ORF307 – Optimization

19. Linear optimization review
• How do mathematicians know when to use "heuristics," or arbitrary measures to make the algorithm work better?

• Why is not being able to warm start a problem if we can start interior point methods with an infeasible solution?
Today’s lecture
Linear optimization review

• Formulations
• Piecewise linear optimization
• Duality
• Sensitivity analysis
• Simplex method
• Interior point methods
Formulations
Linear optimization

\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax \leq b \\
& \quad Dx = f
\end{align*}

- Minimization
- Less-than ineq. constraints
- Equality constraints

\[ x \text{ is feasible if it satisfies the constraints } Ax \leq b \text{ and } Dx = f \]

The feasible set is the set of all feasible points

\[ x^* \text{ is optimal if it is feasible and } c^T x^* \leq c^T x \text{ for all feasible } x \]

The optimal value is \( p^* = c^T x^* \)

Unbounded problem: \( c^T x \) is unbounded below on the feasible set \( (p^* = -\infty) \)

Infeasible problem: feasible set is empty \( (p^* = +\infty) \)
Feasibility problems

\[
\begin{align*}
\text{find} & \quad x \\
\text{subject to} & \quad Ax \leq b \\
& \quad Dx = f \\
\end{align*}
\]

\[
\begin{align*}
\text{minimize} & \quad 0 \\
\text{subject to} & \quad Ax \leq b \\
& \quad Dx = f \\
\end{align*}
\]

Possible results

- \( p^* = 0 \) if constraints are feasible (consistent).
  (Every feasible \( x \) is optimal)
- \( p^* = \infty \) otherwise
Standard form

Definition

minimize $c^T x$
subject to $Ax = b$
$x \geq 0$

• Minimization
• Equality constraints
• Nonnegative variables

Useful to

• develop algorithms
• algebraic manipulations
Piecewise linear optimization
Piecewise-linear minimization

minimize \[ f(x) = \max_{i=1,\ldots,m} (a_i^T x + b_i) \]

minimize \[ t \]

subject to \[ a_i^T x + b_i \leq t, \quad i = 1, \ldots, m \]
Piecewise-linear minimization

minimize \( f(x) = \max_{i=1,\ldots,m} (a_i^T x + b_i) \)

minimize \( t \)

subject to \( a_i^T x + b_i \leq t, \quad i = 1, \ldots, m \)

Matrix notation

minimize \( \tilde{c}^T \tilde{x} \)

subject to \( \tilde{A} \tilde{x} \leq \tilde{b} \)

\[
\tilde{x} = \begin{bmatrix} x \\ t \end{bmatrix}, \quad \tilde{c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} a_1^T & -1 \\ \vdots & \vdots \\ a_m^T & -1 \end{bmatrix}, \quad \tilde{b} = \begin{bmatrix} -b_1 \\ \vdots \\ -b_m \end{bmatrix}
\]
1 and infinity norms reformulations

1-norm minimization:

minimize \( \|Ax - b\|_1 = \sum_i |(Ax - b)_i| \)

Equivalent to:

minimize \( 1^T u \)
subject to \( -u \leq Ax - b \leq u \)

\( \|Ax - b\|_1 \) minimization:

Absolute value of every element \((Ax - b)_i\) is bounded by a component of the vector \(u\)

\( \|Ax - b\|_\infty \) minimization:

minimize \( \|Ax - b\|_\infty = \max_i |(Ax - b)_i| \)

Equivalent to:

minimize \( t \)
subject to \( -t1 \leq Ax - b \leq t1 \)

\( \|Ax - b\|_\infty \) minimization:

Absolute value of every element \((Ax - b)_i\) is bounded by the same scalar \(t\)
Duality
Inequality form LP

minimize $c^T x$
subject to $Ax \leq b$
Inequality form LP

minimize \( c^T x \)
subject to \( Ax \leq b \)

Relax the constraint

\[
g(y) = \min_x c^T x + y^T (Ax - b)
\]
Inequality form LP

minimize \[ c^T x \]
subject to \[ Ax \leq b \]

Relax the constraint
\[ g(y) = \min_{x} c^T x + y^T (Ax - b) \]

Lower bound
\[ g(y) \leq c^T x^* + y^T (Ax^* - b) \leq c^T x^* \]
we must have \[ y \geq 0 \]
Inequality form LP

minimize \( c^T x \)
subject to \( Ax \leq b \)

Relax the constraint
\[
g(y) = \min_x c^T x + y^T (Ax - b)
\]

Lower bound
\[
g(y) \leq c^T x^* + y^T (Ax^* - b) \leq c^T x^*
\]
we must have \( y \geq 0 \)

Lagrangian
\[
L(x, y)
\]
Dual of LP with inequalities

Derivation

Dual function

\[ g(y) = \min_x \left( c^T x + y^T (Ax - b) \right) - b^T y + \min_x \left( c + A^T y \right)^T x \]
Dual of LP with inequalities

Derivation

Dual function

\[ g(y) = \min_x \left( c^T x + y^T (Ax - b) \right) - b^T y + \min_x \left( c + A^T y \right)^T x \]

\[ g(y) = \begin{cases} -b^T y & \text{if } c + A^T y = 0 \text{ (and } y \geq 0) \\ -\infty & \text{otherwise} \end{cases} \]
Dual of LP with inequalities

Derivation

Dual function

\[
g(y) = \min_x (c^T x + y^T (Ax - b)) - b^T y + \min_x (c + A^T y)^T x
\]

\[
g(y) = \begin{cases} 
-b^T y & \text{if } c + A^T y = 0 \quad (\text{and } y \geq 0) \\
-\infty & \text{otherwise}
\end{cases}
\]

Dual problem (find the best bound)

maximize \( g(y) \) = maximize \( -b^T y \)

subject to \( A^T y + c = 0 \)
\( y \geq 0 \)
General forms

Inequality form LP

minimize \( c^T x \)
subject to \( Ax \leq b \)

maximize \( -b^T y \)
subject to \( A^T y + c = 0 \)
\( y \geq 0 \)

Standard form LP

minimize \( c^T x \)
subject to \( Ax = b \)
\( x \geq 0 \)

maximize \( -b^T y \)
subject to \( A^T y + c \geq 0 \)

LP with inequalities and equalities

minimize \( c^T x \)
subject to \( Ax \leq b \)
\( Dx = f \)

maximize \( -b^T y - f^T z \)
subject to \( A^T y + D^T z + c = 0 \)
\( y \geq 0 \)
Weak duality

Theorem
If $x, y$ satisfy:

- $x$ is a feasible solution to the primal problem
- $y$ is a feasible solution to the dual problem

$-b^T y \leq c^T x$
Weak duality

**Theorem**
If $x, y$ satisfy:

- $x$ is a feasible solution to the primal problem
- $y$ is a feasible solution to the dual problem

Then:

$$-b^T y \leq c^T x$$

**Proof**
We know that $Ax \leq b$, $A^T y + c = 0$ and $y \geq 0$. Therefore,

$$0 \leq y^T (b - Ax) = b^T y - y^T Ax = c^T x + b^T y$$
Weak duality

**Theorem**
If $x, y$ satisfy:
- $x$ is a feasible solution to the primal problem
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$$-b^T y \leq c^T x$$

**Proof**
We know that $Ax \leq b$, $A^T y + c = 0$ and $y \geq 0$. Therefore,

$$0 \leq y^T (b - Ax) = b^T y - y^T Ax = c^T x + b^T y$$

**Remark**
- Any dual feasible $y$ gives a **lower bound** on the primal optimal value
- Any primal feasible $x$ gives an **upper bound** on the dual optimal value
- $c^T x + b^T y$ is the **duality gap**
Weak duality

Corollaries

Unboundedness vs feasibility

- Primal unbounded ($p^* = -\infty$) $\Rightarrow$ dual infeasible ($d^* = -\infty$)
- Dual unbounded ($d^* = +\infty$) $\Rightarrow$ primal infeasible ($p^* = +\infty$)
Weak duality
Corollaries

Unboundedness vs feasibility
• Primal unbounded \( (p^* = -\infty) \) \( \Rightarrow \) dual infeasible \( (d^* = -\infty) \)
• Dual unbounded \( (d^* = +\infty) \) \( \Rightarrow \) primal infeasible \( (p^* = +\infty) \)

Optimality condition
If \( x, y \) satisfy:
• \( x \) is a feasible solution to the primal problem
• \( y \) is a feasible solution to the dual problem
• The duality gap is zero, \( i.e., \) \( c^T x + b^T y = 0 \)

Then \( x \) and \( y \) are optimal solutions to the primal and dual problem respectively
Strong duality

<table>
<thead>
<tr>
<th>Primal</th>
<th>Dual</th>
</tr>
</thead>
<tbody>
<tr>
<td>minimize $c^T x$</td>
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<tr>
<td>subject to $Ax = b$</td>
<td>subject to $A^T y + c \geq 0$</td>
</tr>
<tr>
<td>$x \geq 0$</td>
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Theorem

If a linear optimization problem has an optimal solution, then

- so does its dual
- the optimal values of the primal and dual are equal
## Relationship between primal and dual

<table>
<thead>
<tr>
<th>$d^*$</th>
<th>$p^* = +\infty$</th>
<th>$p^*$ finite</th>
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<tbody>
<tr>
<td>$d^* = +\infty$</td>
<td>primal inf.</td>
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Complementary slackness

Primal

minimize $c^T x$

subject to $Ax \leq b$

Dual

maximize $-b^T y$

subject to $A^T y + c = 0$

$y \geq 0$

Theorem
Primal,dual feasible $x, y$ are optimal if and only if

$$y_i (b_i - a_i^T x) = 0, \quad i = 1, \ldots, m$$

i.e., at optimum, $b - Ax$ and $y$ have a complementary sparsity pattern:

$$y_i > 0 \quad \Rightarrow \quad a_i^T x = b_i$$

$$a_i^T x < b_i \quad \Rightarrow \quad y_i = 0$$
Complementary slackness

Primal

\[
\begin{array}{l}
\text{minimize} \quad c^T x \\
\text{subject to} \quad Ax \leq b
\end{array}
\]

Dual

\[
\begin{array}{l}
\text{maximize} \quad -b^T y \\
\text{subject to} \quad A^T y + c = 0 \\
y \geq 0
\end{array}
\]

Proof

The duality gap at primal feasible \( x \) and dual feasible \( y \) can be written as

\[
c^T x + b^T y = (\underbrace{-A^T y}_\text{C}\)^T x + b^T y = (b - Ax)^T y = \sum_{i=1}^{m} y_i (b_i - a_i^T x) = 0
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Complementary slackness

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Proof

The duality gap at primal feasible $x$ and dual feasible $y$ can be written as

$$c^T x + b^T y = (-A^T y)^T x + b^T y = (b - A x)^T y = \sum_{i=1}^{m} y_i (b_i - a_i^T x) = 0$$

Since all the elements of the sum are nonnegative, they must all be 0.
Complementary slackness

Primal
minimize $c^T x$
subject to $Ax \leq b$

Dual
maximize $-b^T y$
subject to $A^T y + c = 0$
$y \geq 0$

Proof
The duality gap at primal feasible $x$ and dual feasible $y$ can be written as

$$c^T x + b^T y = (-A^T y)^T x + b^T y = (b - Ax)^T y = \sum_{i=1}^{m} y_i (b_i - a_i^T x) = 0$$

Since all the elements of the sum are nonnegative, they must all be 0

For feasible $x$ and $y$ complementary slackness = zero duality gap
Farkas lemma

Theorem
Given $A$ and $b$, exactly one of the following statements is true:

1. There exists an $x$ with $Ax = b$, $x \geq 0$
2. There exists a $y$ with $A^Ty \geq 0$, $b^Ty < 0$
Farkas lemma

Proof

1 and 2 cannot be both true (easy)

\[ x \geq 0, \ Ax = b \text{ and } y^T A \geq 0 \quad \rightarrow \quad y^T b = y^T Ax \geq 0 \]
Farkas lemma

Proof

1 and 2 cannot be both true (easy)

\[ x \geq 0, \ Ax = b \text{ and } y^T A \geq 0 \quad \longrightarrow \quad y^T b = y^T Ax \geq 0 \]

1 and 2 cannot be both false (duality)

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\[
\begin{cases}
\text{1. } Ax = b, \ x \geq 0 \\
\text{2. } A^T y \geq 0, \ B^T y < 0
\end{cases}
\]
Farkas lemma

Proof

1 and 2 cannot be both true (easy)

\[ x \geq 0, \ Ax = b \text{ and } y^T A \geq 0 \quad \longrightarrow \quad y^T b = y^T Ax \geq 0 \]

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<td>(x \geq 0)</td>
<td>(y = 0) always feasible</td>
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<td>(d^* \neq -\infty, \ p^* = d^*)</td>
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# Farkas Lemma

**Proof**

1 and 2 cannot be both true (easy)

\[ x \geq 0, \ Ax = b \text{ and } y^T A \geq 0 \quad \Rightarrow \quad y^T b = y^T A x \geq 0 \]

1 and 2 cannot be both false (duality)

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**Alternative 1:** primal feasible \( p^* = d^* = 0 \)

\( b^T y \geq 0 \text{ for all } y \text{ such that } A^T y \geq 0 \)
Farkas lemma

Proof

1 and 2 cannot be both true (easy)

\[ x \geq 0, \ Ax = b \text{ and } y^T A \geq 0 \quad \Rightarrow \quad y^T b = y^T Ax \geq 0 \]

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Alternative 1: primal feasible \(p^* = d^* = 0\)

\(b^T y \geq 0\) for all \(y\) such that \(A^T y \geq 0\)

Alternative 2: primal infeasible \(p^* = d^* = +\infty\)

There exists \(y\) such that \(A^T y \geq 0\) and \(b^T y < 0\)
Farkas lemma

Proof

1 and 2 cannot be both true (easy)

\[ x \geq 0, \ Ax = b \text{ and } y^T A \geq 0 \rightarrow y^T b = y^T Ax \geq 0 \]

1 and 2 cannot be both false (duality)

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Alternative 1: primal feasible \(p^* = d^* = 0\)

\(b^T y \geq 0\) for all \(y\) such that \(A^T y \geq 0\)

Alternative 2: primal infeasible \(p^* = d^* = +\infty\)

There exists \(y\) such that \(A^T y \geq 0\) and \(b^T y < 0\)

\(y\) is an infeasibility certificate
Sensitivity analysis
Changes in problem data

**Goal:** extract information from $x^*, y^*$ about their sensitivity with respect to changes in problem data

**Modified LP**

minimize \( c^T x \)
subject to \( Ax = b + u \)
\( x \geq 0 \)

**Optimal value function**

\[
p^*(u) = \min \{ c^T x \mid Ax = b + u, \ x \geq 0 \}
\]
Changes in problem data

**Goal:** extract information from $x^*, y^*$ about their sensitivity with respect to changes in problem data

**Modified LP**

minimize $c^T x$

subject to $Ax = b + u$

$x \geq 0$

**Optimal value function**

$p^*(u) = \min\{c^T x \mid Ax = b + u, \; x \geq 0\}$

**Assumption:** $p^*(0)$ is finite

**Properties**

- $p^*(u) > -\infty$ everywhere (from global lower bound)
- $p^*(u)$ is piecewise-linear on its domain
Global sensitivity

Dual of modified LP

maximize \(- (b + u)^T y\)
subject to \(A^T y + c \geq 0\)
Global sensitivity

**Dual of modified LP**

maximize \[-(b + u)^T y\]

subject to \[A^T y + c \geq 0\]

**Global lower bound**

Given \(y^*\) a dual optimal solution for \(u = 0\), then

\[p^*(u) \geq -(b + u)^T y^*\]

\[= p^*(0) - u^T y^*\]

(from weak duality and dual feasibility)
Global sensitivity

Dual of modified LP

maximize $-(b + u)^T y$
subject to $A^T y + c \geq 0$

Global lower bound

Given $y^*$ a dual optimal solution for $u = 0$, then

$$p^*(u) \geq -(b + u)^T y^*$$

$$= p^*(0) - u^T y^*$$

(from weak duality and dual feasibility)

It holds for any $u$
Local sensitivity

*u* in neighborhood of the origin

Original LP

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\]

Optimal solution

**Primal**

\[
\begin{align*}
x_i^* &= 0, \quad i \not\in B \\
x_B^* &= A_B^{-1} b
\end{align*}
\]

**Dual**

\[
y^* = -A_B^{-T} c_B
\]
Local sensitivity

\( u \) in neighborhood of the origin

**Original LP**

minimize \( c^T x \)

subject to \( Ax = b \)

\( x \geq 0 \)

**Optimal solution**

Primal

\( x_B^* = A_B^{-1}b \)

Dual

\( y^* = -A_B^{-T}c_B \)

**Modified LP**

minimize \( c^T x \)

subject to \( Ax = b + u \)

\( x \geq 0 \)

**Modified dual**

maximize \( -(b + u)^T y \)

subject to \( A^T y + c \geq 0 \)

Optimal basis does not change
Local sensitivity

$u$ in neighborhood of the origin

Original LP

\[
\text{minimize } \quad c^T x \\
\text{subject to } \quad Ax = b \\
\quad x \geq 0
\]

Optimal solution

Primal

\[
x^*_B = A_B^{-1}b
\]

Dual

\[
y^* = -A_B^{-T}c_B
\]

Modified LP

\[
\text{minimize } \quad c^T x \\
\text{subject to } \quad Ax = b + u \\
\quad x \geq 0
\]

Modified dual

\[
\text{maximize } \quad -(b + u)^T y \\
\text{subject to } \quad A^T y + c \geq 0
\]

Modified optimal solution

\[
x^*_B(u) = A_B^{-1}(b + u) = x^*_B + A_B^{-1}u
\]

\[
y^*(u) = y^*
\]

\[
c = c^T A^T y = c^T A^T (A_B^{-1} b) \geq 0
\]
Derivative of the optimal value function

Modified optimal solution

\[ x^*_B(u) = A_B^{-1}(b + u) = x^*_B + A_B^{-1}u \]

\[ y^*(u) = y^* \]
Derivative of the optimal value function

Modified optimal solution

\[ x^*_B(u) = A_B^{-1}(b + u) = x_B^* + A_B^{-1}u \]
\[ y^*(u) = y^* \]

Optimal value function

\[ p^*(u) = c^T x^*(u) \]
\[ = c^T x^* + c_B^T A_B^{-1}u \]
\[ = p^*(0) - y^{*T}u \] (affine for small \( u \))
Derivative of the optimal value function

Modified optimal solution
\[ x_B^*(u) = A_B^{-1} (b + u) = x_B^* + A_B^{-1} u \]
\[ y^*(u) = y^* \]

Optimal value function
\[ p^*(u) = c^T x^*(u) \]
\[ = c^T x^* + c_B^T A_B^{-1} u \]
\[ = p^*(0) - y^*^T u \] (affine for small \( u \))

Local derivative
\[ \nabla p^*(u) = -y^* \] (\( y^* \) are the shadow prices)
Simplex method
Optimality of extreme points

minimize \quad c^T x
subject to \quad Ax = b
\quad x \geq 0

If
- \( P \) has at least one extreme point
- There exists an optimal solution \( x^* \)

Then, there exists an optimal solution which is an extreme point of \( P \)

We only need to search between extreme points
Equivalence

Theorem

Given a nonempty polyhedron $P = \{x \mid Ax = b, \ x \geq 0\}$

Let $x \in P$

$x$ is a vertex $\iff$ $x$ is an extreme point $\iff$ $x$ is a basic feasible solution
Constructing basic solution

1. Choose any \( m \) independent columns of \( A: \ A_B(1), \ldots, A_B(m) \)
2. Let \( x_i = 0 \) for all \( i \neq B(1), \ldots, B(m) \)
3. Solve \( Ax = b \) for the remaining \( x_B(1), \ldots, x_B(m) \)

Basis matrix

\[
A_B = \begin{bmatrix}
A_B(1) & A_B(2) & \cdots & A_B(m)
\end{bmatrix}
\]

Basis columns

Basic variables

\[
x_B = \begin{bmatrix}
x_B(1) \\
\vdots \\
x_B(m)
\end{bmatrix}
\]

Solve \( A_Bx_B = b \)

If \( x_B \geq 0 \), then \( x \) is a basic feasible solution
Conceptual algorithm

- Start at corner
- Visit neighboring corner that improves the objective
How does the cost change?

Cost improvement

\[ c^T (x + \theta d) - c^T x = \theta c^T d \]
How does the cost change?

Cost improvement

\[ c^T(x + \theta d) - c^T x = \theta c^T d \]

New cost
How does the cost change?

Cost improvement

\[ c^T(x + \theta d) - c^T x = \theta c^T d \]

New cost → Cost improvement → Old cost
How does the cost change?

Cost improvement:

\[ c^T(x + \theta d) - c^Tx = \theta c^Td \]

New cost \quad Old cost

We call \( \bar{c}_j \) the **reduced cost** of (introducing) variable \( x_j \) in the basis:

\[ \bar{c}_j = c^T d = \sum_{i=1}^{n} c_j d_j = c_j + c_B^T d_B = c_j - c_B^T A_B^{-1} A_j \]
Reduced costs

Interpretation
Change in objective/marginal cost of adding $x_j$ to the basis

$$
\bar{c}_j = c_j - c_B^T A_B^{-1} A_j
$$

- $\bar{c}_j > 0$: adding $x_j$ will increase the objective (bad)
- $\bar{c}_j < 0$: adding $x_j$ will decrease the objective (good)
Reduced costs

Interpretation
Change in objective/marginal cost of adding $x_j$ to the basis

$$\bar{c}_j = c_j - c_B^T A_B^{-1} A_j$$

Cost per-unit increase of variable $x_j$

- $\bar{c}_j > 0$: adding $x_j$ will increase the objective (bad)
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\[ \bar{c}_j = c_j - c_B^T A_B^{-1} A_j \]

Cost per-unit increase of variable $x_j$

Cost to change other variables compensating for $x_j$ to enforce $Ax = b$

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- $\bar{c}_j > 0$: adding $x_j$ will increase the objective (bad)
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Reduced costs for basic variables is 0

$$
\bar{c}_B(i) = c_B(i) - c_B^T A_B^{-1} A_B(i) = c_B(i) - c_B^T (A_B^{-1} A_B) e_i
$$

$$
= c_B(i) - c_B^T e_i = c_B(i) - c_B(i) = 0
$$
Optimality conditions

Theorem

Let $x$ be a basic feasible solution associated with basis $B$
Let $\bar{c}$ be the vector of reduced costs.

If $\bar{c} \geq 0$, then $x$ is optimal
Optimality conditions

Theorem

Let $x$ be a basic feasible solution associated with basis $B$
Let $\bar{c}$ be the vector of reduced costs.

\[ \bar{c} \geq 0, \text{ then } x \text{ is optimal} \]

Remark

This is a stopping criterion for the simplex algorithm.

If the neighboring solutions do not improve the cost, we are done.
Single simplex iteration

1. Compute the reduced costs $\bar{c}$
   - Solve $A_B^T p = c_B$
   - $\bar{c} = c - A_T p$

2. If $\bar{c} \geq 0$, $x$ optimal. break

3. Choose $j$ such that $\bar{c}_j < 0$

4. Compute search direction $d$ with
   \[ d_j = 1 \text{ and } A_B d_B = -A_j \]

5. If $d_B \geq 0$, the problem is unbounded
   and the optimal value is $-\infty$. break

6. Compute step length $\theta^* = \min_{\{i \in B \mid d_i < 0\}} \left( -\frac{x_i}{d_i} \right)$

7. Define $y$ such that $y = x + \theta^* d$

8. Get new basis $\bar{B}$ ($i$ exits and $j$ enters)
Single simplex iteration

1. Compute the reduced costs \( \bar{c} \)
   - Solve \( A_B^T p = c_B \)
   - \( \bar{c} = c - A_T p \)
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**Bottleneck**

Two linear systems
**Single simplex iteration**

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   - Solve $A_B^T p = c_B$
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2. If $\bar{c} \geq 0$, $x$ optimal. break

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**Bottleneck**

Two linear systems

---

**Matrix inversion lemma trick**

$\approx n^2$ per iteration

(very cheap)
**Single simplex iteration**

1. Compute the reduced costs \( \bar{c} \)
   - Solve \( A_B^T p = c_B \)
   - \( \bar{c} = c - A^T p \)
2. If \( \bar{c} \geq 0 \), \( x \) optimal. break
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   \( d_j = 1 \) and \( A_B d_B = -A_j \)
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**Bottleneck**

Two linear systems

**Matrix inversion lemma trick**

\( \approx n^2 \) per iteration

(very cheap)

---

**How many iterations do we need?**
Complexity of the simplex method

We do not know any polynomial version of the simplex method, no matter which pivoting rule we pick. Still open research question!
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Still open research question!

Worst-case
There are problem instances where the simplex method will run an exponential number of iterations in terms of the dimensions, e.g. $2^n$
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We do not know any polynomial version of the simplex method, no matter which pivoting rule we pick.

Still open research question!

Worst-case

There are problem instances where the simplex method will run an exponential number of iterations in terms of the dimensions, e.g. $2^n$

Good news: average-case

Practical performance is very good. On average, it stops in $n$ iterations.
Interior point method
Optimality conditions

**Primal**

minimize \( c^T x \)

subject to \( Ax + s = b \)

\( s \geq 0 \)

**Dual**

maximize \( -b^T y \)

subject to \( A^T y + c = 0 \)

\( y \geq 0 \)

**KKT conditions**

\( Ax + s - b = 0 \)

\( A^T y + c = 0 \)

\( s_i y_i = 0, \quad i = 1, \ldots, m \)

\( s, y \geq 0 \)

\[ S = \begin{bmatrix} s_1 \\ & \ddots \\ & & s_m \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 \\ & \ddots \\ & & y_m \end{bmatrix} \]

\[ SY1 = 0 \]
Main idea

\[ h(x, s, y) = \begin{bmatrix} Ax + s - b \\ A^T y + c \\ SY 1 \end{bmatrix} = 0 \]

\[ S = \text{diag}(s) \]

\[ Y = \text{diag}(y) \]

\[ s, y \geq 0 \]

- Apply variants of Newton’s method to solve \( h(x, s, y) = 0 \)
- Enforce \( s, y > 0 \) (strictly) at every iteration
- **Motivation** avoid getting stuck in “corners”
Main idea

\[
h(x, s, y) = \begin{bmatrix} Ax + s - b \\ A^T y + c \\ SY1 \end{bmatrix} \leq 0
\]

\[
S = \text{diag}(s) \\
Y = \text{diag}(y)
\]

\[s, y \geq 0\]

- Apply variants of Newton’s method to solve \(h(x, s, y) = 0\)
- Enforce \(s, y > 0\) (strictly) at every iteration
- **Motivation** avoid getting stuck in “corners”

**Issue**

Pure **Newton’s step** does not allow significant progress towards

\[
h(x, s, y) = 0 \text{ and } x, y \geq 0.
\]
Smoothed optimality conditions

Optimality conditions

\[ Ax + s - b = 0 \]
\[ A^T y + c = 0 \]
\[ s_i y_i = \tau \]
\[ s, y \geq 0 \]

Same optimality conditions for a “smoothed” version of our problem
Central path

minimize \[ c^T x - \tau \sum_{i=1}^{m} \log(s_i) \]
subject to \[ Ax + s = b \]

Set of points \((x^*(\tau), s^*(\tau), y^*(\tau))\) with \(\tau > 0\) such that
\[
Ax + s - b = 0 \\
A^T y + c = 0 \\
s_i y_i = \tau \\
s, y \geq 0
\]
Central path

minimize \( c^T x - \tau \sum_{i=1}^{m} \log(s_i) \)
subject to \( Ax + s = b \)

Set of points \((x^*(\tau), s^*(\tau), y^*(\tau))\)
with \( \tau > 0 \) such that

\[
\begin{align*}
Ax + s - b &= 0 \\
A^T y + c &= 0 \\
s_i y_i &= \tau \\
s, y &\geq 0
\end{align*}
\]

Main idea
Follow central path as \( \tau \to 0 \)
Newton’s method for smoothed optimality conditions

Smoothened optimality conditions

\[ h_\tau(x, s, y) = \begin{bmatrix} Ax + s - b \\ A^T y + c \\ SY1 - \tau1 \end{bmatrix} = 0 \]

\[ s, y \geq 0 \]
Newton’s method for smoothed optimality conditions

Smoothed optimality conditions

\[ h_\tau(x, s, y) = \begin{bmatrix} Ax + s - b \\ A^T y + c \\ SY1 - \tau1 \end{bmatrix} = 0 \]

\[ s, y \geq 0 \]

Linear system

\[
\begin{bmatrix}
  0 & A & I \\
  A^T & 0 & 0 \\
  S & 0 & Y
\end{bmatrix}
\begin{bmatrix}
  \Delta y \\
  \Delta x \\
  \Delta s
\end{bmatrix}
= \begin{bmatrix}
  -r_p \\
  -r_d \\
  -SY + \tau1
\end{bmatrix}
\]

Line search to enforce \( x, s > 0 \)

\[ (x, s, y) \leftarrow (x, s, y) + \alpha(\Delta x, \Delta s, \Delta y) \]
Algorithm step

Linear system

\[
\begin{bmatrix}
0 & A & I \\
A^T & 0 & 0 \\
S & 0 & Y
\end{bmatrix}
\begin{bmatrix}
\Delta y \\
\Delta x \\
\Delta s
\end{bmatrix}
= 
\begin{bmatrix}
-r_p \\
-r_d \\
-SY1 + \sigma \mu 1
\end{bmatrix}
\]

Duality measure

\[\mu = \frac{s^T y}{m}\]

Centering parameter

\[\sigma \in [0, 1]\]
Algorithm step

Linear system

\[
\begin{bmatrix}
0 & A & I \\
A^T & 0 & 0 \\
S & 0 & Y \\
\end{bmatrix}
\begin{bmatrix}
\Delta y \\
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\Delta s \\
\end{bmatrix}
= \begin{bmatrix}
-r_p \\
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-SY \mathbf{1} + \sigma \mu \mathbf{1} \\
\end{bmatrix}
\]

Duality measure

\[
\mu = \frac{s^T y}{m}
\]

Centering parameter

\[
\sigma \in [0, 1]
\]

\[
\sigma = 0 \quad \Rightarrow \quad \text{Newton step}
\]

\[
\sigma = 1 \quad \Rightarrow \quad \text{Centering step towards } (x^*(\mu), s^*(\mu), y^*(\mu))
\]
Algorithm step

Linear system

$$
\begin{bmatrix}
0 & A & I \\
A^T & 0 & 0 \\
S & 0 & Y
\end{bmatrix}
\begin{bmatrix}
\Delta y \\
\Delta x \\
\Delta s
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= 
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Duality measure

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Line search to enforce \(x, s > 0\)

$$
(x, s, y) \leftarrow (x, s, y) + \alpha(\Delta x, \Delta s, \Delta y)
$$
Path-following algorithm idea

Newton step $\sigma = 0$

Centering step $\sigma = 1$

Combined step $x^*$
Path-following algorithm idea

Centering step
Moves towards the central path
and is usually biased towards $s, y > 0$.
No progress on duality measure $\mu$.
Path-following algorithm idea

Centering step
Moves towards the **central path** and is usually biased towards $s, y > 0$. **No progress** on duality measure $\mu$.

Newton step
Moves towards the **zero duality measure** $\mu$. Quickly violates $s, y > 0$. 

Combined step
$x^*$

Newton step
$\sigma = 0$

Centering step
$\sigma = 1$
Path-following algorithm idea

Centering step
Moves towards the **central path** and is usually biased towards $s, y > 0$.
**No progress** on duality measure $\mu$.

Newton step
Moves towards the **zero duality measure** $\mu$. Quickly violates $s, y > 0$.

Combined step
Best of both, with longer steps.
Choosing the centering parameter

Newton direction

$\left( \Delta x_a, \Delta s_a, \Delta y_a \right)$

- The Newton step might quickly hit nonnegativity.
- The centering step might not reduce complementary condition.
Choosing the centering parameter

Newton direction

\[(\Delta x_a, \Delta s_a, \Delta y_a)\]

- The Newton step might quickly hit nonnegativity.
- The centering step might not reduce complementary condition.

Maximum step-size

\[\alpha_p = \max\{\alpha \in [0, 1] \mid s + \alpha \Delta s_a \geq 0\}\]
\[\alpha_d = \max\{\alpha \in [0, 1] \mid y + \alpha \Delta y_a \geq 0\}\]

Centering parameter heuristic (after a Newton step)

\[\mu_a = \frac{(s + \alpha_p \Delta s_a)^T(y + \alpha_d \Delta y_a)}{m}\]
Choosing the centering parameter

Newton direction

\[ (\Delta x_a, \Delta s_a, \Delta y_a) \]

- The Newton step might quickly hit nonnegativity.
- The centering step might not reduce complementary condition.

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Centering parameter heuristic (after a Newton step)

\[ \mu_a = \frac{(s + \alpha_p \Delta s_a)^T(y + \alpha_d \Delta y_a)}{m} \quad \Rightarrow \quad \sigma = \left( \frac{\mu_a}{\mu} \right)^3 \]
Mehrotra predictor-corrector algorithm

Initialization

Given \((x, s, y)\) such that \(s, y > 0\)

1. Termination conditions

\[ r_p = Ax + s - b, \quad r_d = A^T y + c, \quad \mu = (s^T y)/m \]

If \(\|r_p\|, \|r_d\|, \mu\) are small, break

Optimal solution \((x^*, s^*, y^*)\)

2. Newton step (affine scaling)

\[
\begin{bmatrix}
0 & A & I \\
A^T & 0 & 0 \\
S & 0 & Y
\end{bmatrix}
\begin{bmatrix}
\Delta y_a \\
\Delta x_a \\
\Delta s_a
\end{bmatrix}
= 
\begin{bmatrix}
-r_p \\
-r_d \\
-SY1
\end{bmatrix}
\]
Mehrotra predictor-corrector algorithm

3. Centering parameter

\[ \alpha_p = \max \{ \alpha \in [0, 1] \mid s + \alpha \Delta s_a \geq 0 \} \]

\[ \alpha_d = \max \{ \alpha \in [0, 1] \mid y + \alpha \Delta y_a \geq 0 \} \]

\[ \mu_a = \frac{(s + \alpha_p \Delta s_a)^T(y + \alpha_d \Delta y_a)}{m} \]

\[ \sigma = \left( \frac{\mu_a}{\mu} \right)^3 \]

4. Corrected direction

\[
\begin{bmatrix}
0 & A & I \\
A^T & 0 & 0 \\
S & 0 & Y \\
\end{bmatrix}
\begin{bmatrix}
\Delta y \\
\Delta x \\
\Delta s \\
\end{bmatrix}
= 
\begin{bmatrix}
-SY1 \\
-\Delta S_a \Delta Y_a 1 + \sigma \mu 1 \\
-r_p \\
-r_d \\
\end{bmatrix}
\]

Linearization correction

Centering heuristic
Mehrotra predictor-corrector algorithm

5. **Update iterates**

\[
\alpha_p = \max\{\alpha \geq 0 \mid s + \alpha \Delta s \geq 0\}
\]

\[
\alpha_d = \max\{\alpha \geq 0 \mid y + \alpha \Delta y \geq 0\}
\]

\[
(x, s) = (x, s) + \min\{1, \eta \alpha_p\}(\Delta x, \Delta s)
\]

\[
y = y + \min\{1, \eta \alpha_d\}\Delta y
\]

*Avoid corners*

\[
\eta = 1 - \epsilon \approx 0.99
\]
Solving the search equations

Step 2 (Newton) and 4 (Corrected direction) solve equations of the form

\[
\begin{bmatrix}
0 & A & I \\
A^T & 0 & 0 \\
S & 0 & Y \\
\end{bmatrix}
\begin{bmatrix}
\Delta y \\
\Delta x \\
\Delta s \\
\end{bmatrix}
=
\begin{bmatrix}
b_y \\
b_x \\
b_s \\
\end{bmatrix}
\]

(not symmetric)
Solving the search equations

Step 2 (Newton) and 4 (Corrected direction) solve equations of the form

\[
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\Delta s
\end{bmatrix} =
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b_y \\
b_x \\
b_s
\end{bmatrix}
\]

(not symmetric)

Substitute last equation, \( \Delta s = Y^{-1}(b_s - S\Delta y) \), into first

\[
\begin{bmatrix}
-Y^{-1}S & A \\
A^T & 0
\end{bmatrix}
\begin{bmatrix}
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b_x
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Solving the search equations

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\end{bmatrix}
= 
\begin{bmatrix}
b_y - Y^{-1}b_s \\
b_x
\end{bmatrix}
\]

Substitute first equation, \( \Delta y = S^{-1}Y(A\Delta x - b_y + Y^{-1}b_s) \), into second

\[
A^TS^{-1}YA\Delta x = b_x + A^TS^{-1}Yb_y - A^TS^{-1}b_s
\]
Reduced linear system

Coefficient matrix

\[ B = A^T S^{-1} Y A \]

- \( B \) is positive definite
  - if \( A \) has linearly independent columns
- Sparsity pattern of \( B \) is the pattern of \( A^T A \)
  - (independent of \( S^{-1} Y \))

Sparse cholesky factorization

\[ B = PLL^T P^T \]

- Reorder only once to get \( P \)
- One numerical factorization per interior-point iteration \( O(n^3) \)
- Forward/backward substitution twice per iteration \( O(n^2) \)
Reduced linear system

Coefficient matrix

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- Forward/backward substitution twice per iteration \( O(n^2) \)

Per-iteration complexity \( O(n^3) \)
Convergence

Mehrotra’s algorithm

No convergence theory $\rightarrow$ Examples where it diverges (rare!)
Convergence

Mehrotra’s algorithm

No convergence theory \rightarrow \text{Examples where it diverges (rare!)}

Fantastic convergence \textbf{in practice} \rightarrow \text{Fewer than 30 iterations}
Convergence

Mehrotra’s algorithm

No convergence theory → Examples where it **diverges** (rare!)
Fantastic convergence in **practice** → Fewer than 30 iterations

**Theoretical iteration complexity**
Alternative versions (slower than Mehrotra) converge in $O(\sqrt{n})$ iterations
Convergence

Mehrotra’s algorithm

No convergence theory  Examples where it diverges (rare!)
Fantastic convergence in practice  Fewer than 30 iterations

Theoretical iteration complexity
Alternative versions (slower than Mehrotra) converge in $O(\sqrt{n})$ iterations

Average iteration complexity
Average iterations complexity is $O(\log n)$
Convergence

Mehrotra’s algorithm

No convergence theory  →  Examples where it diverges (rare!)
Fantastic convergence in practice  →  Fewer than 30 iterations

Theoretical iteration complexity
Alternative versions (slower than Mehrotra) converge in \( O(\sqrt{n}) \) iterations

Average iteration complexity
Average iterations complexity is \( O(\log n) \)

<table>
<thead>
<tr>
<th>Operations</th>
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<tbody>
<tr>
<td>( O(n^{3.5}) )</td>
</tr>
<tr>
<td>( O(n^3 \log n) )</td>
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</table>
Interior-point vs simplex
Comparison between interior-point method and simplex
Comparison between interior-point method and simplex

Primal simplex

- Primal feasibility
  - Zero duality gap
  - Dual feasibility
### Comparison between interior-point method and simplex

<table>
<thead>
<tr>
<th>Primal simplex</th>
<th>Dual simplex</th>
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<tr>
<td>• Primal feasibility</td>
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- **Exponential worst-case complexity**
  - Requires feasible point
  - Can be warm-started
## Comparison between interior-point method and simplex

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<th>Zero duality gap</th>
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<td>Can be warm-started</td>
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- Primal feasibility
- Dual feasibility
- Zero duality gap
- Primal feasibility
- Dual feasibility
- Zero duality gap
- Interior condition
- Primal feasibility
- Dual feasibility
- Zero duality gap
Comparison between interior-point method and simplex

Primal simplex
- Primal feasibility
- Zero duality gap
- Dual feasibility

Dual simplex
- Dual feasibility
- Zero duality gap
- Primal feasibility

Primal-dual interior-point
- Interior condition
- Primal feasibility
- Dual feasibility
- Zero duality gap

Exponential worst-case complexity
- Requires feasible point
- Can be warm-started

Polynomial worst-case complexity
- Allows infeasible start
- Cannot be warm-started
Which algorithm should I use?

**Dual simplex**
- Small-to-medium problems
- Repeated solves with varying constraints

**Interior-point (barrier)**
- Medium-to-large problems
- Sparse structured problems

How do solvers with multiple options decide?

Concurrent Optimization

Why not both? (crossover)

Interior-point \rightarrow Few simplex steps
Questions
Next lecture

• Integer optimization