• A feasible direction is any direction that "stays in" P, whereas a basic direction is one that points in the direction of a neighboring basic solution. Is there a difference between a feasible direction and a basic direction?

• Why does maximizing the lower bound of the cost make it “better”?
Recap
Optimal objective values

Primal
minimize \( c^T x \)
subject to \( Ax \leq b \)

\( p^* \) is the primal optimal value

Dual
maximize \( -b^T y \)
subject to \( A^T y + c = 0 \)
\( y \geq 0 \)

\( d^* \) is the dual optimal value

Primal infeasible: \( p^* = +\infty \)
Primal unbounded: \( p^* = -\infty \)

Dual infeasible: \( d^* = -\infty \)
Dual unbounded: \( d^* = +\infty \)
Relationship between primal and dual

<table>
<thead>
<tr>
<th></th>
<th>$p^* = +\infty$</th>
<th>$p^*$ finite</th>
<th>$p^* = -\infty$</th>
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</thead>
<tbody>
<tr>
<td>$d^* = +\infty$</td>
<td>primal inf.</td>
<td></td>
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<tr>
<td></td>
<td>dual unb.</td>
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<tr>
<td>$d^*$ finite</td>
<td>optimal values</td>
<td></td>
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<td></td>
<td>equal</td>
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<tr>
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<td>exception</td>
<td></td>
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<tr>
<td></td>
<td>dual inf</td>
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</tbody>
</table>

- Upper-right excluded by **weak duality**
- $(1, 1)$ and $(3, 3)$ proven by **weak duality**
- $(3, 1)$ and $(2, 2)$ proven by **strong duality**
Today’s agenda
More on duality

• Two-person zero-sum games
• Farkas lemma
• Complementary slackness
• KKT conditions
Two-person games
Rock paper scissors

Rules
At count to three declare one of: Rock, Paper, or Scissors

Winners
Identical selection is a draw, otherwise:
• Rock beats ("dulls") scissors
• Scissors beats ("cuts") paper
• Paper beats ("covers") rock

Extremely popular: world RPS society, USA RPS league, etc.
Two-person zero-sum game

• Player 1 (P1) chooses a number \( i \in \{1, \ldots, m\} \) (one of \( m \) actions)
• Player 2 (P2) chooses a number \( j \in \{1, \ldots, n\} \) (one of \( n \) actions)

Two players make their choice independently

Rule

Player 1 pays \( A_{ij} \) to player 2

\[
A \in \mathbb{R}^{m \times n} \text{ is the payoff matrix}
\]

Rock, Paper, Scissors

\[
R = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}
\]

\[
P = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}
\]

\[
S = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}
\]
Mixed (randomized) strategies

Deterministic strategies can be systematically defeated

Randomized strategies
- P1 chooses randomly according to distribution $x$:
  \[ x_i = \text{probability that P1 selects action } i \]
- P2 chooses randomly according to distribution $y$:
  \[ y_j = \text{probability that P2 selects action } j \]

Expected payoff (from P1 P2), if they use mixed-strategies $x$ and $y$,
\[
\sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j A_{ij} = x^T A y
\]
Mixed strategies and probability simplex

**Probability simplex** in $\mathbb{R}^k$

$$P_k = \{p \in \mathbb{R}^k \mid p \geq 0, \quad 1^T p = 1\}$$

**Mixed strategy**

For a game player, a mixed strategy is a distribution over all possible deterministic strategies.

The **set of all mixed strategies** is the probability simplex $\rightarrow x \in P_m, \quad y \in P_n$
Optimal mixed strategies

P1: optimal strategy $x^*$ is the solution of

\[
\text{minimize } \max_{y \in P_n} x^T Ay \\
\text{subject to } x \in P_m
\]

\[
\text{minimize } \max_{j=1,\ldots,n} (A^T x)_j \\
\text{subject to } x \in P_m
\]

Inner problem over deterministic strategies (vertices)

P2: optimal strategy $y^*$ is the solution of

\[
\text{maximize } \min_{x \in P_m} x^T Ay \\
\text{subject to } y \in P_n
\]

\[
\text{maximize } \min_{i=1,\ldots,m} (Ay)_i \\
\text{subject to } y \in P_n
\]

Optimal strategies $x^*$ and $y^*$ can be computed using linear optimization
Minmax theorem

Theorem

\[
\max_{y \in P_n} \min_{x \in P_m} x^T A y = \min_{x \in P_m} \max_{y \in P_n} x^T A y
\]

Proof

The optimal \( x^\ast \) is the solution of

\[
\begin{align*}
\text{minimize} & \quad t \\
\text{subject to} & \quad A^T x \leq t1 \\
& \quad 1^T x = 1 \\
& \quad x \geq 0
\end{align*}
\]

The optimal \( y^\ast \) is the solution of

\[
\begin{align*}
\text{maximize} & \quad w \\
\text{subject to} & \quad A y \geq w1 \\
& \quad 1^T y = 1 \\
& \quad y \geq 0
\end{align*}
\]

The two LPs are \textbf{duals} and by \textbf{strong duality} the equality follows.

\[\square\]
Nash equilibrium

Theorem

\[
\max_{y \in P_n} \min_{x \in P_m} x^T Ay = \min_{x \in P_m} \max_{y \in P_n} x^T Ay
\]

Consequence

The pair of mixed strategies \((x^*, y^*)\) attains the **Nash equilibrium** of the two-person matrix game, i.e.,

\[
x^T Ay^* \geq x^{*T} Ay^* \geq x^T Ay, \quad \forall x \in P_m, \forall y \in P_n
\]
Example

\[ A = \begin{bmatrix}
4 & 2 & 0 & -3 \\
-2 & -4 & -3 & 3 \\
-2 & -3 & 4 & 1 \\
\end{bmatrix} \]

**Optimal deterministic strategies**

\[
\min_i \max_j A_{ij} = 3 > -2 = \max_j \min_i A_{ij}
\]

**Optimal mixed strategies**

\[
x^* = (0.37, 0.33, 0.3), \quad y^* = (0.4, 0, 0.13, 0.47)
\]

**Expected payoff**

\[
x^{*T} A y^* = 0.2
\]
Farkas lemma
Feasibility of polyhedra

\[ P = \{ x \mid Ax = b, \quad x \geq 0 \} \]

How to show that \( P \) is **feasible**?

Easy: we just need to provide an \( x \in P \), i.e., a **certificate**

How to show that \( P \) is **infeasible**?
Farkas lemma

**Theorem**
Given $A$ and $b$, exactly one of the following statements is true:

1. There exists an $x$ with $Ax = b$, $x \geq 0$

2. There exists a $y$ with $A^Ty \geq 0$, $b^Ty < 0$
**Farkas lemma**

**Geometric interpretation**

1. First alternative
   There exists an $x$ with $Ax = b, \ x \geq 0$
   
   $$b = \sum_{i=1}^{n} x_i A_i, \ \ x_i \geq 0, \ i = 1, \ldots, n$$

   $b$ is in the cone generated by the columns of $A$

2. Second alternative
   There exists a $y$ with $A^T y \geq 0, \ b^T y < 0$
   
   $$y^T A_i \geq 0, \ i = 1, \ldots, m, \quad y^T b < 0$$

   The hyperplane $y^T z = 0$

   separates $b$ from $A_1, \ldots, A_n$
Farkas lemma

There exists $x$ with $Ax = b, \ x \geq 0$ \hspace{1cm} OR \hspace{1cm} There exists $y$ with $A^T y \geq 0, \ b^T y < 0$

Proof

1 and 2 cannot be both true (easy)

$$x \geq 0, \ Ax = b \ \text{and} \ y^T A \geq 0 \hspace{1cm} \rightarrow \hspace{1cm} y^T b = y^T Ax \geq 0$$
Farkas lemma

There exists \( x \) with \( Ax = b, \ x \geq 0 \) \hspace{1cm} \text{OR} \hspace{1cm} \text{There exists} \ y \ \text{with} \ A^T y \geq 0, \ b^T y < 0

Proof

1 and 2 cannot be both false (duality)

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</tr>
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<td>subject to ( A^T y \geq 0 )</td>
</tr>
<tr>
<td>( x \geq 0 )</td>
<td>( y = 0 ) always feasible</td>
</tr>
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Strong duality holds

\( d^* \neq -\infty, \ p^* = d^* \)
Farkas lemma

There exists $x$ with $Ax = b$, $x \geq 0$ OR There exists $y$ with $A^T y \geq 0$, $b^T y < 0$

Proof

1 and 2 cannot be both false (duality)

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Alternative 1: primal feasible $p^* = d^* = 0$

$b^T y \geq 0$ for all $y$ such that $A^T y \geq 0$
Farkas lemma

There exists $x$ with $Ax = b$, $x \geq 0$ \quad OR \quad There exists $y$ with $A^T y \geq 0$, $b^T y < 0$

Proof

1 and 2 cannot be both false (duality)

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Alternative 2: primal infeasible $p^* = d^* = +\infty$

There exists $y$ such that $A^T y \geq 0$ and $b^T y < 0$

$y$ is an infeasibility certificate
Farkas lemma

Many variations

There exists $x$ with $Ax = b$, $x \geq 0$

OR

There exists $y$ with $A^T y \geq 0$, $b^T y < 0$

There exists $x$ with $Ax \leq b$, $x \geq 0$

OR

There exists $y$ with $A^T y \geq 0$, $b^T y < 0$, $y \geq 0$

There exists $x$ with $Ax \leq b$

OR

There exists $y$ with $A^T y = 0$, $b^T y < 0$, $y \geq 0
Complementary slackness
Optimality conditions

Primal

minimize $c^T x$

subject to $Ax \leq b$

Dual

maximize $-b^T y$

subject to $A^T y + c = 0$

$y \geq 0$

$x$ and $y$ are **primal** and **dual** optimal if and only if

- $x$ is **primal feasible**: $Ax \leq b$
- $y$ is **dual feasible**: $A^T y + c = 0$ and $y \geq 0$
- The **duality gap** is zero: $c^T x + b^T y = 0$

Can we relate $x$ and $y$ (not only the objective)?
Complementary slackness

Primal
minimize \( c^T x \)
subject to \( Ax \leq b \)

Dual
maximize \( -b^T y \)
subject to \( A^T y + c = 0 \)
\( y \geq 0 \)

Theorem
Primal dual feasible \( x, y \) are optimal if and only if
\[ y_i(b_i - a_i^T x) = 0, \quad i = 1, \ldots, m \]
i.e., at optimum, \( b - Ax \) and \( y \) have a complementary sparsity pattern:
\[ y_i > 0 \quad \Rightarrow \quad a_i^T x = b_i \]
\[ a_i^T x < b_i \quad \Rightarrow \quad y_i = 0 \]
Complementary slackness

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$y \geq 0$

**Proof**

The duality gap at primal feasible $x$ and dual feasible $y$ can be written as

$$c^T x + b^T y = (-A^T y)^T x + b^T y = (b - Ax)^T y = \sum_{i=1}^{m} y_i (b_i - a_i^T x) = 0$$

Since all the elements of the sum are nonnegative, they must all be 0.

For feasible $x$ and $y$ complementary slackness = zero duality gap
Example

minimize $-4x_1 - 5x_2$

subject to

$$
\begin{bmatrix}
-1 & 0 \\
2 & 1 \\
0 & -1 \\
1 & 2 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\end{bmatrix}
\leq
\begin{bmatrix}
0 \\
3 \\
0 \\
3 \\
\end{bmatrix}
$$

Let’s show that feasible $x = (1, 1)$ is optimal

Second and fourth constraints are active at $x \rightarrow y = (0, y_2, 0, y_4)$

$$
A^T y = -c \Rightarrow 
\begin{bmatrix}
2 & 1 \\
1 & 2 \\
\end{bmatrix}
\begin{bmatrix}
y_2 \\
y_4 \\
\end{bmatrix} = 
\begin{bmatrix}
4 \\
5 \\
\end{bmatrix}
$$

and $y_2 \geq 0$, $y_4 \geq 0$

$y = (0, 1, 0, 2)$ satisfies these conditions and proves that $x$ is optimal

Complementary slackness is useful to recover $y^*$ from $x^*$
Geometric interpretation

Example in $\mathbb{R}^2$

Two active constraints at optimum: $a_1^T x^* = b_1, \quad a_2^T x^* = b_2$

Optimal dual solution $y$ satisfies:

$$A^T y + c = 0, \quad y \geq 0, \quad y_i = 0 \text{ for } i \neq \{1, 2\}$$

In other words, $-c = a_1 y_1 + a_2 y_2$ with $y_1, y_2 \geq 0$
KKT Conditions
Lagrangian and duality

Primal

minimize \( c^T x \)
subject to \( Ax \leq b \)

Dual

maximize \( -b^T y \)
subject to \( A^T y + c = 0 \)
\( y \geq 0 \)

Dual function

\[
g(y) = \min_{x} \left( c^T x + y^T (Ax - b) \right)
\]

\[
= -b^T y + \min_{x} \left( c + A^T y \right)^T x
\]

\[
= \begin{cases} 
-b^T y & \text{if } c + A^T y = 0 \\
-\infty & \text{otherwise}
\end{cases}
\]

Lagrangian

\[
L(x, y) = c^T x + y^T (Ax - b)
\]

\[
\nabla_x L(x, y) = c + A^T y = 0
\]
Karush-Kuhn-Tucker conditions
Optimality conditions for linear optimization

**Primal**

minimize \( c^T x \)

subject to \( Ax \leq b \)

**Dual**

maximize \( -b^T y \)

subject to \( A^T y + c = 0 \)

\( y \geq 0 \)

**Primal feasibility**

\( Ax \leq b \)

**Dual feasibility**

\( \nabla_x L(x, y) = A^T y + c = 0 \) and \( y \geq 0 \)

**Complementary slackness**

\( y_i (Ax - b)_i = 0, \quad i = 1, \ldots, m \)
Karush-Kuhn-Tucker conditions

Solving linear optimization problems

**Primal**
- minimize $c^T x$
- subject to $Ax \leq b$

**Dual**
- maximize $-b^T y$
- subject to $A^T y + c = 0$
  - $y \geq 0$

We can solve our optimization problem by solving a system of equations

$$
\nabla_x L(x, y) = A^T y + c = 0 \\
b - Ax \geq 0 \\
y \geq 0 \\
y^T (b - Ax) = 0
$$
Linear optimization duality

Today, we learned to:

• **Interpret** linear optimization duality using game theory
• **Prove** Farkas lemma using duality
• **Geometrically link** primal and dual solutions with complementary slackness
• **Derive** KKT optimality conditions
References

- Bertsimas and Tsitsiklis: Introduction to Linear Optimization
  - Chapter 4: Duality theory
- R. Vanderbei: Linear Programming — Foundations and Extensions
  - Chapter 11: Game Theory
Next lecture

• Sensitivity analysis