A feasible direction is any direction that "stays in" P, whereas a basic direction is one that points in the direction of a neighboring basic solution. Is there a difference between a feasible direction and a basic direction?

Why does maximizing the lower bound of the cost make it “better”?

\[ p^* = c^T x^* \]
Recap
Optimal objective values

Primal
minimize \( c^T x \)
subject to \( Ax \leq b \)

\( p^* \) is the primal optimal value

Dual
maximize \( -b^T y \)
subject to \( A^T y + c = 0 \)
\( y \geq 0 \)

\( d^* \) is the dual optimal value

Primal infeasible: \( p^* = +\infty \)
Primal unbounded: \( p^* = -\infty \)

Dual infeasible: \( d^* = -\infty \)
Dual unbounded: \( d^* = +\infty \)
## Relationship between primal and dual

<table>
<thead>
<tr>
<th></th>
<th>( p^* = +\infty )</th>
<th>( p^* ) finite</th>
<th>( p^* = -\infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d^* = +\infty )</td>
<td>primal inf. dual unb.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( d^* ) finite</td>
<td></td>
<td>optimal values equal</td>
<td></td>
</tr>
<tr>
<td>( d^* = -\infty )</td>
<td>exception</td>
<td></td>
<td>primal unb. dual inf</td>
</tr>
</tbody>
</table>

- Upper-right excluded by **weak duality**
- \((1, 1)\) and \((3, 3)\) proven by **weak duality**
- \((3, 1)\) and \((2, 2)\) proven by **strong duality**
Today’s agenda
More on duality

• Two-person zero-sum games
• Farkas lemma
• Complementary slackness
• KKT conditions
Two-person games
Rock paper scissors

Rules
At count to three declare one of: Rock, Paper, or Scissors

Winners
Identical selection is a draw, otherwise:
• Rock beats (“dulls”) scissors
• Scissors beats (“cuts”) paper
• Paper beats (“covers”) rock

Extremely popular: world RPS society, USA RPS league, etc.
Two-person zero-sum game

• Player 1 (P1) chooses a number $i \in \{1, \ldots, m\}$ (one of $m$ actions)
• Player 2 (P2) chooses a number $j \in \{1, \ldots, n\}$ (one of $n$ actions)

Two players make their choice independently
Two-person zero-sum game

- Player 1 (P1) chooses a number $i \in \{1, \ldots, m\}$ (one of $m$ actions)
- Player 2 (P2) chooses a number $j \in \{1, \ldots, n\}$ (one of $n$ actions)

Two players make their choice independently

Rule

Player 1 pays $A_{ij}$ to player 2

$A \in \mathbb{R}^{m \times n}$ is the payoff matrix

Rock, Paper, Scissors

$$A = \begin{bmatrix} R & P & S \\ \hline R & 0 & 1 & -1 \\ P & -1 & 0 & 1 \\ S & 1 & -1 & 0 \end{bmatrix}$$
Mixed (randomized) strategies

Deterministic strategies can be systematically defeated
Mixed (randomized) strategies

Deterministic strategies can be systematically defeated

Randomized strategies
- P1 chooses randomly according to distribution $x$:
  \[ x_i = \text{probability that P1 selects action } i \]
- P2 chooses randomly according to distribution $y$:
  \[ y_j = \text{probability that P2 selects action } j \]
Mixed (randomized) strategies

Deterministic strategies can be systematically defeated

Randomized strategies
- P1 chooses randomly according to distribution $x$:
  \[ x_i = \text{probability that P1 selects action } i \]
- P2 chooses randomly according to distribution $y$:
  \[ y_j = \text{probability that P2 selects action } j \]

Expected payoff (from P1 P2), if they use mixed-strategies $x$ and $y$,

\[ \sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j A_{ij} = x^T A y \]
Mixed strategies and probability simplex

**Probability simplex** in $\mathbb{R}^k$

$P_k = \{ p \in \mathbb{R}^k \mid p \geq 0, \quad 1^T p = 1 \}$

**Mixed strategy**

For a game player, a mixed strategy is a distribution over all possible deterministic strategies.

The **set of all mixed strategies** is the probability simplex $\longrightarrow x \in P_m, \quad y \in P_n$
Optimal mixed strategies

P1: optimal strategy $x^*$ is the solution of

$$\text{minimize} \quad \max_{y \in P_n} x^T Ay$$

subject to $x \in P_m$

P2: optimal strategy $y^*$ is the solution of

$$\text{maximize} \quad \min_{x \in P_m} x^T Ay$$

subject to $y \in P_n$
Optimal mixed strategies

P1: optimal strategy \( x^* \) is the solution of

\[
\begin{align*}
\text{minimize} & \quad \max_{y \in P_n} x^T A y \\
\text{subject to} & \quad x \in P_m
\end{align*}
\]

\[
\begin{align*}
\text{maximize} & \quad \min_{x \in P_m} x^T A y \\
\text{subject to} & \quad y \in P_n
\end{align*}
\]

P2: optimal strategy \( y^* \) is the solution of

\[
\begin{align*}
\text{minimize} & \quad \max_{j=1,\ldots,n} (A^T x)_j \\
\text{subject to} & \quad x \in P_m
\end{align*}
\]

\[
\begin{align*}
\text{maximize} & \quad \min_{i=1,\ldots,m} (A y)_i \\
\text{subject to} & \quad y \in P_n
\end{align*}
\]
Optimal mixed strategies

P1: optimal strategy $x^*$ is the solution of

minimize $\max_{y \in P_n} x^T Ay$
subject to $x \in P_m$

P2: optimal strategy $y^*$ is the solution of

maximize $\min_{x \in P_m} x^T Ay$
subject to $y \in P_n$

Inner problem over deterministic strategies (vertices)
Optimal mixed strategies

P1: optimal strategy $x^*$ is the solution of

$$\begin{align*}
\text{minimize} & \quad \max_{y \in P_n} x^T A y \\
\text{subject to} & \quad x \in P_m
\end{align*}$$

minimize $\max_{j=1,\ldots,n} (A^T x)_j$
subject to $x \in P_m$

Inner problem over deterministic strategies (vertices)

P2: optimal strategy $y^*$ is the solution of

$$\begin{align*}
\text{maximize} & \quad \min_{x \in P_m} x^T A y \\
\text{subject to} & \quad y \in P_n
\end{align*}$$

maximize $\min_{i=1,\ldots,m} (Ay)_i$
subject to $y \in P_n$

Optimal strategies $x^*$ and $y^*$ can be computed using linear optimization
Minmax theorem

Theorem

\[ \max_{y \in P_n} \min_{x \in P_m} x^T Ay = \min_{x \in P_m} \max_{y \in P_n} x^T Ay \]
Minmax theorem

Theorem

$$\max_{y \in P_n} \min_{x \in P_m} x^T Ay = \min_{x \in P_m} \max_{y \in P_n} x^T Ay$$

Proof

The optimal $x^*$ is the solution of

minimize $t$

subject to $A^T x \leq t1$

$1^T x = 1$

$x \geq 0$
Theorem

\[ \max \min_{y \in P_n} x^T Ay = \min \max_{x \in P_m} x^T Ay \]

Proof

The optimal \( x^* \) is the solution of

\[
\begin{align*}
\text{minimize} & \quad t \\
\text{subject to} & \quad A^T x \leq t \mathbf{1} \\
& \quad 1^T x = 1 \\
& \quad x \geq 0
\end{align*}
\]

The optimal \( y^* \) is the solution of

\[
\begin{align*}
\text{maximize} & \quad w \\
\text{subject to} & \quad A y \geq w \mathbf{1} \\
& \quad 1^T y = 1 \\
& \quad y \geq 0
\end{align*}
\]
Minmax theorem

Theorem

\[
\max_{y \in P_n} \min_{x \in P_m} x^T Ay = \min_{x \in P_m} \max_{y \in P_n} x^T Ay
\]

Proof

The optimal \( x^* \) is the solution of

\[
\text{minimize } t \\
\text{subject to } A^T x \leq t \mathbf{1} \\
1^T x = 1 \\
x \geq 0
\]

The optimal \( y^* \) is the solution of

\[
\text{maximize } w \\
\text{subject to } Ay \geq w \mathbf{1} \\
1^T y = 1 \\
y \geq 0
\]

The two LPs are duals and by strong duality the equality follows.
Nash equilibrium

Theorem

\[
\max_{y \in P_n} \min_{x \in P_m} x^T Ay = \min_{x \in P_m} \max_{y \in P_n} x^T Ay
\]

Consequence

The pair of mixed strategies \((x^*, y^*)\) attains the Nash equilibrium of the two-person matrix game, i.e.,

\[
x^T Ay^* \geq x^{*T} Ay^* \geq x^T Ay, \quad \forall x \in P_m, \forall y \in P_n
\]
Example

\[ A = \begin{bmatrix} 4 & 2 & 0 & -3 \\ -2 & -4 & -3 & 3 \\ -2 & -3 & 4 & 1 \end{bmatrix} \]

Optimal deterministic strategies

\[ \min_i \max_j A_{ij} = 3 > -2 = \max_j \min_i A_{ij} \]
Example

\[ y = (0, 0, 0, 0) \]
\[ x = (0, 1, 0) \]

\[ x \nless \nbigtriangleup A y \]

\[ A = \begin{bmatrix}
4 & 2 & 0 & -3 \\
-2 & -4 & -3 & 3 \\
-2 & -3 & 4 & 1
\end{bmatrix} \]

Optimal deterministic strategies

\[ \min_i \max_j A_{ij} = 3 > -2 = \max_j \min_i A_{ij} \]

Optimal mixed strategies

\[ x^* = (0.37, 0.33, 0.3), \quad y^* = (0.4, 0, 0.13, 0.47) \]

Expected payoff

\[ x^T A y^* = 0.2 \]
Farkas lemma
Feasibility of polyhedra

\[ P = \{x \mid Ax = b, \quad x \geq 0\} \]
Feasibility of polyhedra

\[ P = \{ x \mid Ax = b, \quad x \geq 0 \} \]

How to show that \( P \) is feasible?
Easy: we just need to provide an \( x \in P \), i.e., a certificate
Feasibility of polyhedra

\[ P = \{ x \mid Ax = b, \ x \geq 0 \} \]

How to show that \( P \) is **feasible**?

Easy: we just need to provide an \( x \in P \), i.e., a **certificate**

How to show that \( P \) is **infeasible**?
Farkas lemma

Theorem
Given $A$ and $b$, exactly one of the following statements is true:

1. There exists an $x$ with $Ax = b$, $x \geq 0$

2. There exists a $y$ with $A^T y \geq 0$, $b^T y < 0$
Farkas lemma

Geometric interpretation

1. First alternative
There exists an $x$ with $Ax = b$, $x \geq 0$

$$b = \sum_{i=1}^{n} x_i A_i, \quad x_i \geq 0, \ i = 1, \ldots, n$$

$b$ is in the cone generated by the columns of $A$
Farkas lemma

Geometric interpretation

1. First alternative
There exists an $x$ with $Ax = b$, $x \geq 0$

$$b = \sum_{i=1}^{n} x_i A_i, \quad x_i \geq 0, \quad i = 1, \ldots, n$$

$b$ is in the cone generated by the columns of $A$

2. Second alternative
There exists a $y$ with $A^T y \geq 0$, $b^T y < 0$

$$y^T A_i \geq 0, \quad i = 1, \ldots, m, \quad y^T b < 0$$

The hyperplane $y^T z = 0$ separates $b$ from $A_1, \ldots, A_n$
Farkas lemma

There exists $x$ with $Ax = b, \ x \geq 0$ \quad OR \quad There exists $y$ with $A^Ty \geq 0, (b^Ty < 0)$

Proof

1 and 2 cannot be both true (easy)
Farkas lemma

There exists $x$ with $Ax = b, \ x \geq 0$ \quad \text{OR} \quad \text{There exists } y \text{ with } A^T y \geq 0, \ b^T y < 0$

Proof

1 and 2 cannot be both false (duality)

<table>
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<tr>
<th>Primal</th>
<th>Dual</th>
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<tr>
<td>minimize ( 0 )</td>
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</tr>
<tr>
<td>subject to ( Ax = b )</td>
<td>subject to ( A^T y \geq 0 )</td>
</tr>
<tr>
<td>( x \geq 0 )</td>
<td></td>
</tr>
</tbody>
</table>
## Farkas lemma

There exists $x$ with $Ax = b$, $x \geq 0$ \hspace{1cm} OR \hspace{1cm} There exists $y$ with $A^Ty \geq 0$, $b^Ty < 0$

### Proof

1 and 2 cannot be both false (duality)

<table>
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<td>subject to $A^Ty \geq 0$</td>
</tr>
<tr>
<td>$x \geq 0$</td>
<td>$y = 0$ always feasible</td>
</tr>
</tbody>
</table>

**Strong duality holds**

$d^* \neq -\infty$, $p^* = d^*$
Farkas lemma

There exists $x$ with $Ax = b$, $x \geq 0$ \textbf{OR} There exists $y$ with $A^T y \geq 0$, $b^T y < 0$

Proof

1 and 2 cannot be both false (duality)

\begin{align*}
\text{Primal} & \quad \begin{cases}
\text{minimize} & 0 \\
\text{subject to} & Ax = b \\
& x \geq 0
\end{cases} \\
\text{Dual} & \quad \begin{cases}
\text{maximize} & -b^T y \\
\text{subject to} & A^T y \geq 0
\end{cases}
\end{align*}

\textbf{Alternative 1:} primal feasible $p^* = d^* = 0$

$b^T y \geq 0$ for all $y$ such that $A^T y \geq 0$
Farkas lemma

There exists \( x \) with \( Ax = b, \ x \geq 0 \) \hspace{1cm} OR \hspace{1cm} There exists \( y \) with \( A^T y \geq 0, \ b^T y < 0 \)

Proof

1 and 2 cannot be both false (duality)

Primal

\begin{align*}
\text{minimize} & \quad 0 \\
\text{subject to} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}

Dual

\begin{align*}
\text{maximize} & \quad -b^T y \\
\text{subject to} & \quad A^T y \geq 0
\end{align*}

\[ \tilde{y} = \alpha y \]
\[ \alpha > 0 \]

Alternative 2: primal infeasible \( p^* = d^* = +\infty \)

There exists \( y \) such that \( A^T y \geq 0 \) and \( b^T y < 0 \)
Farkas lemma

There exists $x$ with $Ax = b, \ x \geq 0$ \hspace{1cm} OR \hspace{1cm} There exists $y$ with $A^T y \geq 0, \ b^T y < 0$

Proof

1 and 2 cannot be both false (duality)

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<td>$y$</td>
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**Alternative 2:** primal infeasible $p^* = d^* = +\infty$

There exists $y$ such that $A^T y \geq 0$ and $b^T y < 0$

$y$ is an infeasibility certificate
Farkas lemma

Many variations

There exists \( x \) with \( Ax = b, \ x \geq 0 \)

\[ OR \]

There exists \( y \) with \( A^T y \geq 0, \ b^T y < 0 \)

\-----------------------------

There exists \( x \) with \( Ax \leq b, \ x \geq 0 \)

\[ OR \]

There exists \( y \) with \( A^T y \geq 0, \ b^T y < 0, \ y \geq 0 \)

\-----------------------------

There exists \( x \) with \( Ax \leq b \)

\[ OR \]

There exists \( y \) with \( A^T y = 0, \ b^T y < 0, \ y \geq 0 \)
Complementary slackness
Optimality conditions

**Primal**
- minimize $c^T x$
- subject to $Ax \leq b$

**Dual**
- maximize $-b^T y$
- subject to $A^T y + c = 0$
  - $y \geq 0$
Optimality conditions

**Primal**
- minimize \( c^T x \)
- subject to \( Ax \leq b \)

**Dual**
- maximize \(-b^T y\)
- subject to \( A^T y + c = 0\)  
  \(y \geq 0\)

\(x\) and \(y\) are **primal** and **dual** optimal if and only if

- **\(x\) is primal feasible:** \(Ax \leq b\)
- **\(y\) is dual feasible:** \(A^T y + c = 0\) and \(y \geq 0\)
- The **duality gap** is zero: \(c^T x + b^T y = 0\)
Optimality conditions

Primal

\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax \leq b
\end{align*}

Dual

\begin{align*}
\text{maximize} & \quad -b^T y \\
\text{subject to} & \quad A^T y + c = 0 \\
& \quad y \geq 0
\end{align*}

\(x\) and \(y\) are \textbf{primal} and \textbf{dual} optimal if and only if

\begin{itemize}
\item \(x\) is \textbf{primal feasible}: \(Ax \leq b\)
\item \(y\) is \textbf{dual feasible}: \(A^T y + c = 0\) and \(y \geq 0\)
\item The \textbf{duality gap} is zero: \(c^T x + b^T y = 0\)
\end{itemize}

Can we relate \(x\) and \(y\) (not only the objective)?
Complementary slackness

Primal

minimize \( c^T x \)

subject to \( Ax \leq b \)

Dual

maximize \(-b^T y\)

subject to \( A^T y + c = 0 \)
\( y \geq 0 \)

Theorem
Primal, dual feasible \( x, y \) are optimal if and only if

\[ y_i(b_i - a_i^T x) = 0, \quad i = 1, \ldots, m \]

i.e., at optimum, \( b - Ax \) and \( y \) have a complementary sparsity pattern:

\[ y_i > 0 \quad \Rightarrow \quad a_i^T x = b_i \]
\[ a_i^T x < b_i \quad \Rightarrow \quad y_i = 0 \]
Complementary slackness

Primal
minimize \( c^T x \)
subject to \( Ax \leq b \)

Dual
maximize \( -b^T y \)
subject to \( A^T y + c = 0 \), \( y \geq 0 \)

Proof
The duality gap at primal feasible \( x \) and dual feasible \( y \) can be written as
\[
c^T x + b^T y = (-A^T y)^T x + b^T y = (b - Ax)^T y = \sum_{i=1}^m y_i (b_i - a_i^T x) = 0
\]
Complementary slackness

Primal

minimize $c^T x$

subject to $Ax \leq b$

Dual

maximize $-b^T y$

subject to $A^T y + c = 0$

$y \geq 0$

Proof

The duality gap at primal feasible $x$ and dual feasible $y$ can be written as

$$c^T x + b^T y = (-A^T y)^T x + b^T y = (b - Ax)^T y = \sum_{i=1}^{m} y_i (b_i - a_i^T x) = 0$$

Since all the elements of the sum are nonnegative, they must all be $0$.

$\blacksquare$
Complementary slackness

Primal

minimize \( c^T x \)
subject to \( Ax \leq b \)

Dual

maximize \(-b^T y\)
subject to \( A^T y + c = 0\)
\( y \geq 0 \)

Proof

The duality gap at primal feasible \( x \) and dual feasible \( y \) can be written as

\[
c^T x + b^T y = (-A^T y)^T x + b^T y = (b - Ax)^T y = \sum_{i=1}^{m} y_i (b_i - a_i^T x) = 0
\]

Since all the elements of the sum are nonnegative, they must all be 0.

For feasible \( x \) and \( y \) complementary slackness = zero duality gap
Example

minimize $-4x_1 - 5x_2$

subject to

$$\begin{bmatrix}
-1 & 0 \\
2 & 1 \\
0 & -1 \\
1 & 2
\end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$

Let’s show that feasible $x = (1, 1)$ is optimal
Example

minimize $-4x_1 - 5x_2$

subject to

Let's show that feasible $x = (1, 1)$ is optimal

Second and fourth constraints are active at $x$ $\longrightarrow$ $y = (0, y_2, 0, y_4)$

$A^T y = -c$ $\Rightarrow$ $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_2 \\ y_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

and $y_2 \geq 0$, $y_4 \geq 0$
Example

minimize $-4x_1 - 5x_2$

subject to

$$\begin{bmatrix}
-1 & 0 \\
2 & 1 \\
0 & -1 \\
1 & 2
\end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 3 \\ 0 \\ 3 \end{bmatrix}$$

Let’s show that feasible $x = (1, 1)$ is optimal

Second and fourth constraints are active at $x$ $\quad \rightarrow \quad y = (0, y_2, 0, y_4)$

$A^T y = -c \quad \Rightarrow \quad \begin{bmatrix}
2 & 1 \\
1 & 2
\end{bmatrix} \begin{bmatrix} y_2 \\ y_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

and $y_2 \geq 0, \quad y_4 \geq 0$

$y = (0, 1, 0, 2)$ satisfies these conditions and proves that $x$ is optimal
Example

minimize \(-4x_1 - 5x_2\)

subject to

\[
\begin{bmatrix}
-1 & 0 \\
2 & 1 \\
0 & -1 \\
1 & 2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\leq
\begin{bmatrix}
0 \\
3 \\
0 \\
3
\end{bmatrix}
\]

Let’s show that feasible \(x = (1, 1)\) is optimal

Second and fourth constraints are active at \(x\) \(\rightarrow\) \(y = (0, y_2, 0, y_4)\)

\[
A^Ty = -c \quad \Rightarrow \quad \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_2 \\ y_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}
\]

and \(y_2 \geq 0, \quad y_4 \geq 0\)

\(y = (0, 1, 0, 2)\) satisfies these conditions and proves that \(x\) is optimal

Complementary slackness is useful to recover \(y^*\) from \(x^*\)
Geometric interpretation

Example in $\mathbb{R}^2$

Two active constraints at optimum: $a_1^T x^* = b_1$, $a_2^T x^* = b_2$
Geometric interpretation
Example in $\mathbb{R}^2$

Two active constraints at optimum: $a_1^T x^* = b_1$, $a_2^T x^* = b_2$

Optimal dual solution $y$ satisfies:

$$A^T y + c = 0, \quad y \geq 0, \quad y_i = 0 \text{ for } i \neq \{1, 2\}$$

In other words, $-c = a_1 y_1 + a_2 y_2$ with $y_1, y_2 \geq 0$
KKT Conditions
Lagrangian and duality

Primal
minimize $c^T x$
subject to $Ax \leq b$

Dual
maximize $-b^T y$
subject to $A^T y + c = 0$
$y \geq 0$
Lagrangian and duality

Primal
minimize $c^T x$
subject to $Ax \leq b$

Dual
maximize $-b^T y$
subject to $A^T y + c = 0$
$y \geq 0$

Dual function

$$g(y) = \min_{x} \left( c^T x + y^T (Ax - b) \right)$$

$$= -b^T y + \min_{x} \left( c + A^T y \right)^T x$$

$$= \begin{cases} 
-b^T y & \text{if } c + A^T y = 0 \\
-\infty & \text{otherwise}
\end{cases}$$
Lagrangian and duality

**Primal**

minimize \( c^T x \)

subject to \( Ax \leq b \)

**Dual function**

\[
g(y) = \min_x (c^T x + y^T (Ax - b)) 
\]

\[
= -b^T y + \min_x (c + A^T y)^T x 
\]

\[
= \begin{cases} 
  -b^T y & \text{if } c + A^T y = 0 \\
  -\infty & \text{otherwise}
\end{cases}
\]

**Dual**

maximize \(-b^T y\)

subject to \(A^T y + c = 0\)

\(y \geq 0\)

**Lagrangian**

\[
L(x, y) = c^T x + y^T (Ax - b)
\]
Lagrangian and duality

Primal

minimize \( c^T x \)

subject to \( Ax \leq b \)

Dual function

\[
g(y) = \min_x (c^T x + y^T (Ax - b))
\]

\[
= -b^T y + \min_x (c + A^T y)^T x
\]

\[
= \begin{cases} 
-b^T y & \text{if } c + A^T y = 0 \\
-\infty & \text{otherwise}
\end{cases}
\]

Lagrangian

\[
L(x, y) = c^T x + y^T (Ax - b)
\]

\[
\nabla_x L(x, y) = c + A^T y = 0
\]
Karush-Kuhn-Tucker conditions
Optimality conditions for linear optimization

<table>
<thead>
<tr>
<th>Primal</th>
<th>Dual</th>
</tr>
</thead>
<tbody>
<tr>
<td>minimize $c^T x$</td>
<td>maximize $-b^T y$</td>
</tr>
<tr>
<td>subject to $Ax \leq b$</td>
<td>subject to $A^T y + c = 0$</td>
</tr>
<tr>
<td>$y \geq 0$</td>
<td>$y \geq 0$</td>
</tr>
</tbody>
</table>

Primal feasibility

Dual feasibility

Complementary slackness

$$Ax \leq b$$

$$\nabla_x L(x, y) = A^T y + c = 0 \quad \text{and} \quad y \geq 0$$

$$y_i (Ax - b)_i = 0, \quad i = 1, \ldots, m$$
Karush-Kuhn-Tucker conditions

Solving linear optimization problems

Primal
minimize $c^T x$
subject to $Ax \leq b$

Dual
maximize $-b^T y$
subject to $A^T y + c = 0$
$y \geq 0$

We can solve our optimization problem by solving a system of equations

$$
\nabla_x L(x, y) = A^T y + c = 0 \quad \text{(Stationarity)}
$$
$$
b - Ax \geq 0 \quad \text{(Primal feasibility)}
$$
$$
y \geq 0 \quad \text{(Dual feasibility)}
$$
$$
y^T (b - Ax) = 0 \quad \text{(Complementary slackness)}
$$
Linear optimization duality

Today, we learned to:

- **Interpret** linear optimization duality using game theory
- **Prove** Farkas lemma using duality
- **Geometrically link** primal and dual solutions with complementary slackness
- **Derive** KKT optimality conditions
References

• Bertsimas and Tsitsiklis: Introduction to Linear Optimization
  • Chapter 4: Duality theory
• R. Vanderbei: Linear Programming — Foundations and Extensions
  • Chapter 11: Game Theory
Next lecture

- Sensitivity analysis