Given a nonempty polyhedron \( P = \{ x \mid Ax = b, \ x \geq 0 \} \)

Let \( x \in P \)

\( x \) is a vertex \( \iff \) \( x \) is an extreme point \( \iff \) \( x \) is a basic feasible solution
Constructing basic solution

1. Choose any \( m \) independent columns of \( A \): \( A_{B(1)}, \ldots, A_{B(m)} \)
2. Let \( x_i = 0 \) for all \( i \neq B(1), \ldots, B(m) \)
3. Solve \( Ax = b \) for the remaining \( x_{B(1)}, \ldots, x_{B(m)} \)

\[
A_B = \begin{bmatrix}
A_{B(1)} & A_{B(2)} & \cdots & A_{B(m)}
\end{bmatrix}, \quad x_B = \begin{bmatrix}
x_{B(1)} \\
\vdots \\
x_{B(m)}
\end{bmatrix} \quad \rightarrow \quad \text{Solve} \quad A_B x_B = b
\]

If \( x_B \geq 0 \), then \( x \) is a basic feasible solution
Conceptual algorithm

- Start at corner
- Visit neighboring corner that improves the objective
How does the cost change?

Cost improvement

\[ c^T (x + \theta d) - c^T x = \theta c^T d \]
How does the cost change?

Cost improvement

\[ c^T (x + \theta d) - c^T x = \theta c^T d \]
How does the cost change?

Cost improvement

\[ c^T (x + \theta d) - c^T x = \theta c^T d \]

New cost

Old cost
How does the cost change?

Cost improvement

\[ c^T (x + \theta d) - c^T x = \theta c^T d \]

New cost \hspace{2cm} Old cost

We call \( \bar{c}_j \) the reduced cost of (introducing) variable \( x_j \) in the basis

\[
\bar{c}_j = c^T d = \sum_{i=1}^{n} c_j d_j = c_j + c_B^T d_B = c_j - c_B^T A_B^{-1} A_j
\]
Reduced costs

Interpretation
Change in objective/marginal cost of adding $x_j$ to the basis

$$\tilde{c}_j = c_j - c_B^T A_B^{-1} A_j$$

- $\tilde{c}_j > 0$: adding $x_j$ will increase the objective (bad)
- $\tilde{c}_j < 0$: adding $x_j$ will decrease the objective (good)
Reduced costs

**Interpretation**

Change in objective/marginal cost of adding $x_j$ to the basis

\[ \bar{c}_j = c_j - c_B^T A_B^{-1} A_j \]

Cost per-unit increase of variable $x_j$

- $\bar{c}_j > 0$: adding $x_j$ will increase the objective (bad)
- $\bar{c}_j < 0$: adding $x_j$ will decrease the objective (good)
**Reduced costs**

**Interpretation**
Change in objective/marginal cost of adding $x_j$ to the basis

\[
\tilde{c}_j = c_j - \mathbf{c}_B^T A_B^{-1} A_j
\]

Cost per-unit increase of variable $x_j$

Cost to change other variables compensating for $x_j$ to enforce $Ax = b$

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**Reduced costs**

**Interpretation**
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Cost per-unit increase of variable $x_j$

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- $\bar{c}_j > 0$: adding $x_j$ will increase the objective (bad)
- $\bar{c}_j < 0$: adding $x_j$ will decrease the objective (good)

**Reduced costs for basic variables is 0**

\[
\bar{c}_B(i) = c_B(i) - c_B^T A_B^{-1} A_B(i) = c_B(i) - c_B^T (A_B^{-1} A_B) e_i
\]
\[
= c_B(i) - c_B^T e_i = c_B(i) - c_B(i) = 0
\]
Optimality conditions

Theorem

Let \( x \) be a basic feasible solution associated with basis \( B \)
Let \( \bar{c} \) be the vector of reduced costs.

If \( \bar{c} \geq 0 \), then \( x \) is optimal
Optimality conditions

Theorem

Let $x$ be a basic feasible solution associated with basis $B$. Let $\bar{c}$ be the vector of reduced costs.

If $\bar{c} \geq 0$, then $x$ is optimal

Remark

This is a stopping criterion for the simplex algorithm.

If the neighboring solutions do not improve the cost, we are done.
Single simplex iteration

1. Compute the reduced costs $\bar{c}$
   - Solve $A_B^T p = c_B$
   - $\bar{c} = c - A_T p$

2. If $\bar{c} \geq 0$, $x$ optimal. break

3. Choose $j$ such that $\bar{c}_j < 0$

4. Compute search direction $d$ with $d_j = 1$ and $A_B d_B = -A_j$

5. If $d_B \geq 0$, the problem is unbounded and the optimal value is $-\infty$. break

6. Compute step length $\theta^* = \min_{\{i \in B | d_i < 0\}} \left( -\frac{x_i}{d_i} \right)$

7. Define $y$ such that $y = x + \theta^* d$

8. Get new basis $\bar{B}$ ($i$ exits and $j$ enters)
# Single simplex iteration

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   - Solve $A_B^T p = c_B$
   - $\bar{c} = c - A^T p$

2. If $\bar{c} \geq 0$, $x$ is **optimal. break**

3. Choose $j$ such that $\bar{c}_j < 0$

4. Compute search direction $d$ with $d_j = 1$ and $A_B d_B = -A_j$

5. If $d_B \geq 0$, the problem is **unbounded** and the optimal value is $-\infty$. **break**

6. Compute step length $\theta^* = \min_{\{i \in B | d_i < 0\}} \left( -\frac{x_i}{d_i} \right)$

7. Define $y$ such that $y = x + \theta^* d$

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---

**Bottleneck**

**Two linear systems**
Single simplex iteration

1. Compute the reduced costs $\bar{c}$
   - Solve $A_B^T p = c_B$
   - $\bar{c} = c - A^T p$
2. If $\bar{c} \geq 0$, $x$ optimal. break
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Bottleneck
Two linear systems

Matrix inversion lemma trick
$\approx n^2$ per iteration
(very cheap)
Single simplex iteration

1. Compute the reduced costs $\bar{c}$
   • Solve $A_B^T p = c_B$
   • $\bar{c} = c - A^T p$

2. If $\bar{c} \geq 0$, $x$ optimal. break

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4. Compute search direction $d$ with
   $d_j = 1$ and $A_B d_B = -A_j$

5. If $d_B \geq 0$, the problem is unbounded
   and the optimal value is $-\infty$. break

6. Compute step length $\theta^* = \min_{\{i \in B | d_i < 0\}} \left( -\frac{x_i}{d_i} \right)$

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8. Get new basis $\bar{B}$ ($i$ exits and $j$ enters)

Bottleneck
Two linear systems

Matrix inversion lemma trick
$\approx n^2$ per iteration
(very cheap)

How many iterations do we need?
We do not know any polynomial version of the simplex method, no matter which pivoting rule we pick. Still open research question!
Complexity of the simplex method

We do not know any polynomial version of the simplex method, no matter which pivoting rule we pick.

Still open research question!

Worst-case

There are problem instances where the simplex method will run an exponential number of iterations in terms of the dimensions, e.g. $2^n$
Complexity of the simplex method

We do not know any polynomial version of the simplex method, no matter which pivoting rule we pick.

Still open research question!

Worst-case
There are problem instances where the simplex method will run an exponential number of iterations in terms of the dimensions, e.g. $2^n$

Good news: average-case
Practical performance is very good. On average, it stops in $n$ iterations.
Interior point method
Optimality conditions

**Primal**

minimize $c^T x$

subject to $Ax + s = b$

$s \geq 0$

**Dual**

maximize $-b^T y$

subject to $A^T y + c = 0$

$y \geq 0$

**KKT conditions**

$$Ax + s - b = 0$$

$$A^T y + c = 0$$

$s_i y_i = 0, \quad i = 1, \ldots, m$

$s, y \geq 0$

$$S = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_m \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$$\implies SY 1 = 0$$
Main idea

\[
 h(x, s, y) = \begin{bmatrix}
 Ax + s - b \\
 A^T y + c \\
 SY 1
\end{bmatrix} = 0 \quad S = \text{diag}(s) \\
 Y = \text{diag}(y) \\
 s, y \geq 0
\]

- Apply variants of Newton’s method to solve \( h(x, s, y) = 0 \)
- Enforce \( s, y > 0 \) (strictly) at every iteration
- Motivation avoid getting stuck in “corners”
Main idea

\[
h(x, s, y) = \begin{bmatrix}
Ax + s - b \\
ATy + c \\
SY1
\end{bmatrix} = 0
\]

\[
S = \text{diag}(s) \\
Y = \text{diag}(y)
\]

\[s, y \geq 0\]

- Apply variants of Newton’s method to solve \(h(x, s, y) = 0\)
- Enforce \(s, y > 0\) (strictly) at every iteration
- **Motivation** avoid getting stuck in “corners”

**Issue**

Pure **Newton’s step** does not allow significant progress towards

\[h(x, s, y) = 0 \text{ and } x, y \geq 0.\]
Smoothed optimality conditions

Optimality conditions

\[ Ax + s - b = 0 \]
\[ A^T y + c = 0 \]
\[ s_i y_i = \tau \quad \text{Same } \tau \text{ for every pair} \]
\[ s, y \geq 0 \]

Same optimality conditions for a “smoothed” version of our problem
Central path

minimize \quad c^T x - \tau \sum_{i=1}^{m} \log(s_i)

subject to \quad Ax + s = b

Set of points \((x^*(\tau), s^*(\tau), y^*(\tau))\)

with \(\tau > 0\) such that

\[
Ax + s - b = 0
\]

\[
A^T y + c = 0
\]

\[
s_i y_i = \tau
\]

\[
s, y \geq 0
\]
Central path

minimize \( c^T x - \tau \sum_{i=1}^m \log(s_i) \)
subject to \( Ax + s = b \)

Set of points \((x^*(\tau), s^*(\tau), y^*(\tau))\)
with \(\tau > 0\) such that
\[
\begin{align*}
Ax + s - b &= 0 \\
A^T y + c &= 0 \\
s_i y_i &= \tau \\
s, y &\geq 0
\end{align*}
\]

Main idea
Follow central path as \(\tau \to 0\)
Newton’s method for smoothed optimality conditions

Smoothed optimality conditions

\[ h_\tau(x, s, y) = \begin{bmatrix} Ax + s - b \\ A^T y + c \\ SY1 - \tau1 \end{bmatrix} = 0 \]

\[ s, y \geq 0 \]
Newton’s method for smoothed optimality conditions

Smoothed optimality conditions

\[ h_\tau(x, s, y) = \begin{bmatrix} Ax + s - b \\ A^T y + c \\ SY1 - \tau1 \end{bmatrix} = 0 \]

\( s, y \geq 0 \)

Linear system

\[
\begin{bmatrix}
0 & A & I \\
A^T & 0 & 0 \\
S & 0 & Y
\end{bmatrix}
\begin{bmatrix}
\Delta y \\
\Delta x \\
\Delta s
\end{bmatrix} =
\begin{bmatrix}
-r_p \\
-r_d \\
-SY + \tau1
\end{bmatrix}
\]

**Line search** to enforce \( x, s > 0 \)

\((x, s, y) \leftarrow (x, s, y) + \alpha(\Delta x, \Delta s, \Delta y)\)
Algorithm step

Linear system

\[
\begin{bmatrix}
0 & A & I \\
A^T & 0 & 0 \\
S & 0 & Y
\end{bmatrix}
\begin{bmatrix}
\Delta y \\
\Delta x \\
\Delta s
\end{bmatrix}
= 
\begin{bmatrix}
-r_p \\
-r_d \\
-SY1 + \sigma \mu 1
\end{bmatrix}
\]

Duality measure

\[
\mu = \frac{s^T y}{m}
\]

Centering parameter

\[
\sigma \in [0, 1]
\]
Algorithm step

Linear system

\[
\begin{bmatrix}
0 & A & I \\
A^T & 0 & 0 \\
S & 0 & Y
\end{bmatrix}
\begin{bmatrix}
\Delta y \\
\Delta x \\
\Delta s
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=
\begin{bmatrix}
-r_p \\
-r_d \\
-SY1 + \sigma \mu 1
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\]

Duality measure

\[
\mu = \frac{s^T y}{m}
\]

Centering parameter

\[
\sigma \in [0, 1]
\]

\[
\sigma = 0 \quad \Rightarrow \quad \text{Newton step}
\]

\[
\sigma = 1 \quad \Rightarrow \quad \text{Centering step towards } (x^*(\mu), s^*(\mu), y^*(\mu))
\]
Algorithm step

Linear system

\[
\begin{bmatrix}
0 & A & I \\
A^T & 0 & 0 \\
S & 0 & Y
\end{bmatrix}
\begin{bmatrix}
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Duality measure

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\]

Line search to enforce \( x, s > 0 \)

\[
(x, s, y) \leftarrow (x, s, y) + \alpha(\Delta x, \Delta s, \Delta y)
\]
Path-following algorithm idea

Newton step
\( \sigma = 0 \)

Centering step
\( \sigma = 1 \)

Combined step

\( x^* \)
Path-following algorithm idea

Centering step
Moves towards the central path and is usually biased towards $s, y > 0$.

No progress on duality measure $\mu$
Path-following algorithm idea

Centering step
Moves towards the central path and is usually biased towards \( s, y > 0 \).
No progress on duality measure \( \mu \).

Newton step
Moves towards the zero duality measure \( \mu \). Quickly violates \( s, y > 0 \).
Path-following algorithm idea

Centering step
Moves towards the central path
and is usually biased towards $s, y > 0$.

No progress on duality measure $\mu$

Newton step
Moves towards the zero duality measure $\mu$. Quickly violates $s, y > 0$.

Combined step
Best of both, with longer steps.
Choosing the centering parameter

Newton direction

\((\Delta x_a, \Delta s_a, \Delta y_a)\)

- The Newton step might quickly hit nonnegativity.
- The centering step might not reduce complementary condition.
Choosing the centering parameter

Newton direction

\[ (\Delta x_a, \Delta s_a, \Delta y_a) \]

- The Newton step might quickly hit nonnegativity.
- The centering step might not reduce complementary condition.

Maximum step-size

\[ \alpha_p = \max\{\alpha \in [0, 1] \mid s + \alpha \Delta s_a \geq 0\} \]
\[ \alpha_d = \max\{\alpha \in [0, 1] \mid y + \alpha \Delta y_a \geq 0\} \]

Centering parameter heuristic (after a Newton step)

\[ \mu_a = \frac{(s + \alpha_p \Delta s_a)^T (y + \alpha_d \Delta y_a)}{m} \]
Choosing the centering parameter

Newton direction

\((\Delta x_a, \Delta s_a, \Delta y_a)\)

- The Newton step might quickly hit nonnegativity.
- The centering step might not reduce complementary condition.

Maximum step-size

\[\alpha_p = \max\{\alpha \in [0, 1] \mid s + \alpha \Delta s_a \geq 0\}\]
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Centering parameter heuristic (after a Newton step)

\[\mu_a = \left(\frac{(s + \alpha_p \Delta s_a)^T(y + \alpha_d \Delta y_a)}{m}\right) \quad \Rightarrow \quad \sigma = \left(\frac{\mu_a}{\mu}\right)^3\]
Mehrotra predictor-corrector algorithm

Initialization
Given \((x, s, y)\) such that \(s, y > 0\)

1. Termination conditions

   \[ r_p = Ax + s - b, \quad r_d = A^T y + c, \quad \mu = (s^T y) / m \]
   
   If \(\|r_p\|, \|r_d\|, \mu\) are small, break Optimal solution \((x^*, s^*, y^*)\)

2. Newton step (affine scaling)

\[
\begin{bmatrix}
0 & A & I \\
A^T & 0 & 0 \\
S & 0 & Y
\end{bmatrix}
\begin{bmatrix}
\Delta y_a \\
\Delta x_a \\
\Delta s_a
\end{bmatrix} =
\begin{bmatrix}
-r_p \\
-r_d \\
-SY 1
\end{bmatrix}
\]
Mehrotra predictor-corrector algorithm

3. Centering parameter

\[
\alpha_p = \max\{\alpha \in [0, 1] \mid s + \alpha \Delta s_a \geq 0\}
\]

\[
\alpha_d = \max\{\alpha \in [0, 1] \mid y + \alpha \Delta y_a \geq 0\}
\]

\[
\mu_a = \frac{(s + \alpha_p \Delta s_a)^T (y + \alpha_d \Delta y_a)}{m}
\]

\[
\sigma = \left(\frac{\mu_a}{\mu}\right)^3
\]

4. Corrected direction

\[
\begin{bmatrix}
0 & A & I \\
A^T & 0 & 0 \\
S & 0 & Y
\end{bmatrix}
\begin{bmatrix}
\Delta y \\
\Delta x \\
\Delta s
\end{bmatrix}
= 
\begin{bmatrix}
-SY1 - \Delta S_a \Delta Y_a 1 + \sigma \mu 1
\end{bmatrix}
\]

Linearization correction

Centering heuristic
5. Update iterates

\[ \alpha_p = \max\{\alpha \geq 0 \mid s + \alpha \Delta s \geq 0\} \]
\[ \alpha_d = \max\{\alpha \geq 0 \mid y + \alpha \Delta y \geq 0\} \]

\[ (x, s) = (x, s) + \min\{1, \eta\alpha_p\}(\Delta x, \Delta s) \]
\[ y = y + \min\{1, \eta\alpha_d\}\Delta y \]

Avoid corners

\[ \eta = 1 - \epsilon \approx 0.99 \]
Solving the search equations

Step 2 (Newton) and 4 (Corrected direction) solve equations of the form

\[
\begin{bmatrix}
0 & A & I \\
A^T & 0 & 0 \\
S & 0 & Y
\end{bmatrix}
\begin{bmatrix}
\Delta y \\
\Delta x \\
\Delta s
\end{bmatrix}
=
\begin{bmatrix}
b_y \\
b_x \\
b_s
\end{bmatrix}
\]

(not symmetric)
Solving the search equations

Step 2 (Newton) and 4 (Corrected direction) solve equations of the form

\[
\begin{bmatrix}
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\Delta x \\
\Delta s \\
\end{bmatrix}
=
\begin{bmatrix}
b_y \\
b_x \\
b_s \\
\end{bmatrix}
\]

(not symmetric)

Substitute last equation, \(\Delta s = Y^{-1}(b_s - S\Delta y)\), into first

\[
\begin{bmatrix}
-Y^{-1}S & A \\
A^T & 0 \\
\end{bmatrix}
\begin{bmatrix}
\Delta y \\
\Delta x \\
\end{bmatrix}
=
\begin{bmatrix}
b_y - Y^{-1}b_s \\
b_x \\
\end{bmatrix}
\]
Solving the search equations

Step 2 (Newton) and 4 (Corrected direction) solve equations of the form (not symmetric)

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A^T & 0
\end{bmatrix}
\begin{bmatrix}
\Delta y \\
\Delta x
\end{bmatrix}
=
\begin{bmatrix}
b_y - Y^{-1}b_s \\
b_x
\end{bmatrix}
\]

Substitute first equation, \( \Delta y = S^{-1}Y(A\Delta x - b_y + Y^{-1}b_s) \), into second

\[
A^T S^{-1}Y A \Delta x = b_x + A^T S^{-1}Y b_y - A^T S^{-1}b_s
\]
Reduced linear system

Coefficient matrix

\[ B = A^T S^{-1} Y A \]

- \( B \) is **positive definite** if \( A \) has linearly independent columns
- Sparsity pattern of \( B \) is the **pattern** of \( A^T A \) (independent of \( S^{-1} Y \))

Sparse cholesky factorization

\[ B = PLL^T P^T \]

- Reorder only once to get \( P \)
- One numerical factorization per interior-point iteration \( O(n^3) \)
- Forward/backward substitution twice per iteration \( O(n^2) \)
Reduced linear system

Coefficient matrix

\[ B = A^T S^{-1} Y A \]

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Sparse cholesky factorization

\[ B = P L L^T P^T \]

- Reorder only once to get \( P \)
- One numerical factorization per interior-point iteration \( O(n^3) \)
- Forward/backward substitution twice per iteration \( O(n^2) \)
Convergence

Mehrotra’s algorithm

No convergence theory  Examples where it diverges (rare!)
Convergence

Mehrotra’s algorithm

No convergence theory \rightarrow \text{Examples where it diverges (rare!)}
Fantastic convergence \text{in practice} \rightarrow \text{Fewer than 30 iterations}
Convergence

Mehrotra’s algorithm

No convergence theory → Examples where it diverges (rare!)
Fantastic convergence in practice → Fewer than 30 iterations

Theoretical iteration complexity
Alternative versions (slower than Mehrotra) converge in $O(\sqrt{n})$ iterations
Convergence

Mehrotra’s algorithm

No convergence theory → Examples where it **diverges** (rare!)
Fantastic convergence **in practice** → Fewer than 30 iterations

**Theoretical iteration complexity**
Alternative versions (slower than Mehrotra)
converge in $O(\sqrt{n})$ iterations

**Average iteration complexity**
Average iterations complexity is $O(\log n)$
Convergence

Mehrotra’s algorithm

No convergence theory → Examples where it **diverges** (rare!)
Fantastic convergence in **practice** → Fewer than 30 iterations

**Theoretical iteration complexity**
Alternative versions (slower than Mehrotra) converge in $O(\sqrt{n})$ iterations

**Operations**

$n^3.5$

**Average iteration complexity**
Average iterations complexity is $O(\log n)$

$O(n^3 \log n)$
Interior-point vs simplex
Comparison between interior-point method and simplex
Comparison between interior-point method and simplex

Primal simplex

- Primal feasibility
  - Zero duality gap
  - Dual feasibility
Comparison between interior-point method and simplex

Primal simplex

- Primal feasibility
  - Zero duality gap
  - Dual feasibility

Dual simplex

- Dual feasibility
  - Zero duality gap
  - Primal feasibility
## Comparison between interior-point method and simplex

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**Exponential worst-case complexity**

- Requires feasible point
- Can be warm-started
### Comparison between interior-point method and simplex

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**Exponential worst-case complexity**

- Requires feasible point
- Can be warm-started
### Comparison between interior-point method and simplex

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- **Primal feasibility**
- **Dual feasibility**
- **Zero duality gap**
- **Primal feasibility**
- **Dual feasibility**
- **Zero duality gap**

- **Requires feasible point**
- **Can be warm-started**
- **Allows infeasible start**
- **Cannot be warm-started**

- **Exponential worst-case complexity**
- **Polynomial worst-case complexity**
Which algorithm should I use?

**Dual simplex**
- Small-to-medium problems
- Repeated solves with varying constraints

**Interior-point (barrier)**
- Medium-to-large problems
- Sparse structured problems

How do solvers with multiple options decide?

Concurrent Optimization

Why not both? (crossover)

Interior-point \(\rightarrow\) Few simplex steps
Questions
Next lecture

• Integer optimization