Ed Forum

• what are the general ways for relaxing an LP?

• how creating the duality problem can be useful in practical applications?
Recap
Weak and strong duality
## Optimal objective values

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$p^*$ is the primal optimal value

$d^*$ is the dual optimal value

Primal infeasible: $p^* = +\infty$

Primal unbounded: $p^* = -\infty$

Dual infeasible: $d^* = -\infty$

Dual unbounded: $d^* = +\infty$
Weak duality

Theorem
If \( x, y \) satisfy:

- \( x \) is a feasible solution to the primal problem
- \( y \) is a feasible solution to the dual problem

\[
-b^T y \leq c^T x
\]

Proof
We know that \( Ax \leq b, A^T y + c = 0 \) and \( y \geq 0 \). Therefore,

\[
0 \leq y^T (b - Ax) = b^T y - y^T Ax = c^T x + b^T y
\]

Remark
- Any dual feasible \( y \) gives a lower bound on the primal optimal value
- Any primal feasible \( x \) gives an upper bound on the dual optimal value
- \( c^T x + b^T y \) is the duality gap
Weak duality

Corollaries

Unboundedness vs feasibility

- Primal unbounded \((p^* = -\infty)\) \(\Rightarrow\) dual infeasible \((d^* = -\infty)\)
- Dual unbounded \((d^* = +\infty)\) \(\Rightarrow\) primal infeasible \((p^* = +\infty)\)

Optimality condition

If \(x, y\) satisfy:

- \(x\) is a feasible solution to the primal problem
- \(y\) is a feasible solution to the dual problem
- The duality gap is zero, i.e., \(c^T x + b^T y = 0\)

Then \(x\) and \(y\) are optimal solutions to the primal and dual problem respectively
Strong duality

**Theorem**
If a linear optimization problem has an optimal solution, so does its dual, and the optimal value of primal and dual are equal

\[ d^* = p^* \]
Strong duality

Constructive proof

Given a primal optimal solution \( x^* \) we will construct a dual optimal solution \( y^* \)

Apply simplex to problem in **standard form**

\[
\begin{align*}
\text{minimize } & \quad c^T x \\
\text{subject to } & \quad A x = b \\
& \quad x \geq 0
\end{align*}
\]

- optimal basis \( B \)
- optimal solution \( x^* \) with \( A_B x_B^* = b \)
- reduced costs \( \bar{c} = c - A^T A_B^{-T} c_B \geq 0 \)

Define \( y^* \) such that \( y^* = -A_B^{-T} c_B \). Therefore, \( A^T y^* + c \geq 0 \) (\( y^* \) dual feasible).

\[
- b^T y^* = - b^T (-A_B^{-T} c_B) = c_B^T (A_B^{-1} b) = c_B^T x_B^* = c^T x^*
\]

By weak duality theorem corollary, \( y^* \) is an optimal solution of the dual. Therefore, \( d^* = p^* \).
Exception to strong duality

Primal

minimize \( x \)
subject to \( 0 \cdot x \leq -1 \)

Optimal value is \( p^* = +\infty \)

Dual

maximize \( y \)
subject to \( 0 \cdot y + 1 = 0 \)
\( y \geq 0 \)

Optimal value is \( d^* = -\infty \)

Both primal and dual infeasible
# Relationship between primal and dual

<table>
<thead>
<tr>
<th>$d^* = +\infty$</th>
<th>$p^* = +\infty$</th>
<th>$p^*$ finite</th>
<th>$p^* = -\infty$</th>
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<tr>
<td>primal inf.</td>
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<td>dual unb.</td>
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- Upper-right excluded by **weak duality**
- $(1, 1)$ and $(3, 3)$ proven by **weak duality**
- $(3, 1)$ and $(2, 2)$ proven by **strong duality**
Example
Production problem

maximize \( x_1 + 2x_2 \)  \text{Profits}
subject to
\begin{align*}
  x_1 &\leq 100 \\
  2x_2 &\leq 200 \\
  x_1 + x_2 &\leq 150 \\
  x_1, x_2 &\geq 0
\end{align*}  \text{Resources}

Dualize

1. Transform in inequality form

minimize \( c^T x \)
subject to \( Ax \leq b \)
\[
\begin{bmatrix}
  1 & 0 \\
  0 & 2 \\
  1 & 1 \\
  -1 & 0 \\
  0 & -1 \\
\end{bmatrix}
\]
\[
\begin{bmatrix}
  100 \\
  200 \\
  150 \\
  0 \\
  0
\end{bmatrix}
\]

2. Derive dual

maximize \( -b^T y \)
subject to \( A^T y + c = 0 \)
\[
y \geq 0
\]
Production problem

Dualized

maximize \(-b^T y\)
subject to \(A^T y + c = 0\)
\(y \geq 0\)

\(c = (-1, -2)\)
\(A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \)
\(b = (100, 200, 150, 0, 0)\)

Fill-in data

minimize \(100y_1 + 200y_2 + 150y_3\)
subject to \(y_1 + y_3 - y_4 = 1\)
\(2y_2 + y_3 - y_5 = 2\)
\(y_1, y_2, y_3, y_4, y_5 \geq 0\)

Eliminate variables

minimize \(100y_1 + 200y_2 + 150y_3\)
subject to \(y_1 + y_3 \geq 1\)
\(2y_2 + y_3 \geq 2\)
\(y_1, y_2, y_3 \geq 0\)
Production problem

The dual

minimize \[ 100y_1 + 200y_2 + 150y_3 \]
subject to \[ y_1 + y_3 \geq 1 \]
\[ 2y_2 + y_3 \geq 2 \]
\[ y_1, y_2, y_3 \geq 0 \]

Interpretation

• **Sell all your resources** at a fair (minimum) price
• Selling must be more convenient than producing:
  - Product 1 (price 1, needs \(1\times\) resource 1 and 3): \(y_1 + y_3 \geq 1\)
  - Product 2 (price 2, needs \(2\times\) resource 2 and \(1\times\) resource 3): \(2y_2 + y_3 \geq 2\)
Today’s agenda

More on duality

• Two-person zero-sum games
• Farkas lemma
• Complementary slackness
• KKT conditions
Two-person games
Rock paper scissors

Rules
At count to three declare one of: Rock, Paper, or Scissors

Winners
Identical selection is a draw, otherwise:
• Rock beats (“dulls”) scissors
• Scissors beats (“cuts”) paper
• Paper beats (“covers”) rock

Extremely popular: world RPS society, USA RPS league, etc.
Two-person zero-sum game

- Player 1 (P1) chooses a number $i \in \{1, \ldots, m\}$ (one of $m$ actions)
- Player 2 (P2) chooses a number $j \in \{1, \ldots, n\}$ (one of $n$ actions)

Two players make their choice independently

**Rule**

Player 1 pays $A_{ij}$ to player 2

$A \in \mathbb{R}^{m \times n}$ is the payoff matrix

**Rock, Paper, Scissors**

$$
A = \begin{bmatrix}
0 & 1 & -1 \\
-1 & 0 & 1 \\
1 & -1 & 0
\end{bmatrix}
$$
Mixed (randomized) strategies

Deterministic strategies can be systematically defeated

Randomized strategies
- P1 chooses randomly according to distribution $x$:
  \[ x_i = \text{probability that P1 selects action } i \]
- P2 chooses randomly according to distribution $y$:
  \[ y_j = \text{probability that P2 selects action } j \]

Expected payoff (from P1 P2), if they use mixed-strategies $x$ and $y$,
\[
\sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j A_{ij} = x^T A y
\]
Mixed strategies and probability simplex

**Probability simplex in** $\mathbb{R}^k$

$P_k = \{p \in \mathbb{R}^k \mid p \geq 0, \quad 1^T p = 1\}$

**Mixed strategy**

For a game player, a mixed strategy is a distribution over all possible deterministic strategies.

The **set of all mixed strategies** is the probability simplex $\rightarrow x \in P_m, \quad y \in P_n$
Optimal mixed strategies

P1: optimal strategy $x^*$ is the solution of

\[
\begin{align*}
\text{minimize} & \quad \max_{y \in P_n} x^T Ay \\
\text{subject to} & \quad x \in P_m
\end{align*}
\]

\[\rightarrow\]

\[
\begin{align*}
\text{minimize} & \quad \max_{j=1,\ldots,n} (A^T x)_j \\
\text{subject to} & \quad x \in P_m
\end{align*}
\]

Inner problem over deterministic strategies (vertices)

P2: optimal strategy $y^*$ is the solution of

\[
\begin{align*}
\text{maximize} & \quad \min_{x \in P_m} x^T Ay \\
\text{subject to} & \quad y \in P_n
\end{align*}
\]

\[\rightarrow\]

\[
\begin{align*}
\text{maximize} & \quad \min_{i=1,\ldots,m} (Ay)_i \\
\text{subject to} & \quad y \in P_n
\end{align*}
\]

Optimal strategies $x^*$ and $y^*$ can be computed using linear optimization
Minmax theorem

Theorem

\[
\max_{y \in P_n} \min_{x \in P_m} x^T Ay = \min_{x \in P_m} \max_{y \in P_n} x^T Ay
\]

Proof

The optimal \(x^*\) is the solution of

- minimize \(t\)
- subject to \(A^T x \leq t1\)
- \(1^T x = 1\)
- \(x \geq 0\)

The optimal \(y^*\) is the solution of

- maximize \(w\)
- subject to \(Ay \geq w1\)
- \(1^T y = 1\)
- \(y \geq 0\)

The two LPs are duals and by strong duality the equality follows.
Nash equilibrium

Theorem

\[
\max_{y \in P_n} \min_{x \in P_m} x^T Ay = \min_{x \in P_m} \max_{y \in P_n} x^T Ay
\]

Consequence
The pair of mixed strategies \((x^*, y^*)\) attains the Nash equilibrium of the two-person matrix game, i.e.,

\[
x^T Ay^* \geq x^{*T} Ay^* \geq x^* T Ay, \quad \forall x \in P_m, \forall y \in P_n
\]
Example

\[
A = \begin{bmatrix}
4 & 2 & 0 & -3 \\
-2 & -4 & -3 & 3 \\
-2 & -3 & 4 & 1 \\
\end{bmatrix}
\]

Optimal deterministic strategies

\[
\min_i \max_j A_{ij} = 3 > -2 = \max_j \min_i A_{ij}
\]

Optimal mixed strategies

\[
x^* = (0.37, 0.33, 0.3), \quad \quad y^* = (0.4, 0, 0.13, 0.47)
\]

Expected payoff

\[
x^{*T} A y^* = 0.2
\]
Farkas lemma
Feasibility of polyhedra

\[ P = \{x \mid Ax = b, \quad x \geq 0\} \]

How to show that \( P \) is feasible?
Easy: we just need to provide an \( x \in P \), i.e., a certificate

How to show that \( P \) is infeasible?
Farkas lemma

**Theorem**
Given $A$ and $b$, exactly one of the following statements is true:

1. There exists an $x$ with $Ax = b$, $x \geq 0$
2. There exists a $y$ with $A^T y \geq 0$, $b^T y < 0$
Farkas lemma

Geometric interpretation

1. First alternative
   There exists an $x$ with $Ax = b, x \geq 0$
   $$b = \sum_{i=1}^{n} x_i A_i, \quad x_i \geq 0, \quad i = 1, \ldots, n$$
   $b$ is in the cone generated by the columns of $A$

2. Second alternative
   There exists a $y$ with $A^T y \geq 0, b^T y < 0$
   $$y^T A_i \geq 0, \quad i = 1, \ldots, m, \quad y^T b < 0$$
   The hyperplane $y^T z = 0$
   separates $b$ from $A_1, \ldots, A_n$
Farkas lemma

There exists $x$ with $Ax = b$, $x \geq 0$ \hspace{1cm} \textbf{OR} \hspace{1cm} \text{There exists } y \text{ with } A^Ty \geq 0, \ b^T y < 0$

Proof

1 and 2 cannot be both true (easy)

\[ x \geq 0, \ Ax = b \text{ and } y^TA \geq 0 \hspace{1cm} \rightarrow \hspace{1cm} y^Tb = y^TAx \geq 0 \]
**Farkas lemma**

There exists \( x \) with \( Ax = b, \ x \geq 0 \) \quad \text{OR} \quad \text{There exists} \ y \ \text{with} \ A^T y \geq 0, \ b^T y < 0

**Proof**

1 and 2 cannot be both false (duality)

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<tr>
<td>( x \geq 0 )</td>
<td>[ y = 0 \text{ always feasible} ]</td>
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Strong duality holds

\( d^* \neq -\infty, \ p^* = d^* \)
**Farkas lemma**

There exists $x$ with $Ax = b$, $x \geq 0$  \quad \text{OR} \quad \text{There exists } y \text{ with } A^T y \geq 0, \ b^T y < 0$

**Proof**

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**Alternative 1:** primal feasible $p^* = d^* = 0$

$b^T y \geq 0$ for all $y$ such that $A^T y \geq 0$
Farkas lemma

There exists $x$ with $Ax = b$, $x \geq 0$ \text{ OR } \text{ There exists } y \text{ with } A^T y \geq 0, \quad b^T y < 0

**Proof**

1 and 2 cannot be both false (duality)

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**Alternative 2:** primal infeasible $p^* = d^* = +\infty$

There exists $y$ such that $A^T y \geq 0$ and $b^T y < 0$

$y$ is an infeasibility certificate
Farkas lemma

Many variations

There exists \( x \) with \( Ax = b, \ x \geq 0 \)

\[
\text{OR} \quad \text{There exists } y \text{ with } A^T y \geq 0, \ b^T y < 0
\]

There exists \( x \) with \( Ax \leq b, \ x \geq 0 \)

\[
\text{OR} \quad \text{There exists } y \text{ with } A^T y \geq 0, \ b^T y < 0, \ y \geq 0
\]

There exists \( x \) with \( Ax \leq b \)

\[
\text{OR} \quad \text{There exists } y \text{ with } A^T y = 0, \ b^T y < 0, \ y \geq 0
\]
Complementary slackness
Optimality conditions

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$x$ and $y$ are **primal** and **dual** optimal if and only if

- $x$ is **primal feasible**: $Ax \leq b$
- $y$ is **dual feasible**: $A^T y + c = 0$ and $y \geq 0$
- The **duality gap** is zero: $c^T x + b^T y = 0$

Can we relate $x$ and $y$ (not only the objective)?
Complementary slackness

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**Theorem**

Primal, dual feasible $x$, $y$ are optimal if and only if

$$y_i(b_i - a_i^T x) = 0, \quad i = 1, \ldots, m$$

i.e., at optimum, $b - Ax$ and $y$ have a **complementary sparsity** pattern:

$$y_i > 0 \quad \Rightarrow \quad a_i^T x = b_i$$

$$a_i^T x < b_i \quad \Rightarrow \quad y_i = 0$$
Complementary slackness

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Proof

The duality gap at primal feasible $x$ and dual feasible $y$ can be written as

$$c^T x + b^T y = (-A^T y)^T x + b^T y = (b - Ax)^T y = \sum_{i=1}^{m} y_i (b_i - a_i^T x) = 0$$

Since all the elements of the sum are nonnegative, they must all be 0.

For feasible $x$ and $y$ complementary slackness = zero duality gap
Example

minimize $-4x_1 - 5x_2$

subject to

$$\begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$

Let’s show that feasible $x = (1, 1)$ is optimal

Second and fourth constraints are active at $x ightarrow y = (0, y_2, 0, y_4)$

$$A^T y = -c \quad \Rightarrow \quad \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_2 \\ y_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

and $y_2 \geq 0, \quad y_4 \geq 0$

$y = (0, 1, 0, 2)$ satisfies these conditions and proves that $x$ is optimal

Complementary slackness is useful to recover $y^*$ from $x^*$
**Geometric interpretation**

**Example in \( \mathbb{R}^2 \)**

Two active constraints at optimum:  
\[ a_1^T x^* = b_1, \quad a_2^T x^* = b_2 \]

Optimal dual solution \( y \) satisfies:

\[ A^T y + c = 0, \quad y \geq 0, \quad y_i = 0 \text{ for } i \neq \{1, 2\} \]

In other words,  
\[-c = a_1 y_1 + a_2 y_2 \text{ with } y_1, y_2 \geq 0\]
KKT Conditions
### Lagrangian and duality

**Primal**

- minimize \( c^T x \)
- subject to \( Ax \leq b \)

**Dual function**

- \( g(y) = \min_x \left( c^T x + y^T (Ax - b) \right) \)
- \( = -b^T y + \min_x \left( c + A^T y \right)^T x \)
- \( = \begin{cases} 
- b^T y & \text{if } c + A^T y = 0 \\
- \infty & \text{otherwise}
\end{cases} \)

**Dual**

- maximize \( -b^T y \)
- subject to \( A^T y + c = 0 \)
  - \( y \geq 0 \)

**Lagrangian**

- \( L(x, y) = c^T x + y^T (Ax - b) \)
- \( \nabla_x L(x, y) = c + A^T y = 0 \)
Karush-Kuhn-Tucker conditions
Optimality conditions for linear optimization

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Primal feasibility

$$Ax \leq b$$

Dual feasibility

$$\nabla_x L(x, y) = A^T y + c = 0 \quad \text{and} \quad y \geq 0$$

Complementary slackness

$$y_i (Ax - b)_i = 0, \quad i = 1, \ldots, m$$
Karush-Kuhn-Tucker conditions

Solving linear optimization problems

**Primal**
- minimize $c^T x$
- subject to $Ax \leq b$

**Dual**
- maximize $-b^T y$
- subject to $A^T y + c = 0$
  - $y \geq 0$

We can solve our optimization problem by solving a system of equations

\[
\nabla_x L(x, y) = A^T y + c = 0
\]
\[
b - Ax \geq 0
\]
\[
y \geq 0
\]
\[
y^T (b - Ax) = 0
\]
Linear optimization duality

Today, we learned to:

- **Interpret** linear optimization duality using game theory
- **Prove** Farkas lemma using duality
- **Geometrically link** primal and dual solutions with complementary slackness
- **Derive** KKT optimality conditions
References

- Bertsimas and Tsitsiklis: Introduction to Linear Optimization
  - Chapter 4: Duality theory
- R. Vanderbei: Linear Programming — Foundations and Extensions
  - Chapter 11: Game Theory
Next lecture

- Sensitivity analysis