ORF307 — Optimization

14. Duality II
how exactly do we apply Lagrange multipliers in the context of linear programming to find the best lower bounds?

can we interpret the primal as the dual problem and vice versa? Especially since they are solving for the same thing as stated in the Strong duality theorem.
Recap
Weak and strong duality
Optimal objective values

**Primal**

- minimize \( c^T x \)
- subject to \( Ax \leq b \)

\[ p^* \text{ is the primal optimal value} \]

**Dual**

- maximize \(-b^T y\)
- subject to \( A^T y + c = 0 \)
\[ y \geq 0 \]

\[ d^* \text{ is the dual optimal value} \]

Primal infeasible: \( p^* = +\infty \)
Primal unbounded: \( p^* = -\infty \)

Dual infeasible: \( d^* = -\infty \)
Dual unbounded: \( d^* = +\infty \)
Weak duality

**Theorem**
If \( x, y \) satisfy:
- \( x \) is a feasible solution to the primal problem
- \( y \) is a feasible solution to the dual problem

\[ -b^T y \leq c^T x \]

**Proof**
We know that \( Ax \leq b, A^T y + c = 0 \) and \( y \geq 0 \). Therefore,

\[ 0 \leq y^T (b - Ax) = b^T y - y^T Ax = c^T x + b^T y \]

**Remark**
- Any dual feasible \( y \) gives a **lower bound** on the primal optimal value
- Any primal feasible \( x \) gives an **upper bound** on the dual optimal value
- \( c^T x + b^T y \) is the **duality gap**
Weak duality

Corollaries

Unboundedness vs feasibility
• Primal unbounded \((p^* = -\infty)\) \(\implies\) dual infeasible \((d^* = -\infty)\)
• Dual unbounded \((d^* = +\infty)\) \(\implies\) primal infeasible \((p^* = +\infty)\)

Optimality condition
If \(x, y\) satisfy:
• \(x\) is a feasible solution to the primal problem
• \(y\) is a feasible solution to the dual problem
• The duality gap is zero, i.e., \(c^T x + b^T y = 0\)

Then \(x\) and \(y\) are optimal solutions to the primal and dual problem respectively
Strong duality

**Theorem**
If a linear optimization problem has an optimal solution, so does its dual, and the optimal value of primal and dual are equal

\[ d^* = p^* \]
## Exception to strong duality

<table>
<thead>
<tr>
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<th>Dual</th>
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<tbody>
<tr>
<td>minimize $x$</td>
<td>maximize $y$</td>
</tr>
<tr>
<td>subject to $0 \cdot x \leq -1$</td>
<td>subject to $0 \cdot y + 1 = 0$</td>
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<tr>
<td></td>
<td>$y \geq 0$</td>
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Optimal value is $p^* = +\infty$ \hspace{1cm} Optimal value is $d^* = -\infty$

Both primal and dual infeasible
## Relationship between primal and dual

<table>
<thead>
<tr>
<th>$\mathbf{d}^* = +\infty$</th>
<th>$p^* = +\infty$</th>
<th>$p^*$ finite</th>
<th>$p^* = -\infty$</th>
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<tr>
<td>primal inf.</td>
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<td>dual unb.</td>
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<table>
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<th>$d^*$ finite</th>
<th></th>
<th>optimal values equal</th>
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| $d^* = -\infty$              | exception         |                       | primal unbounded dual inf |

- Upper-right excluded by **weak duality**
- $(1, 1)$ and $(3, 3)$ proven by **weak duality**
- $(3, 1)$ and $(2, 2)$ proven by **strong duality**
Example
Production problem

maximize \( x_1 + 2x_2 \) \hspace{1cm} \text{Profits}
subject to
\begin{align*}
x_1 &\leq 100 \\
2x_2 &\leq 200 \\
x_1 + x_2 &\leq 150 \\
x_1, x_2 &\geq 0
\end{align*} \hspace{1cm} \text{Resources}

Dualize

1. Transform in inequality form

\begin{align*}
c &= (-1, -2) \\
A &= \begin{bmatrix}
1 & 0 \\
0 & 2 \\
1 & 1 \\
-1 & 0 \\
0 & -1
\end{bmatrix} \\
b &= (100, 200, 150, 0, 0)
\end{align*}

2. Derive dual

minimize \( c^T x \) subject to \( Ax \leq b \)
maximize \( -b^T y \) subject to \( A^T y + c = 0 \) \( y \geq 0 \)
Production problem

Dualized

maximize \[-b^T y\]
subject to \[A^T y + c = 0\]
\[y \geq 0\]

\[c = (-1, -2)\]
\[A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}\]
\[b = (100, 200, 150, 0, 0)\]

Fill-in data

minimize \[100y_1 + 200y_2 + 150y_3\]
subject to \[y_1 + y_3 - y_4 = 1\]
\[2y_2 + y_3 - y_5 = 2\]
\[y_1, y_2, y_3, y_4, y_5 \geq 0\]

Eliminate variables

minimize \[100y_1 + 200y_2 + 150y_3\]
subject to \[y_1 + y_3 \geq 1\]
\[2y_2 + y_3 \geq 2\]
\[y_1, y_2, y_3 \geq 0\]
Production problem

The dual

minimize \[ 100y_1 + 200y_2 + 150y_3 \]
subject to \[ y_1 + y_3 \geq 1 \]
\[ 2y_2 + y_3 \geq 2 \]
\[ y_1, y_2, y_3 \geq 0 \]

Interpretation

• **Sell all your resources** at a fair (minimum) price
• Selling must be **more convenient than producing**:
  - Product 1 (price 1, needs 1 × resource 1 and 3): \[ y_1 + y_3 \geq 1 \]
  - Product 2 (price 2, needs 2 × resource 2 and 1 × resource 3): \[ 2y_2 + y_3 \geq 2 \]
Today’s agenda

More on duality

• Two-person zero-sum games
• Farkas lemma
• Complementary slackness
• KKT conditions
Two-person games
Rock paper scissors

Rules
At count to three declare one of: Rock, Paper, or Scissors

Winners
Identical selection is a draw, otherwise:
• Rock beats ("dulls") scissors
• Scissors beats ("cuts") paper
• Paper beats ("covers") rock

Extremely popular: world RPS society, USA RPS league, etc.
Two-person zero-sum game

- Player 1 (P1) chooses a number $i \in \{1, \ldots, m\}$ (one of $m$ actions)
- Player 2 (P2) chooses a number $j \in \{1, \ldots, n\}$ (one of $n$ actions)

Two players make their choice independently

**Rule**

Player 1 pays $A_{ij}$ to player 2

$A \in \mathbb{R}^{m \times n}$ is the **payoff matrix**

**Rock, Paper, Scissors**

$$
A = \begin{bmatrix}
R & P & S \\
R & 0 & 1 & -1 \\
P & -1 & 0 & 1 \\
S & 1 & -1 & 0 \\
\end{bmatrix}
$$
Mixed (randomized) strategies

Deterministic strategies can be systematically defeated

Randomized strategies
- P1 chooses randomly according to distribution $x$: 
  \[ x_i = \text{probability that P1 selects action } i \]
- P2 chooses randomly according to distribution $y$: 
  \[ y_j = \text{probability that P2 selects action } j \]

Expected payoff (from P1 P2), if they use mixed-strategies $x$ and $y$, 
\[ \sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j A_{ij} = x^T Ay \]
Mixed strategies and probability simplex

**Probability simplex** in $\mathbb{R}^k$

$$P_k = \{ p \in \mathbb{R}^k \mid p \geq 0, \quad 1^T p = 1 \}$$

**Mixed strategy**

For a game player, a mixed strategy is a distribution over all possible deterministic strategies.

The **set of all mixed strategies** is the probability simplex $x \in P_m, \quad y \in P_n$
Optimal mixed strategies

P1: optimal strategy $x^*$ is the solution of

\[
\begin{align*}
\text{minimize} & \quad \max_{y \in P_n} x^T A y \\
\text{subject to} & \quad x \in P_m
\end{align*}
\]

P2: optimal strategy $y^*$ is the solution of

\[
\begin{align*}
\text{maximize} & \quad \min_{x \in P_m} x^T A y \\
\text{subject to} & \quad y \in P_n
\end{align*}
\]

Inner problem over deterministic strategies (vertices)

Optimal strategies $x^*$ and $y^*$ can be computed using linear optimization
Minmax theorem

Theorem

$$\max_{y \in P_n} \min_{x \in P_m} x^T Ay = \min_{x \in P_m} \max_{y \in P_n} x^T Ay$$

Proof

The optimal \(x^*\) is the solution of

- minimize \(t\)
- subject to \(A^T x \leq t1\)
- \(1^T x = 1\)
- \(x \geq 0\)

The optimal \(y^*\) is the solution of

- maximize \(w\)
- subject to \(Ay \geq w1\)
- \(1^T y = 1\)
- \(y \geq 0\)

The two LPs are duals and by strong duality the equality follows.
Nash equilibrium

Theorem

$$
\max_{y \in P_n} \min_{x \in P_m} x^T A y = \min_{x \in P_m} \max_{y \in P_n} x^T A y
$$

Consequence

The pair of mixed strategies \((x^*, y^*)\) attains the Nash equilibrium of the two-person matrix game, i.e.,

$$
x^T A y^* \geq x^{*T} A y^* \geq x^{*T} A y, \quad \forall x \in P_m, \forall y \in P_n
$$
Example

\[
A = \begin{bmatrix}
4 & 2 & 0 & -3 \\
-2 & -4 & -3 & 3 \\
-2 & -3 & 4 & 1
\end{bmatrix}
\]

Optimal deterministic strategies

\[
\min_i \max_j A_{ij} = 3 > -2 = \max_j \min_i A_{ij}
\]

Optimal mixed strategies

\[
x^* = (0.37, 0.33, 0.3), \quad y^* = (0.4, 0, 0.13, 0.47)
\]

Expected payoff

\[
x^{*T} A y^* = 0.2
\]
Farkas lemma
Feasibility of polyhedra

\[ P = \{ x \mid Ax = b, \quad x \geq 0 \} \]

How to show that \( P \) is feasible?
Easy: we just need to provide an \( x \in P \), i.e., a certificate

How to show that \( P \) is infeasible?
Farkas lemma

Theorem
Given $A$ and $b$, exactly one of the following statements is true:

1. There exists an $x$ with $Ax = b$, $x \geq 0$
2. There exists a $y$ with $A^T y \geq 0$, $b^T y < 0$
Farkas lemma

Geometric interpretation

1. First alternative

There exists an \( x \) with \( Ax = b \), \( x \geq 0 \)

\[
b = \sum_{i=1}^{n} x_i A_i, \quad x_i \geq 0, \quad i = 1, \ldots, n
\]

\( b \) is in the cone generated by the columns of \( A \)

2. Second alternative

There exists a \( y \) with \( A^T y \geq 0 \), \( b^T y < 0 \)

\[
y^T A_i \geq 0, \quad i = 1, \ldots, m, \quad y^T b < 0
\]

The hyperplane \( y^T z = 0 \)

separates \( b \) from \( A_1, \ldots, A_n \)
Farkas lemma

There exists $x$ with $Ax = b$, $x \geq 0$ \hspace{1cm} OR \hspace{1cm} There exists $y$ with $A^T y \geq 0$, $b^T y < 0$

Proof

1 and 2 cannot be both true (easy)

$x \geq 0$, $Ax = b$ and $y^T A \geq 0$ \hspace{1cm} $\rightarrow$ \hspace{1cm} $y^T b = y^T Ax \geq 0$
Farkas lemma

There exists \( x \) with \( Ax = b, \ x \geq 0 \) \hspace{1cm} OR \hspace{1cm} There exists \( y \) with \( A^Ty \geq 0, \ b^Ty < 0 \)

Proof

1 and 2 cannot be both false (duality)

\[
\text{Primal} \quad \begin{align*}
\text{minimize} & \quad 0 \\
\text{subject to} & \quad Ax = b \\
& \quad x \geq 0
\end{align*} \quad \text{Dual} \quad \begin{align*}
\text{maximize} & \quad -b^Ty \\
\text{subject to} & \quad A^Ty \geq 0
\end{align*}
\]

Strong duality holds \( d^* \neq -\infty, \ p^* = d^* \)
Farkas lemma

There exists \( x \) with \( Ax = b, \ x \geq 0 \) \hspace{1cm} \text{OR} \hspace{1cm} \text{There exists } y \text{ with } A^T y \geq 0, \ b^T y < 0

Proof

1 and 2 cannot be both false (duality)

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Alternative 1: primal feasible \( p^* = d^* = 0 \)

\( b^T y \geq 0 \) for all \( y \) such that \( A^T y \geq 0 \)
Farkas lemma

There exists $x$ with $Ax = b$, $x \geq 0$ \quad \text{OR} \quad \text{There exists } y \text{ with } A^Ty \geq 0, b^Ty < 0

Proof

1 and 2 cannot be both false (duality)

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Alternative 2: primal infeasible $p^* = d^* = +\infty$

There exists $y$ such that $A^Ty \geq 0$ and $b^Ty < 0$

$y$ is an infeasibility certificate
Farkas lemma

Many variations

There exists $x$ with $Ax = b$, $x \geq 0$

OR

There exists $y$ with $A^T y \geq 0$, $b^T y < 0$

There exists $x$ with $Ax \leq b$, $x \geq 0$

OR

There exists $y$ with $A^T y \geq 0$, $b^T y < 0$, $y \geq 0$

There exists $x$ with $Ax \leq b$

OR

There exists $y$ with $A^T y = 0$, $b^T y < 0$, $y \geq 0$
Complementary slackness
Optimality conditions

**Primal**
- minimize $c^T x$
- subject to $Ax \leq b$

**Dual**
- maximize $-b^T y$
- subject to $A^T y + c = 0$
  
  $y \geq 0$

$x$ and $y$ are **primal** and **dual** optimal if and only if
- $x$ is **primal feasible**: $Ax \leq b$
- $y$ is **dual feasible**: $A^T y + c = 0$ and $y \geq 0$
- The **duality gap** is zero: $c^T x + b^T y = 0$

Can we relate $x$ and $y$ (not only the objective)?
# Complementary slackness

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<td>maximize $-b^T y$</td>
</tr>
<tr>
<td>subject to $Ax \leq b$</td>
<td>subject to $A^T y + c = 0$  $y \geq 0$</td>
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**Theorem**

Primal, dual feasible $x, y$ are optimal if and only if

$$y_i(b_i - a_i^T x) = 0, \quad i = 1, \ldots, m$$

i.e., at optimum, $b - Ax$ and $y$ have a **complementary sparsity** pattern:

- $y_i > 0 \quad \Rightarrow \quad a_i^T x = b_i$
- $a_i^T x < b_i \quad \Rightarrow \quad y_i = 0$
Complementary slackness

**Primal**

minimize $c^T x$

subject to $Ax \leq b$

**Dual**

maximize $-b^T y$

subject to $A^T y + c = 0$

$y \geq 0$

**Proof**

The duality gap at primal feasible $x$ and dual feasible $y$ can be written as

$$c^T x + b^T y = (-A^T y)^T x + b^T y = (b - Ax)^T y = \sum_{i=1}^{m} y_i (b_i - a_i^T x) = 0$$

Since all the elements of the sum are nonnegative, they must all be 0

For feasible $x$ and $y$ complementary slackness = zero duality gap
Example

minimize \[-4x_1 - 5x_2\]

subject to

\[
\begin{bmatrix}
-1 & 0 \\
2 & 1 \\
0 & -1 \\
1 & 2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} \leq
\begin{bmatrix}
0 \\
3 \\
0
\end{bmatrix}
\]

Let’s show that feasible \(x = (1, 1)\) is optimal.

Second and fourth constraints are active at \(x \rightarrow y = (0, y_2, 0, y_4)\)

\[
A^T y = -c \quad \Rightarrow \quad \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_2 \\ y_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}
\]

and \(y_2 \geq 0, \ y_4 \geq 0\)

\(y = (0, 1, 0, 2)\) satisfies these conditions and proves that \(x\) is optimal.

Complementary slackness is useful to recover \(y^*\) from \(x^*\).
Geometric interpretation

Example in $\mathbb{R}^2$

Two active constraints at optimum: $a_1^T x^* = b_1, \quad a_2^T x^* = b_2$

Optimal dual solution $y$ satisfies:

$$A^T y + c = 0, \quad y \geq 0, \quad y_i = 0 \text{ for } i \neq \{1, 2\}$$

In other words, $-c = a_1 y_1 + a_2 y_2$ with $y_1, y_2 \geq 0$
KKT Conditions
Lagrangian and duality

Primal

minimize \( c^T x \)
subject to \( Ax \leq b \)

Dual function

\[
g(y) = \min_x \left( c^T x + y^T (Ax - b) \right)
= -b^T y + \min_x \left( c + A^T y \right)^T x
= \begin{cases} 
-b^T y & \text{if } c + A^T y = 0 \\
-\infty & \text{otherwise}
\end{cases}
\]

Lagrangian

\[
L(x, y) = c^T x + y^T (Ax - b)
\]

\[
\nabla_x L(x, y) = c + A^T y = 0
\]

Dual

maximize \( -b^T y \)
subject to \( A^T y + c = 0 \)
\( y \geq 0 \)
Karush-Kuhn-Tucker conditions

Optimality conditions for linear optimization

Primal
minimize $c^T x$
subject to $Ax \leq b$

Dual
maximize $-b^T y$
subject to $A^T y + c = 0$
$y \geq 0$

Primal feasibility
$Ax \leq b$

Dual feasibility
$\nabla_x L(x, y) = A^T y + c = 0$ and $y \geq 0$

Complementary slackness
$y_i (Ax - b)_i = 0, \quad i = 1, \ldots, m$
Karush-Kuhn-Tucker conditions

Solving linear optimization problems

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</tr>
<tr>
<td></td>
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</table>

We can solve our optimization problem by solving a system of equations

\[
\nabla_x L(x, y) = A^T y + c = 0 \\
b - Ax \geq 0 \\
y \geq 0 \\
y^T (b - Ax) = 0
\]
Linear optimization duality

Today, we learned to:

• **Interpret** linear optimization duality using game theory

• **Prove** Farkas lemma using duality

• **Geometrically link** primal and dual solutions with complementary slackness

• **Derive** KKT optimality conditions
References

• Bertsimas and Tsitsiklis: Introduction to Linear Optimization
  • Chapter 4: Duality theory
• R. Vanderbei: Linear Programming — Foundations and Extensions
  • Chapter 11: Game Theory
Next lecture

• Sensitivity analysis