Ed Forum

• A feasible direction is any direction that "stays in" P, whereas a basic direction is one that points in the direction of a neighboring basic solution. Is there a difference between a feasible direction and a basic direction?

• Why does maximizing the lower bound of the cost make it “better”? 

Recap
Optimal objective values

Primal
minimize $c^T x$
subject to $Ax \leq b$

$p^*$ is the primal optimal value

Dual
maximize $-b^T y$
subject to $A^T y + c = 0$
y $\geq 0$

d* is the dual optimal value

Primal infeasible: $p^* = +\infty$
Primal unbounded: $p^* = -\infty$

Dual infeasible: $d^* = -\infty$
Dual unbounded: $d^* = +\infty$
Relationship between primal and dual

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<tr>
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<th>( p^* = +\infty )</th>
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- Upper-right excluded by **weak duality**
- \((1, 1)\) and \((3, 3)\) proven by **weak duality**
- \((3, 1)\) and \((2, 2)\) proven by **strong duality**
Today’s agenda
More on duality

• Two-person zero-sum games
• Farkas lemma
• Complementary slackness
• KKT conditions
Two-person games
Rock paper scissors

Rules
At count to three declare one of: Rock, Paper, or Scissors

Winners
Identical selection is a draw, otherwise:
• Rock beats (“dulls”) scissors
• Scissors beats (“cuts”) paper
• Paper beats (“covers”) rock

Extremely popular: world RPS society, USA RPS league, etc.
Two-person zero-sum game

- Player 1 (P1) chooses a number $i \in \{1, \ldots, m\}$ (one of $m$ actions)
- Player 2 (P2) chooses a number $j \in \{1, \ldots, n\}$ (one of $n$ actions)

Two players make their choice independently

**Rule**

Player 1 pays $A_{ij}$ to player 2

$A \in \mathbb{R}^{m \times n}$ is the **payoff matrix**

**Rock, Paper, Scissors**

\[
A = \begin{bmatrix}
R & P & S \\
0 & 1 & -1 \\
-1 & 0 & 1 \\
1 & -1 & 0
\end{bmatrix}
\]
Mixed (randomized) strategies

Deterministic strategies can be systematically defeated

Randomized strategies

- P1 chooses randomly according to distribution $x$:
  $$x_i = \text{probability that P1 selects action } i$$

- P2 chooses randomly according to distribution $y$:
  $$y_j = \text{probability that P2 selects action } j$$

Expected payoff (from P1 P2), if they use mixed-strategies $x$ and $y$, 

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j A_{ij} = x^T A y$$
Mixed strategies and probability simplex

**Probability simplex** in $\mathbb{R}^k$

$P_k = \{ p \in \mathbb{R}^k \mid p \geq 0, \quad 1^T p = 1 \}$

**Mixed strategy**

For a game player, a mixed strategy is a distribution over all possible deterministic strategies.

The set of all mixed strategies is the probability simplex $\rightarrow x \in P_m, \quad y \in P_n$
Optimal mixed strategies

P1: optimal strategy $x^*$ is the solution of

\[
\text{minimize} \quad \max_{y \in P_n} x^T A y \\
\text{subject to} \quad x \in P_m
\]

P2: optimal strategy $y^*$ is the solution of

\[
\text{maximize} \quad \min_{x \in P_m} x^T A y \\
\text{subject to} \quad y \in P_n
\]

Optimal strategies $x^*$ and $y^*$ can be computed using linear optimization.
Minmax theorem

Theorem

\[
\max_{y \in P_n} \min_{x \in P_m} x^T Ay = \min_{x \in P_m} \max_{y \in P_n} x^T Ay
\]

Proof

The optimal \( x^* \) is the solution of

minimize \( t \)

subject to

\( A^T x \leq t1 \)
\( 1^T x = 1 \)
\( x \geq 0 \)

The optimal \( y^* \) is the solution of

maximize \( w \)

subject to

\( Ay \geq w1 \)
\( 1^T y = 1 \)
\( y \geq 0 \)

The two LPs are **duals** and by **strong duality** the equality follows.
Nash equilibrium

Theorem

\[
\max_{y \in P_n} \min_{x \in P_m} x^T A y = \min_{x \in P_m} \max_{y \in P_n} x^T A y
\]

Consequence

The pair of mixed strategies \((x^*, y^*)\) attains the Nash equilibrium of the two-person matrix game, i.e.,

\[
x^T A y^* \geq x^{*T} A y^* \geq x^{*T} A y, \quad \forall x \in P_m, \ \forall y \in P_n
\]
Example

\[
A = \begin{bmatrix}
4 & 2 & 0 & -3 \\
-2 & -4 & -3 & 3 \\
-2 & -3 & 4 & 1
\end{bmatrix}
\]

Optimal deterministic strategies
\[
\min_i \max_j A_{ij} = 3 > -2 = \max_j \min_i A_{ij}
\]

Optimal mixed strategies
\[
x^* = (0.37, 0.33, 0.3), \quad y^* = (0.4, 0, 0.13, 0.47)
\]

Expected payoff
\[
x^T A y^* = 0.2
\]
Farkas lemma
Feasibility of polyhedra

\[ P = \{ x \mid Ax = b, \quad x \geq 0 \} \]

How to show that \( P \) is **feasible**?
Easy: we just need to provide an \( x \in P \), i.e., a **certificate**

How to show that \( P \) is **infeasible**?
Farkas lemma

Theorem
Given $A$ and $b$, exactly one of the following statements is true:

1. There exists an $x$ with $Ax = b$, $x \geq 0$
2. There exists a $y$ with $A^T y \geq 0$, $b^T y < 0$
Farkas lemma

Geometric interpretation

1. First alternative
   There exists an \( x \) with \( Ax = b, x \geq 0 \)
   \[ b = \sum_{i=1}^{n} x_i A_i, \quad x_i \geq 0, \quad i = 1, \ldots, n \]
   \( b \) is in the cone generated by the columns of \( A \)

2. Second alternative
   There exists a \( y \) with \( A^T y \geq 0, b^T y < 0 \)
   \[ y^T A_i \geq 0, \quad i = 1, \ldots, m, \quad y^T b < 0 \]
   The hyperplane \( y^T z = 0 \) separates \( b \) from \( A_1, \ldots, A_n \)
Farkas lemma

There exists $x$ with $Ax = b, \ x \geq 0$ \hspace{1cm} \textbf{OR} \hspace{1cm} \text{There exists } y \text{ with } A^T y \geq 0, \ b^T y < 0$

Proof

1 and 2 cannot be both true (easy)

\[ x \geq 0, \ Ax = b \text{ and } y^T A \geq 0 \hspace{1cm} \rightarrow \hspace{1cm} y^T b = y^T Ax \geq 0 \]
Farkas lemma

There exists $x$ with $Ax = b$, $x \geq 0$ \text{ OR } There exists $y$ with $A^T y \geq 0$, $b^T y < 0$

Proof

1 and 2 cannot be both false (duality)

**Primal**

| minimize | 0 |
| subject to | $Ax = b$, $x \geq 0$ |

**Dual**

| maximize $-b^T y$ |
| subject to | $A^T y \geq 0$ |

$y = 0$ always feasible

**Strong duality holds**

$d^* \neq -\infty$, $p^* = d^*$
Farkas lemma

There exists \( x \) with \( Ax = b, \ x \geq 0 \) \ OR \ There exists \( y \) with \( A^T y \geq 0, \ b^T y < 0 \)

Proof

1 and 2 cannot be both false (duality)

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Alternative 1: primal feasible \( p^* = d^* = 0 \)

\( b^T y \geq 0 \) for all \( y \) such that \( A^T y \geq 0 \)
Farkas lemma

There exists \( x \) with \( Ax = b, \ x \geq 0 \) \ OR \ There exists \( y \) with \( A^T y \geq 0, \ b^T y < 0 \)

Proof

1 and 2 cannot be both false (duality)

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**Alternative 2:** primal infeasible \( p^* = d^* = +\infty \)

There exists \( y \) such that \( A^T y \geq 0 \) and \( b^T y < 0 \)

\( y \) is an infeasibility certificate
Farkas lemma
Many variations

There exists $x$ with $Ax = b$, $x \geq 0$

or

There exists $y$ with $A^T y \geq 0$, $b^T y < 0$

---------------------------

There exists $x$ with $Ax \leq b$, $x \geq 0$

or

There exists $y$ with $A^T y \geq 0$, $b^T y < 0$, $y \geq 0$

---------------------------

There exists $x$ with $Ax \leq b$

or

There exists $y$ with $A^T y = 0$, $b^T y < 0$, $y \geq 0$
Complementary slackness
## Optimality conditions

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\( x \) and \( y \) are **primal** and **dual** optimal if and only if

- \( x \) is **primal feasible**: \( Ax \leq b \)
- \( y \) is **dual feasible**: \( A^T y + c = 0 \) and \( y \geq 0 \)
- The **duality gap** is zero: \( c^T x + b^T y = 0 \)

Can we **relate** \( x \) and \( y \) (not only the objective)?
Complementary slackness

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Theorem
Primal,dual feasible $x, y$ are optimal if and only if

$$y_i (b_i - a_i^T x) = 0, \quad i = 1, \ldots, m$$

i.e., at optimum, $b - Ax$ and $y$ have a complementary sparsity pattern:

$$y_i > 0 \quad \Rightarrow \quad a_i^T x = b_i$$

$$a_i^T x < b_i \quad \Rightarrow \quad y_i = 0$$
Complementary slackness

Primal

minimize \( c^T x \)
subject to \( Ax \leq b \)

Dual

maximize \( -b^T y \)
subject to \( A^T y + c = 0 \)
\( y \geq 0 \)

Proof

The duality gap at primal feasible \( x \) and dual feasible \( y \) can be written as
\[
c^T x + b^T y = (-A^T y)^T x + b^T y = (b - Ax)^T y = \sum_{i=1}^{m} y_i (b_i - a_i^T x) = 0
\]

Since all the elements of the sum are nonnegative, they must all be 0.

For feasible \( x \) and \( y \) complementary slackness = zero duality gap
Example

minimize \(-4x_1 - 5x_2\)

subject to

\[
\begin{bmatrix}
-1 & 0 \\
2 & 1 \\
0 & -1 \\
1 & 2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\leq
\begin{bmatrix}
0 \\
3 \\
0 \\
3
\end{bmatrix}
\]

Let’s show that feasible \(x = (1, 1)\) is optimal

Second and fourth constraints are active at \(x \rightarrow y = (0, y_2, 0, y_4)\)

\[A^T y = -c \Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_2 \\ y_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}\]

and \(y_2 \geq 0, \ y_4 \geq 0\)

\(y = (0, 1, 0, 2)\) satisfies these conditions and proves that \(x\) is optimal

Complementary slackness is useful to recover \(y^*\) from \(x^*\)
Geometric interpretation
Example in $\mathbb{R}^2$

Two active constraints at optimum: $a_1^T x^* = b_1, \quad a_2^T x^* = b_2$

Optimal dual solution $y$ satisfies:

$$A^T y + c = 0, \quad y \geq 0, \quad y_i = 0 \text{ for } i \neq \{1, 2\}$$

In other words, $-c = a_1 y_1 + a_2 y_2$ with $y_1, y_2 \geq 0$
KKT Conditions
Lagrangian and duality

**Primal**

- minimize $c^T x$
- subject to $Ax \leq b$

**Dual function**

$$g(y) = \min_x (c^T x + y^T (Ax - b))$$

$$= -b^T y + \min_x (c + A^T y)^T x$$

$$= \begin{cases} -b^T y & \text{if } c + A^T y = 0 \\ -\infty & \text{otherwise} \end{cases}$$

**Lagrangian**

$$L(x, y) = c^T x + y^T (Ax - b)$$

$$\nabla_x L(x, y) = c + A^T y = 0$$

**Dual**

- maximize $-b^T y$
- subject to $A^T y + c = 0$
- $y \geq 0$
## Karush-Kuhn-Tucker conditions

Optimality conditions for linear optimization

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| subject to $Ax \leq b$ | subject to $A^T y + c = 0$
| | $y \geq 0$ |

### Primal feasibility

$Ax \leq b$

### Dual feasibility

$\nabla_x L(x, y) = A^T y + c = 0$ and $y \geq 0$

### Complementary slackness

$y_i(Ax - b)_i = 0, \quad i = 1, \ldots, m$
Karush-Kuhn-Tucker conditions

Solving linear optimization problems

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We can solve our optimization problem by solving a system of equations

$$\nabla_x L(x, y) = A^T y + c = 0$$

$$b - Ax \geq 0$$

$$y \geq 0$$

$$y^T (b - Ax) = 0$$
Linear optimization duality

Today, we learned to:

• **Interpret** linear optimization duality using game theory
• **Prove** Farkas lemma using duality
• **Geometrically link** primal and dual solutions with complementary slackness
• **Derive** KKT optimality conditions
References

• Bertsimas and Tsitsiklis: Introduction to Linear Optimization
  • Chapter 4: Duality theory
• R. Vanderbei: Linear Programming — Foundations and Extensions
  • Chapter 11: Game Theory
Next lecture

- Sensitivity analysis