The Voice of Optimization

Bartolomeo Stellato

joint work with Dimitris Bertsimas

MIT MANAGEMENT SLOAN SCHOOL

INFORMS Annual Meeting 2019

Inventory management

$$\begin{array}{ll} \text{minimize} & \sum_{t=0}^{T-1} h(x_t) + o(u_t) \\ \text{subject to} & x_{t+1} = x_t + u_t - d_t \\ & x_0 = x_{\text{init}} \\ & 0 \leq u_t \leq M \end{array}$$

Inventory management

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Inventory

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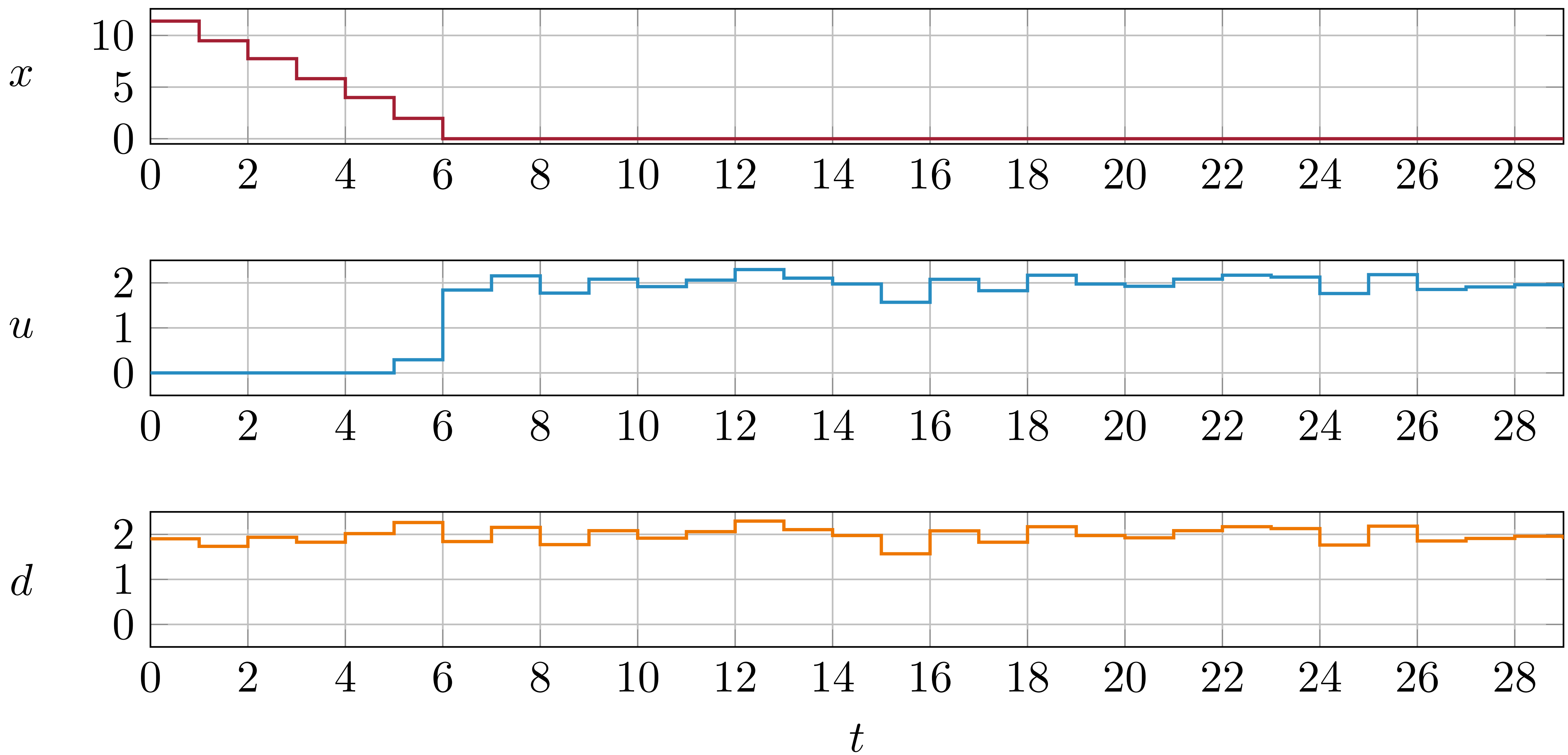
Parameters

How do x and u depend on the parameters?



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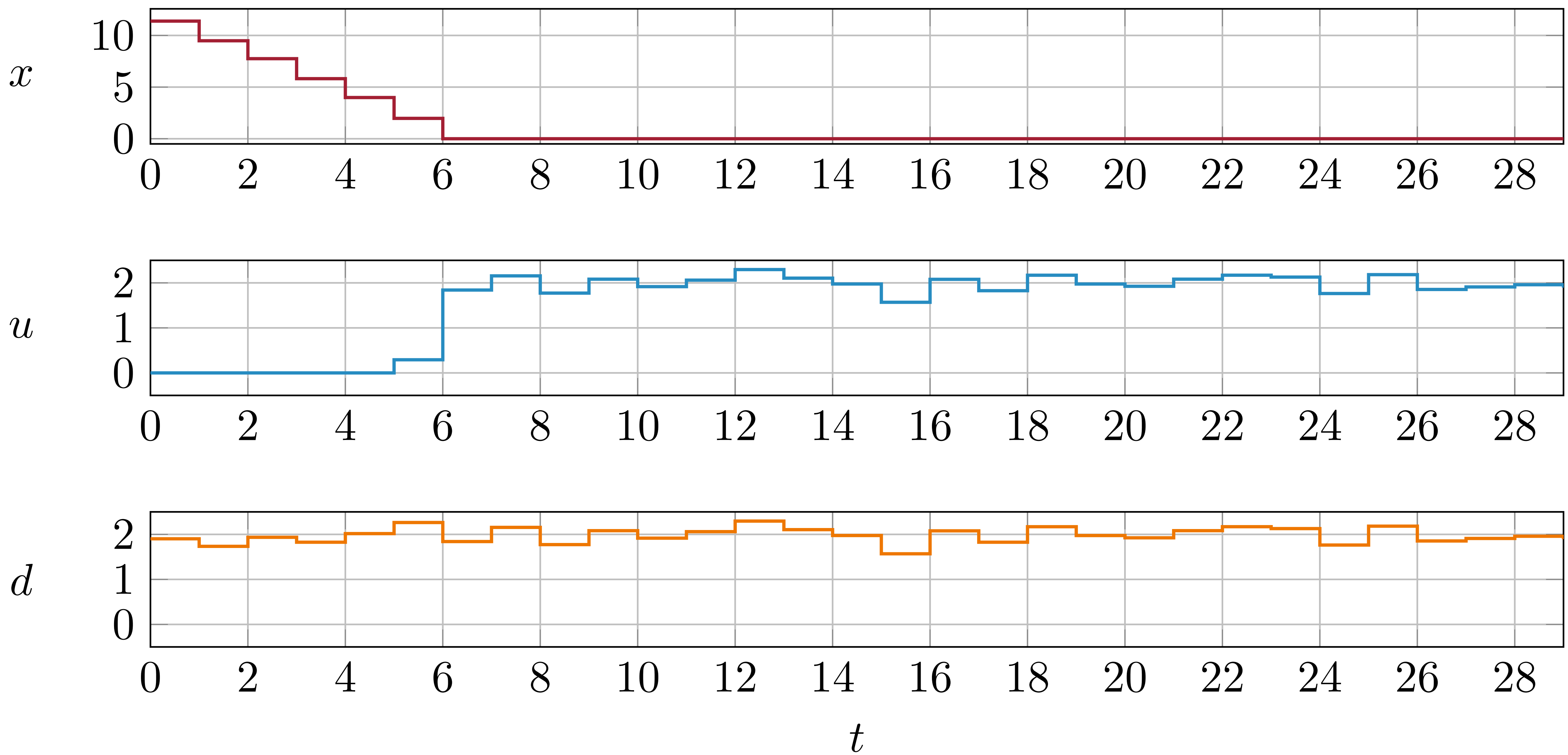
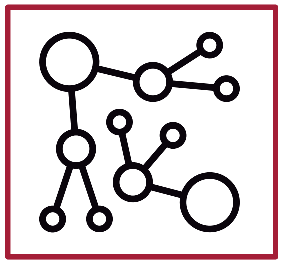


x_{init} ? d_t ?

The decision-making workflow

Data



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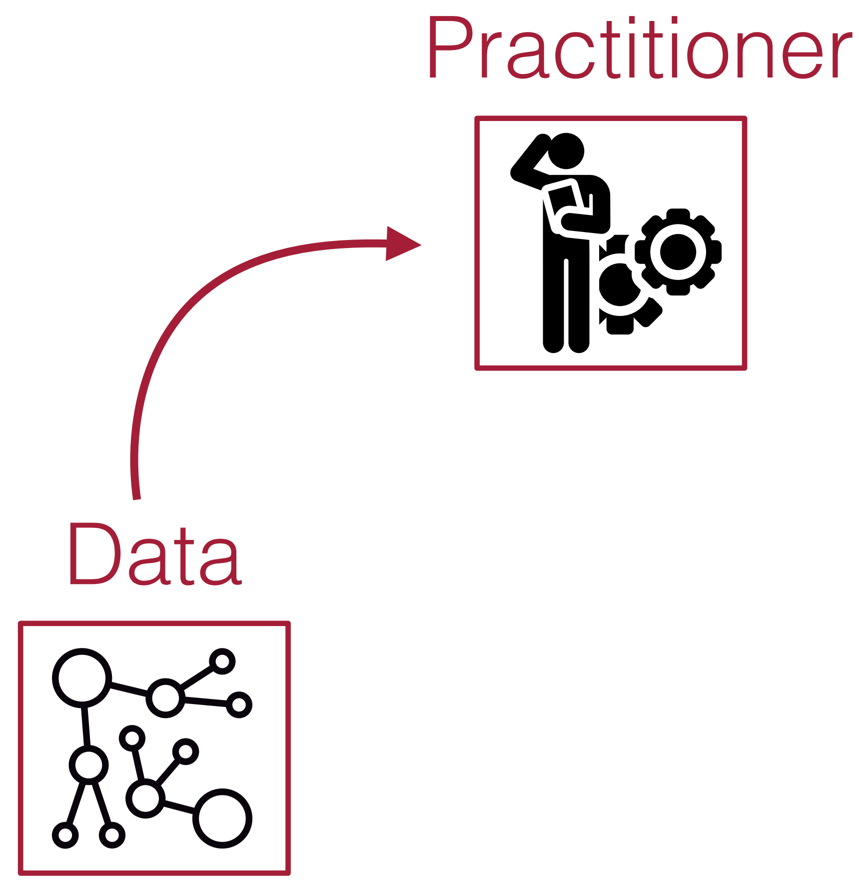

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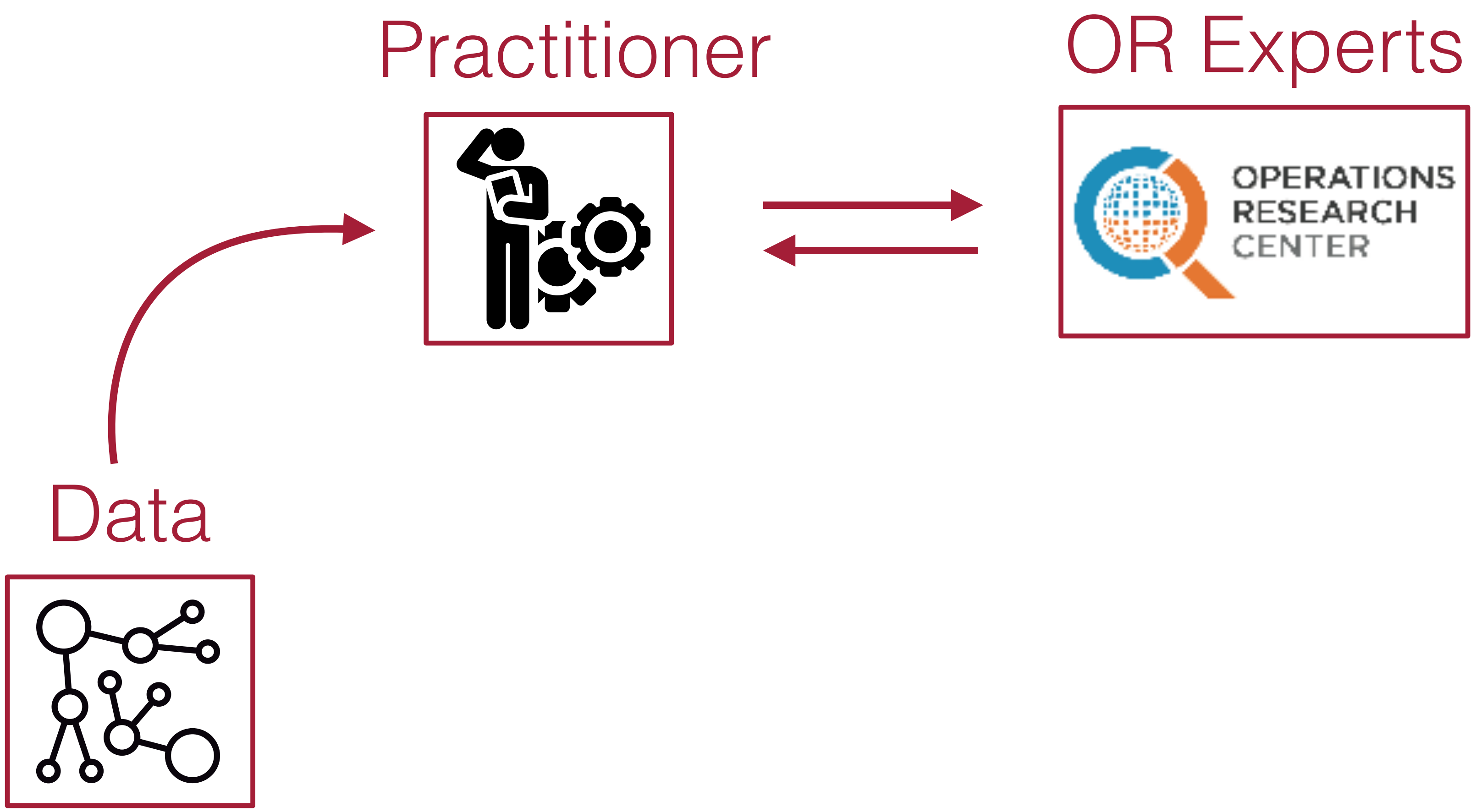

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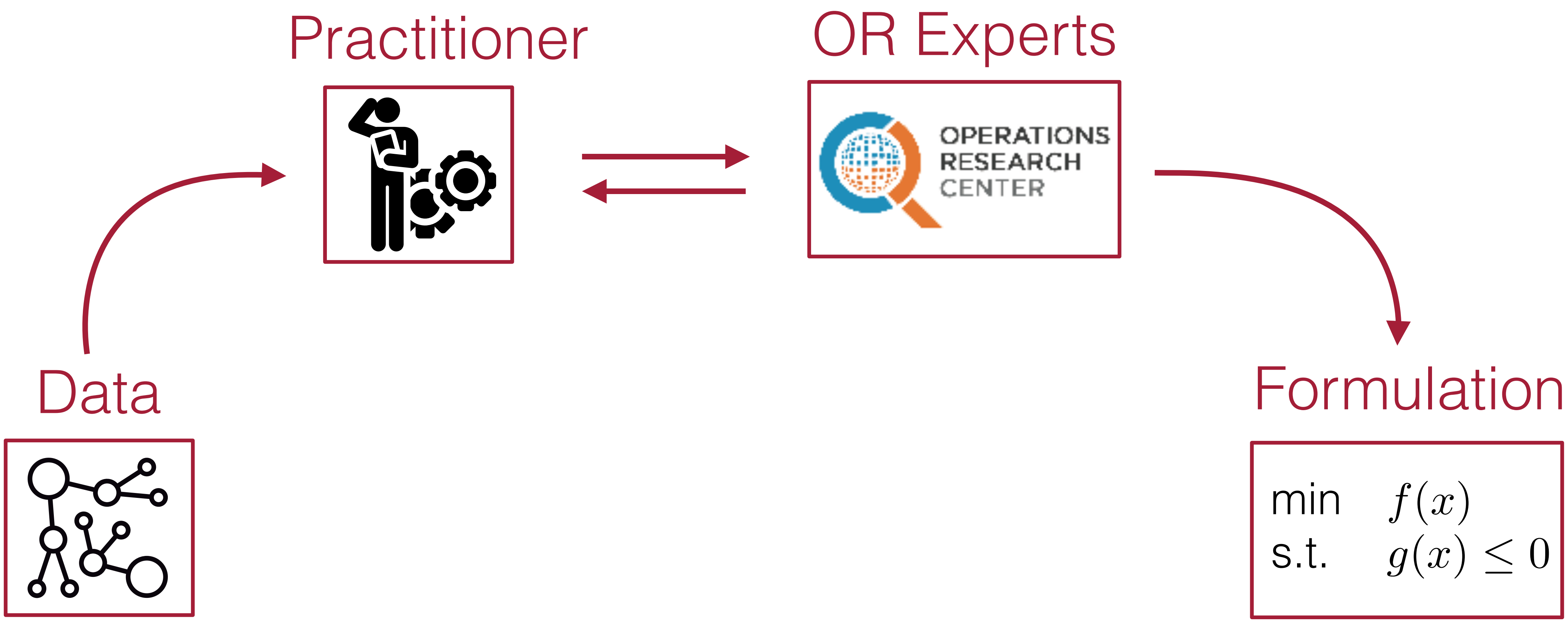

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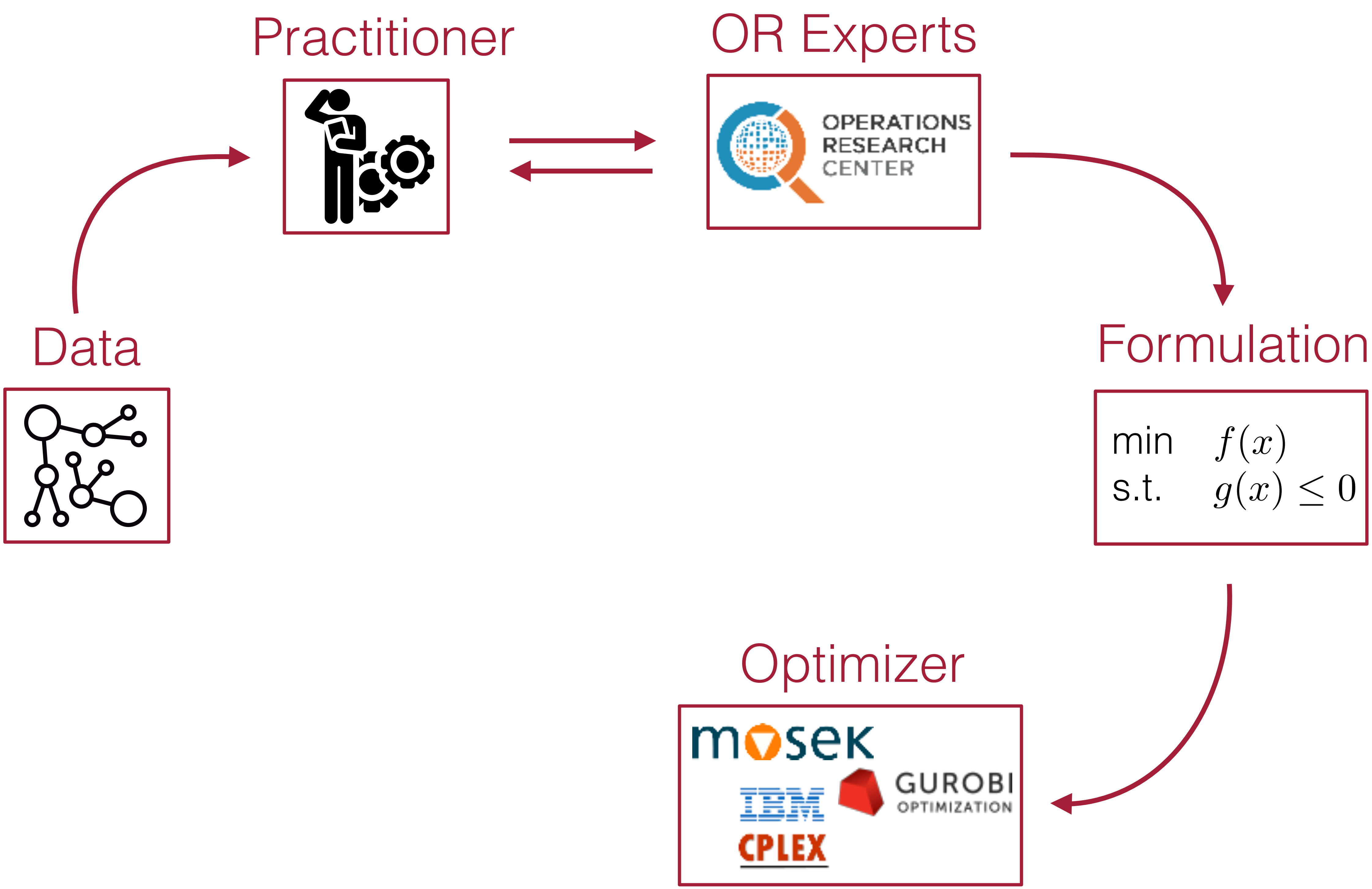

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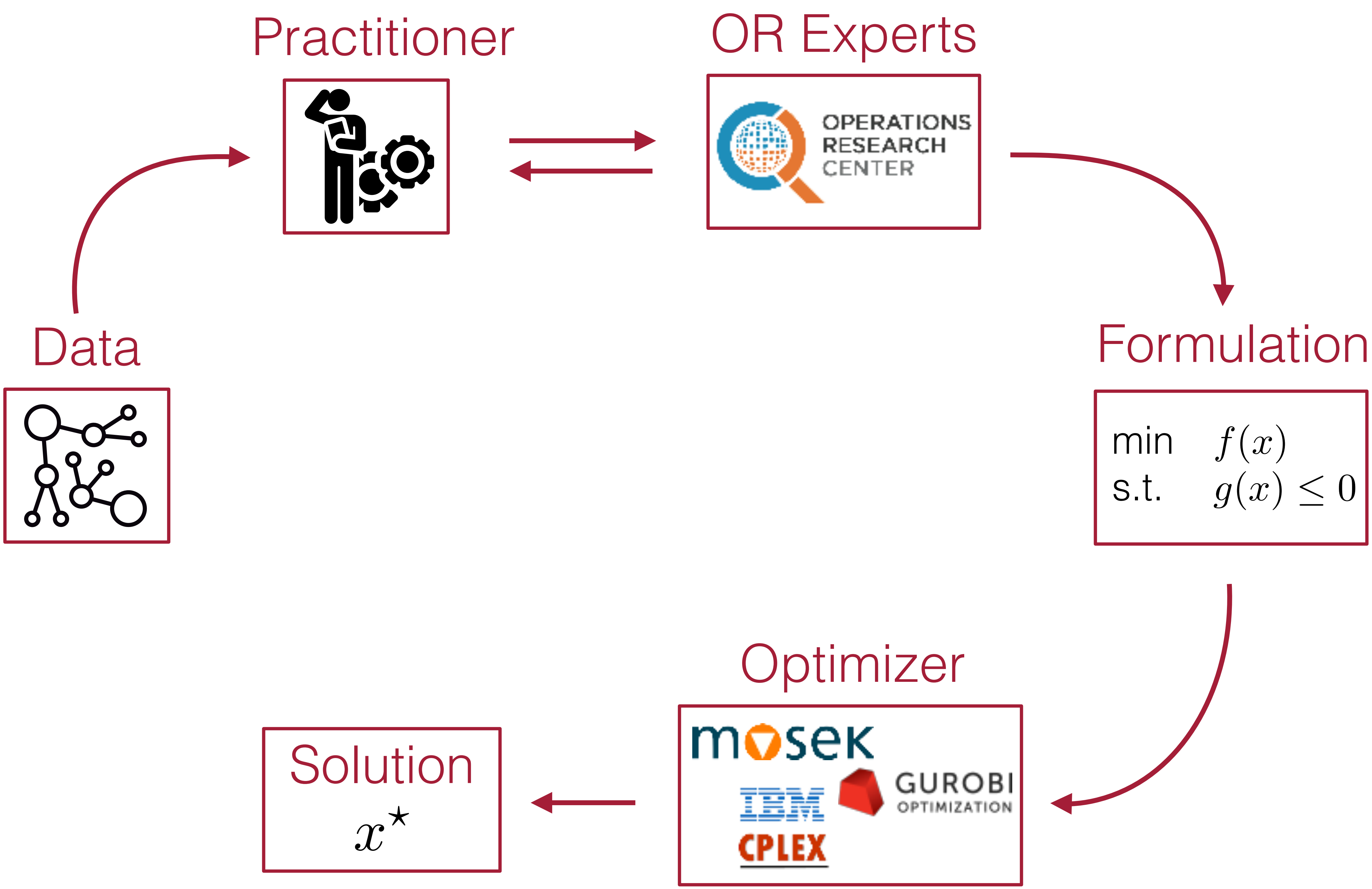

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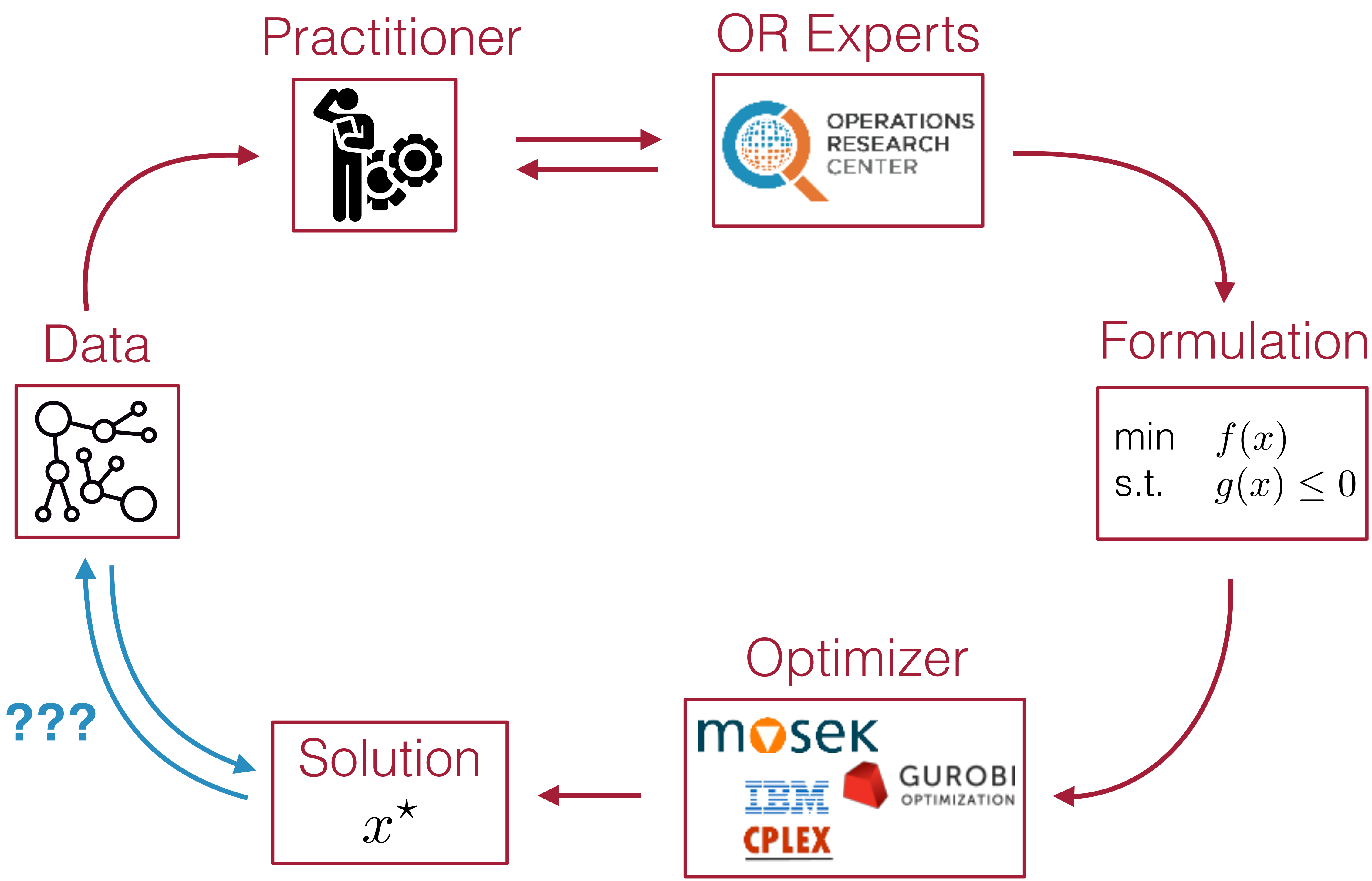

Traditional optimization

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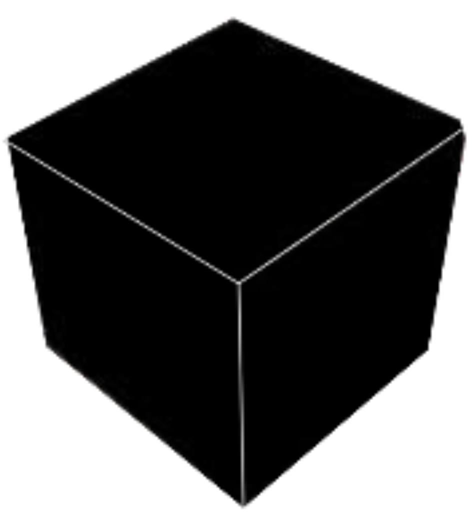
$$\theta \longrightarrow \boxed{\text{Optimization}} \longrightarrow x^\star$$



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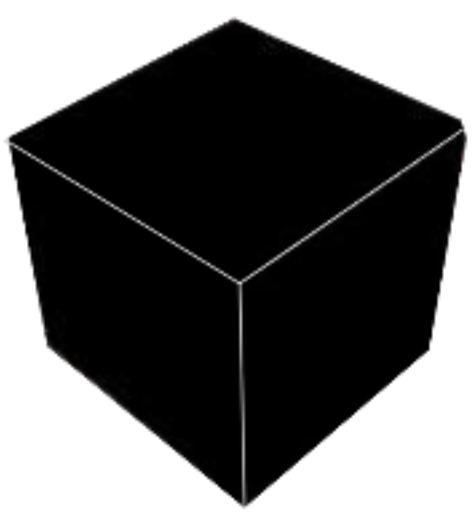
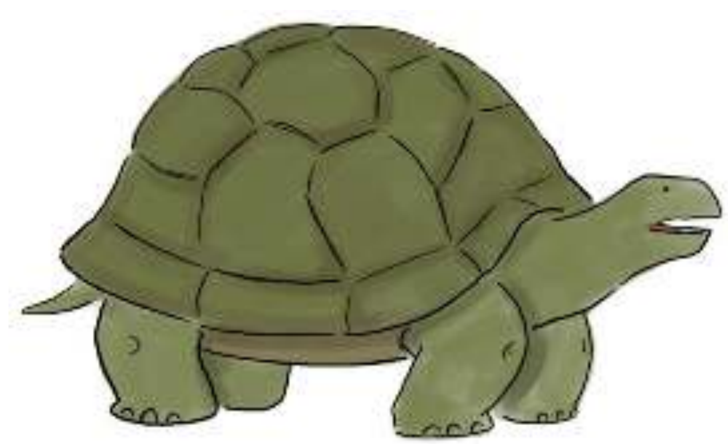
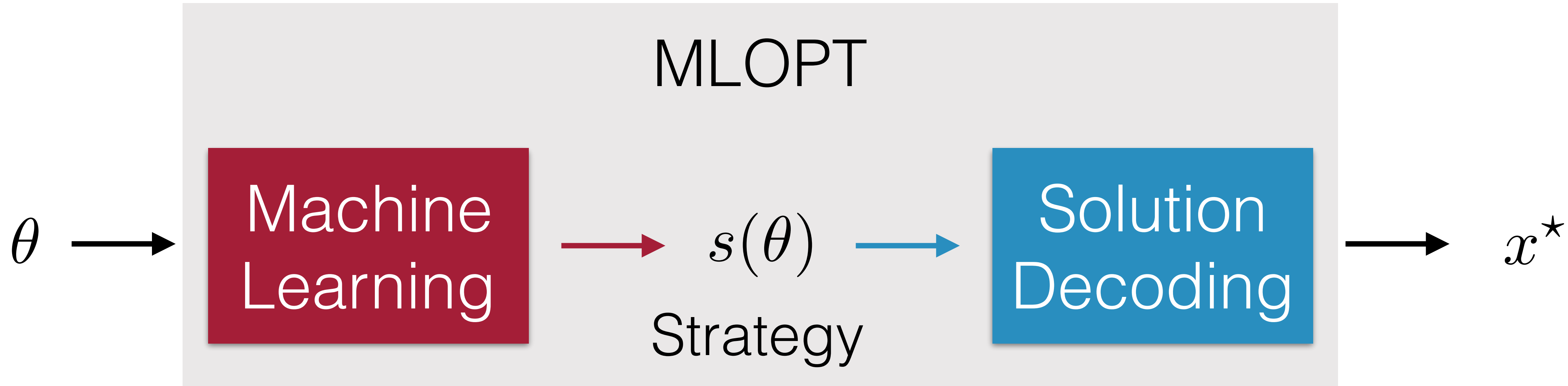




Machine Learning Optimizer



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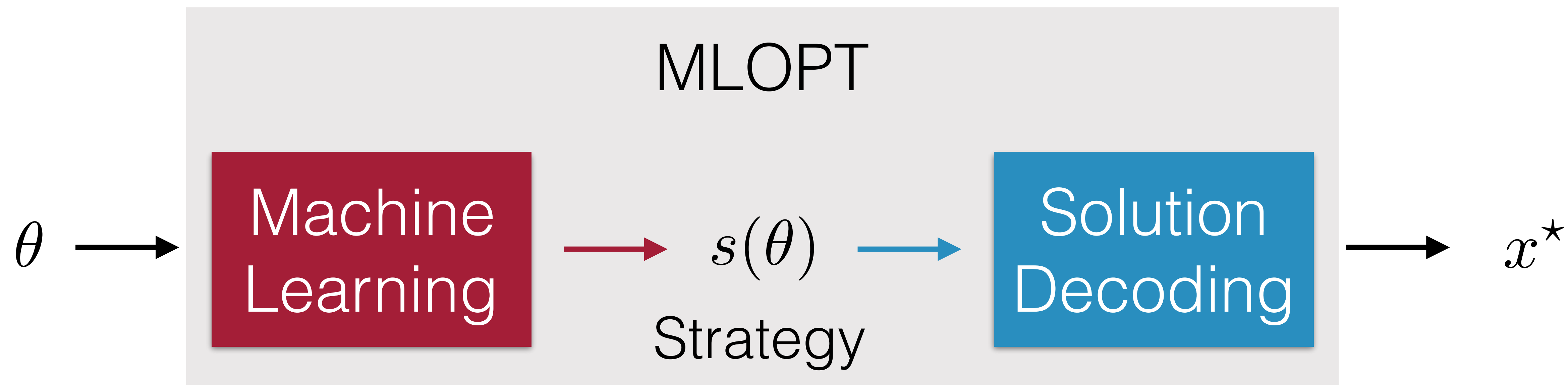

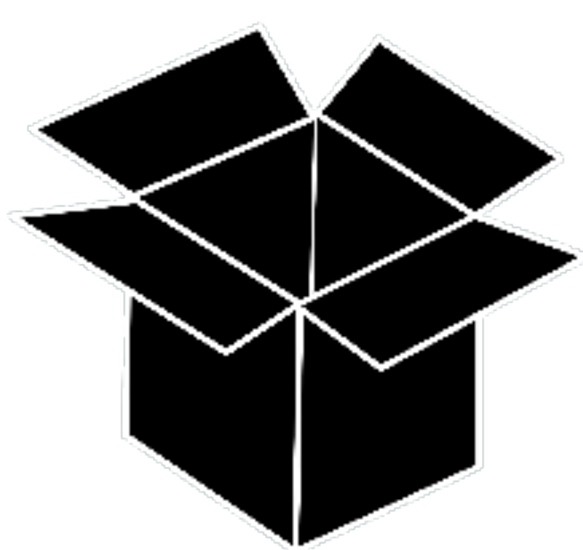

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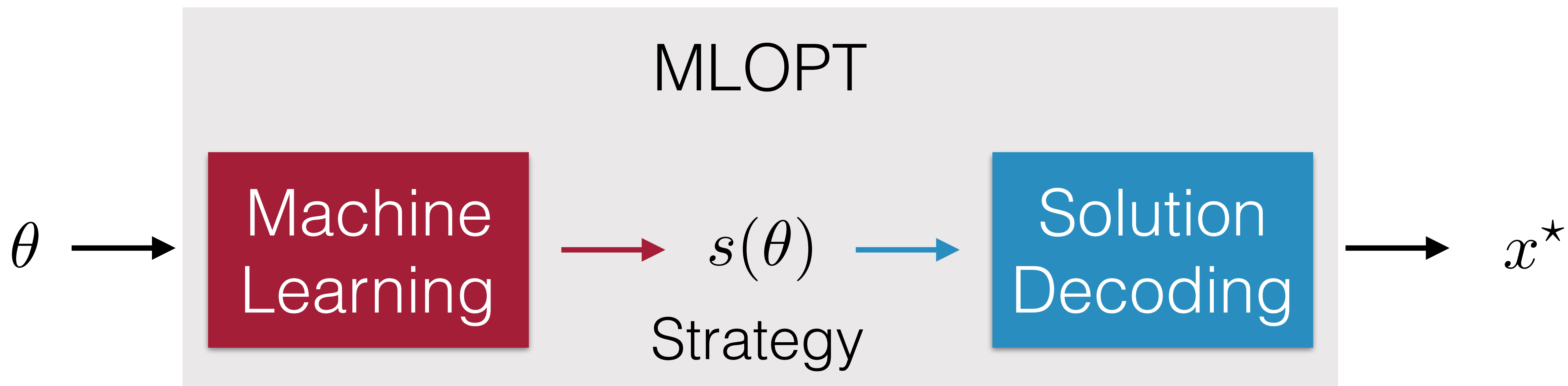
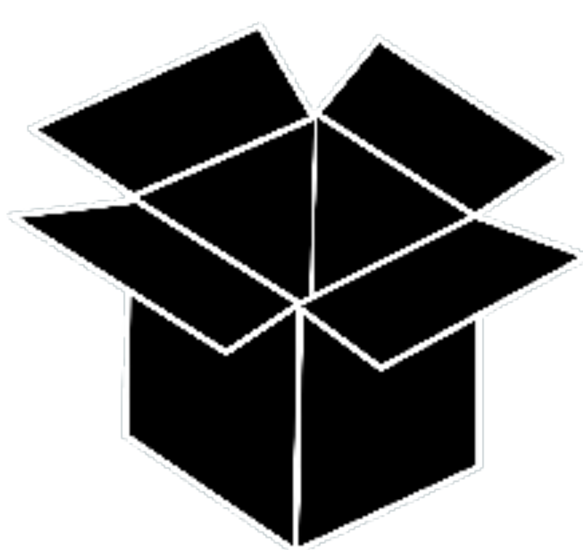
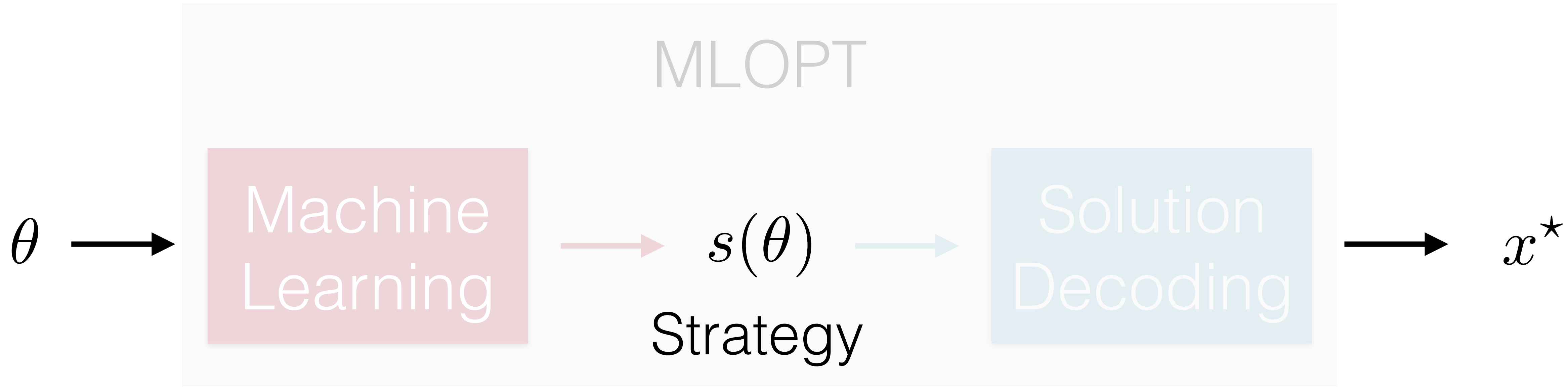


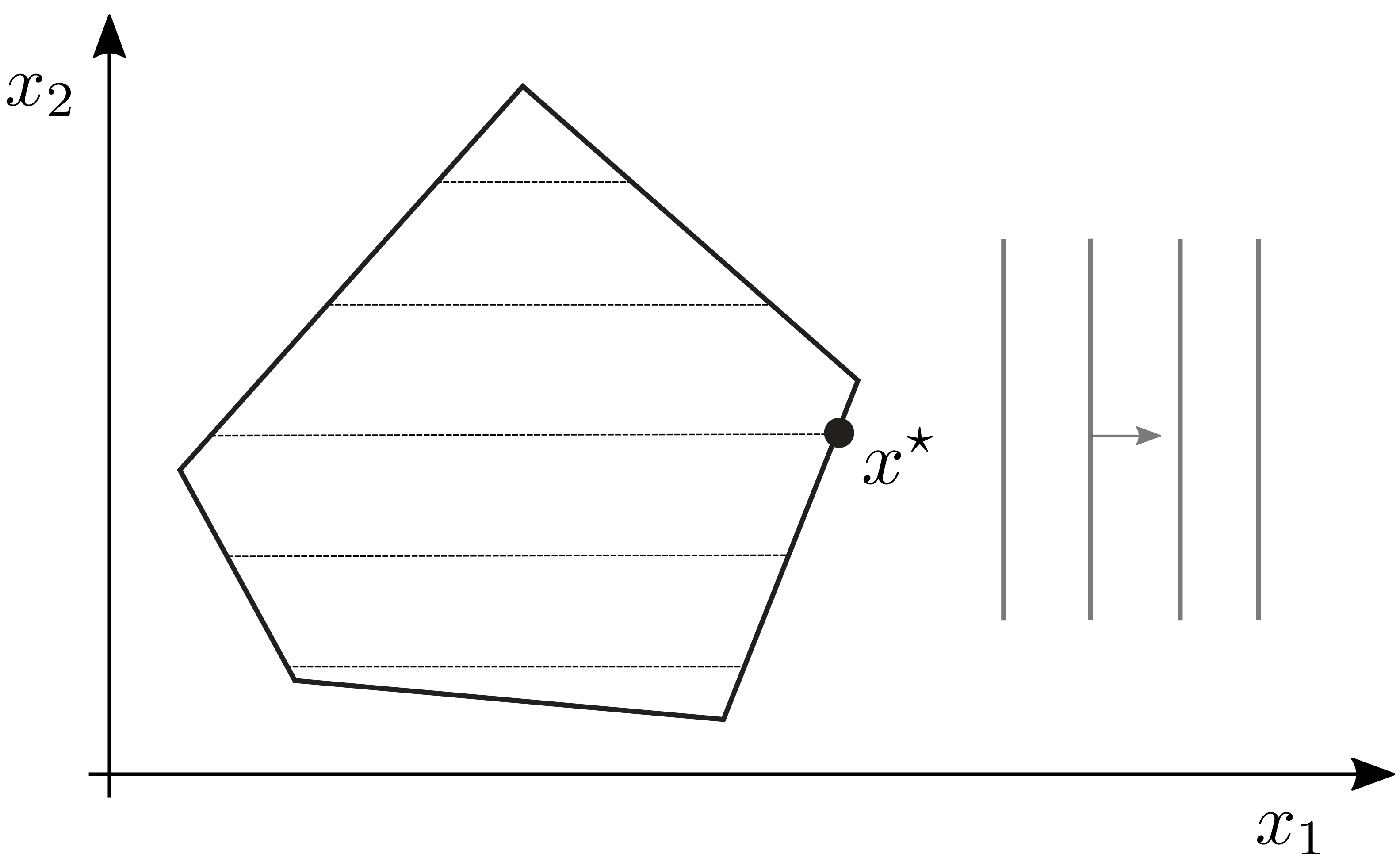


What is a strategy?



Strategies for Mixed-Integer Optimization

$$
\begin{array}{ll}
\text{minimize} & c(\theta)^T x \\
\text{subject to} & A(\theta)x \leq b(\theta) \\
& x_{\mathcal{I}} \in \mathbf{Z}^d
\end{array}
$$



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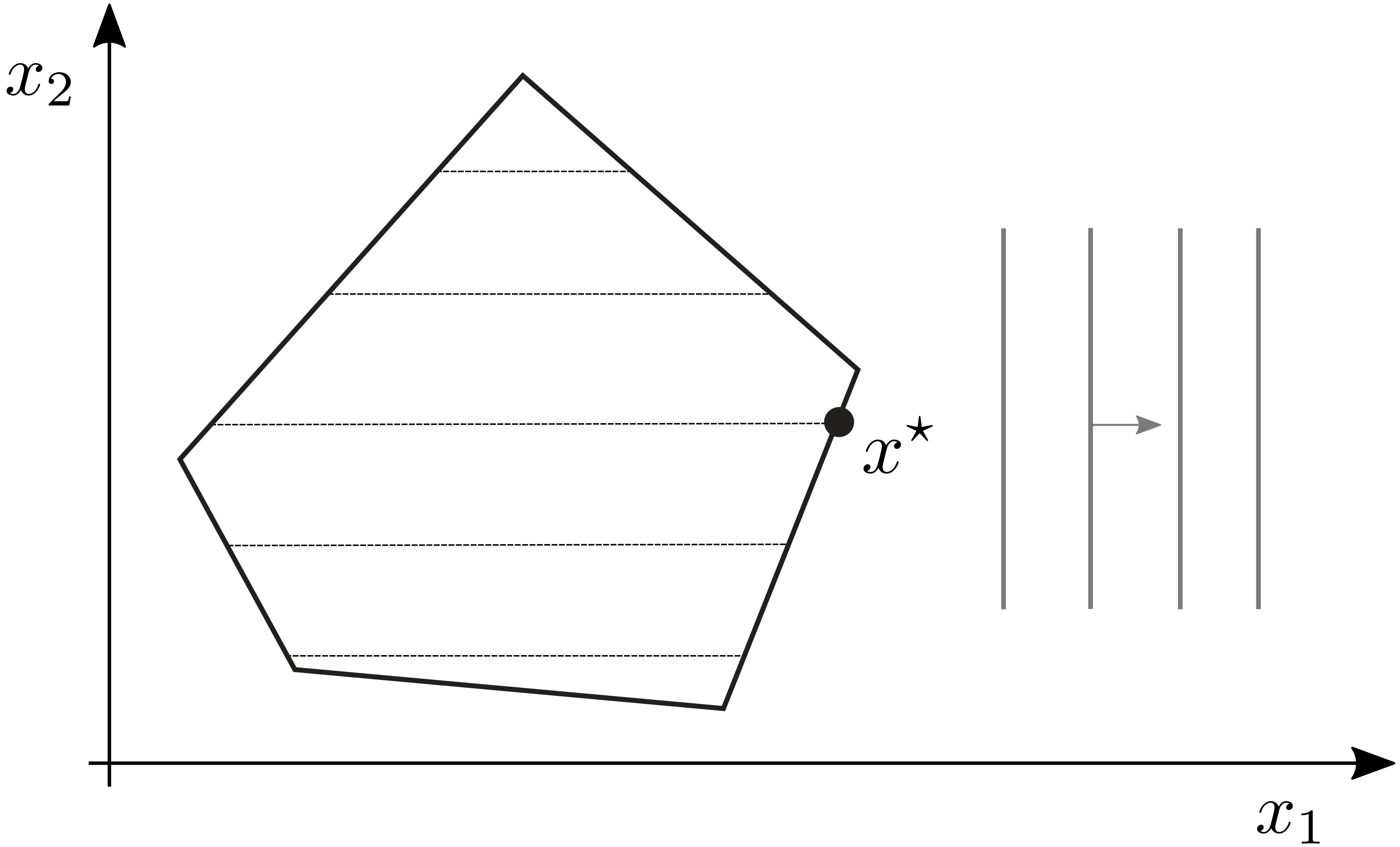
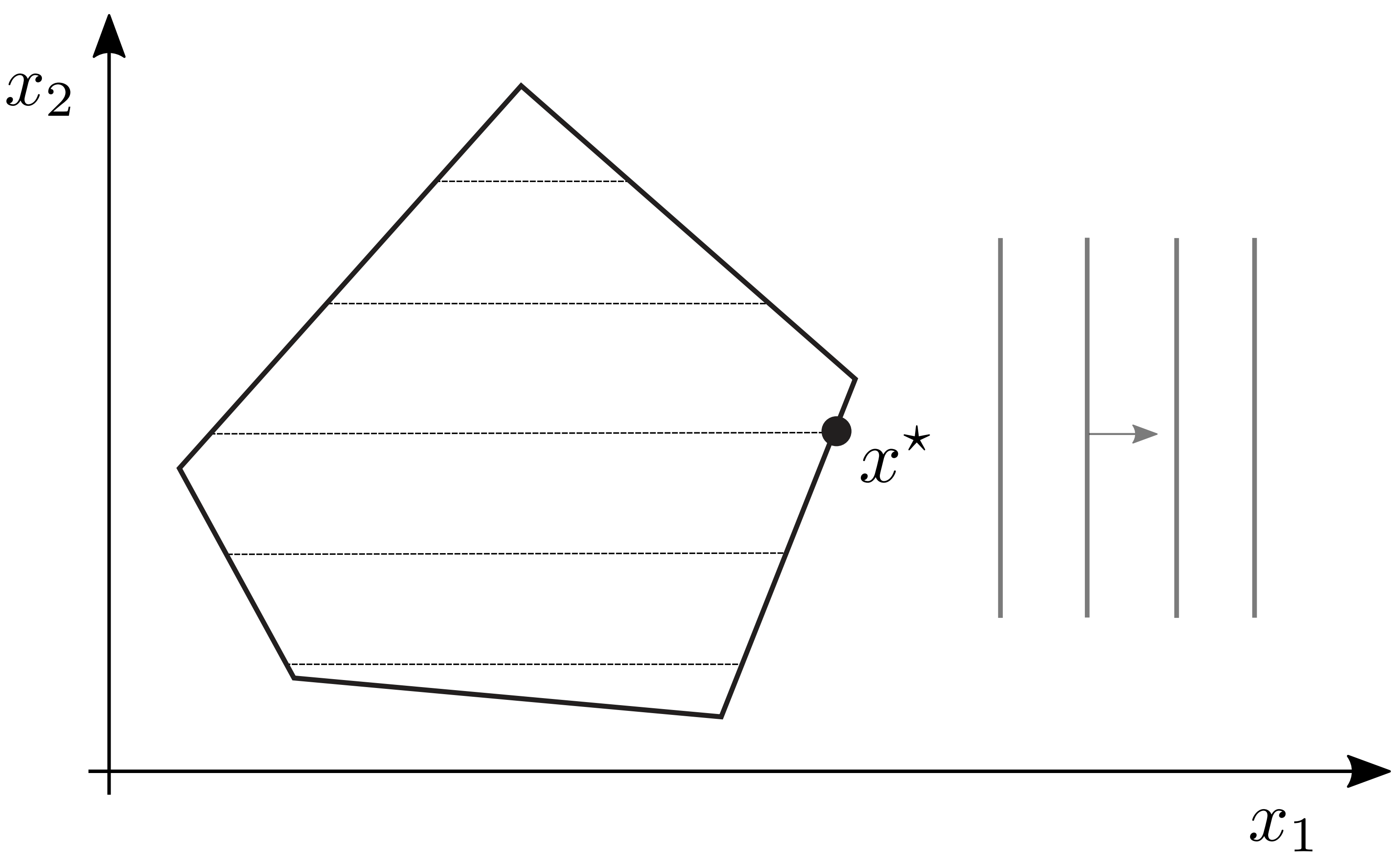
Parameters



Strategies for Mixed-Integer Optimization

$$s(\theta) = (\mathcal{T}(\theta), x^\star_{\mathcal{I}}(\theta))$$



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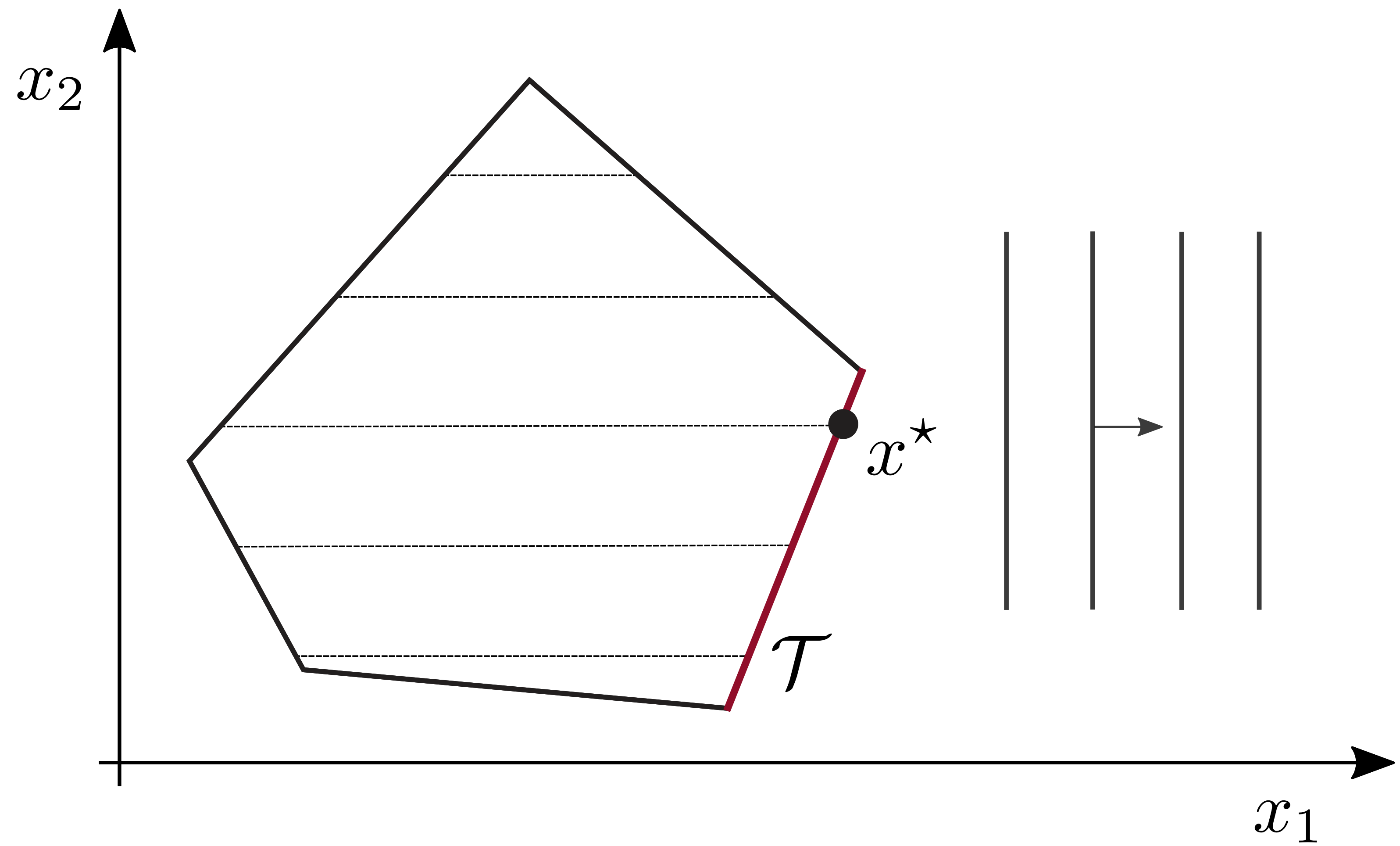
Tight constraints



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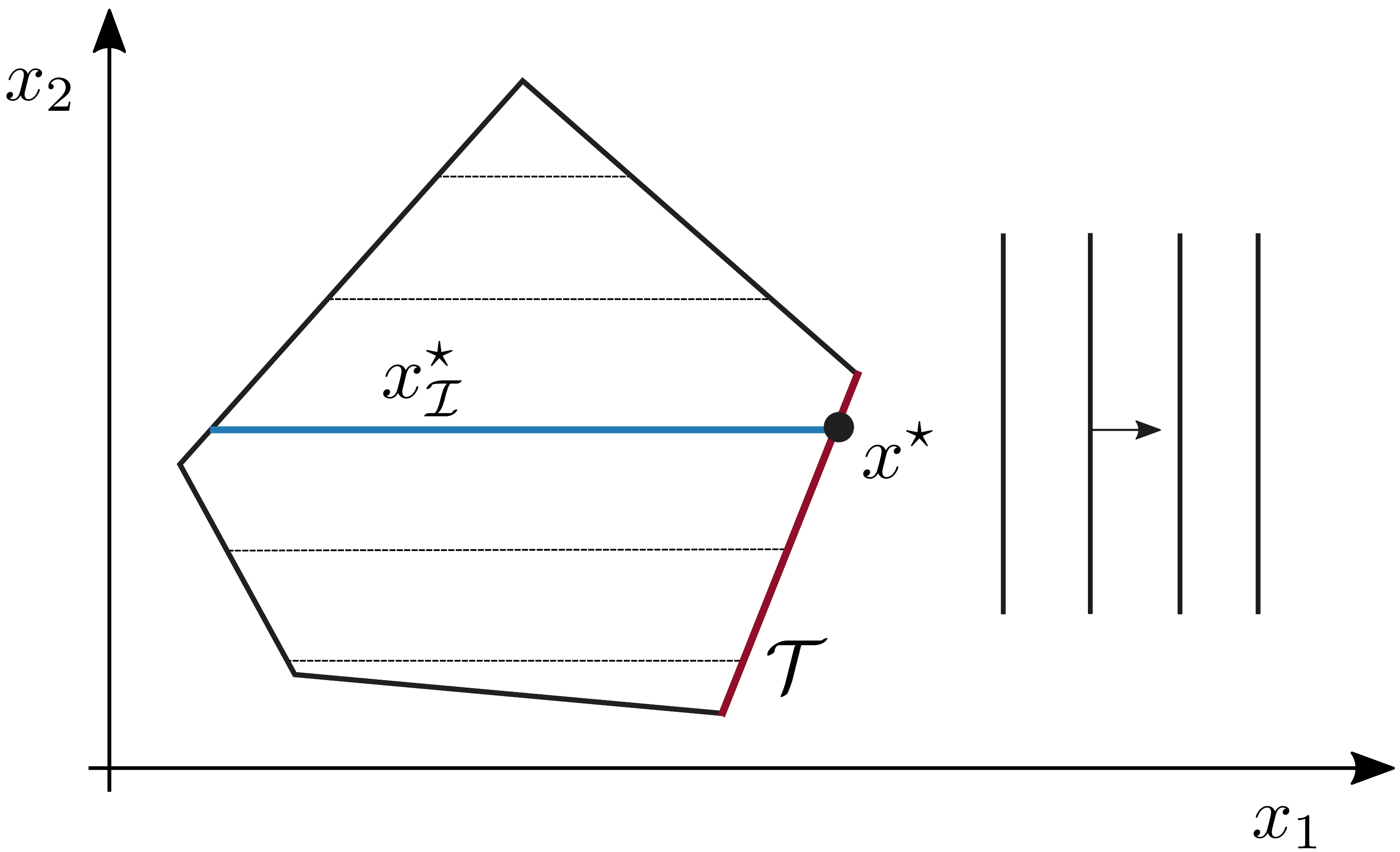
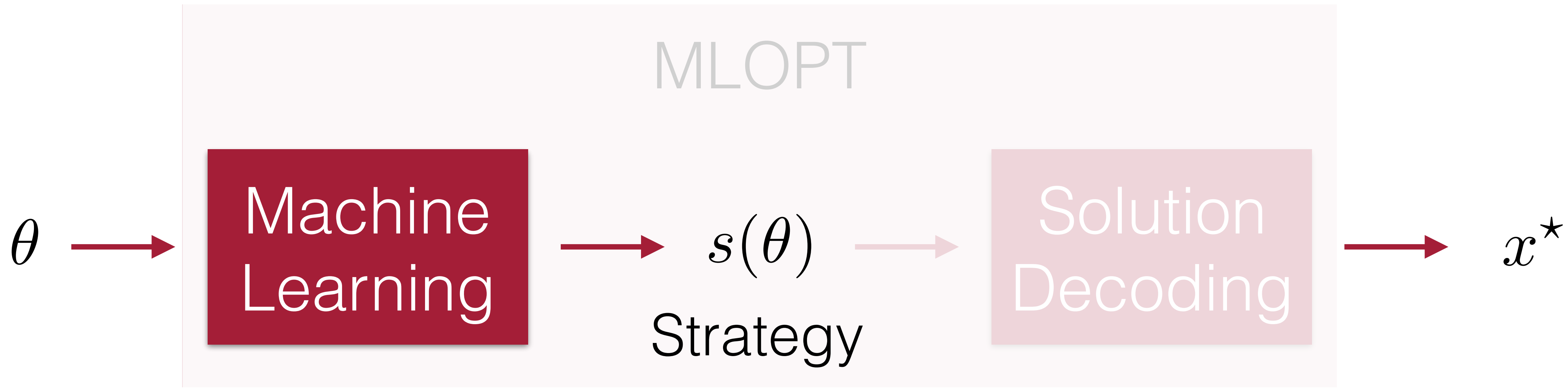
Tight constraints

Integer values



Learning the strategies



Selecting strategies is a classification problem

$\theta \longrightarrow$ **Machine Learning** $\longrightarrow \hat{s}(\theta)$

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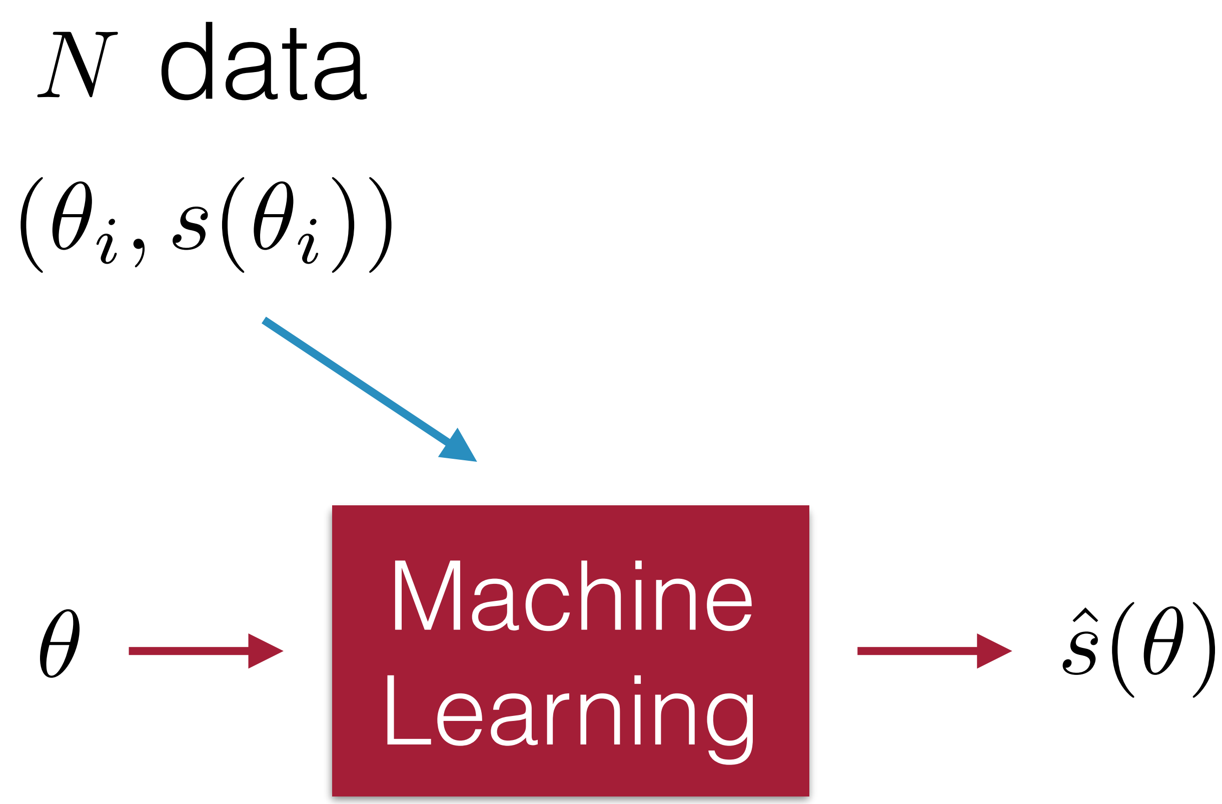

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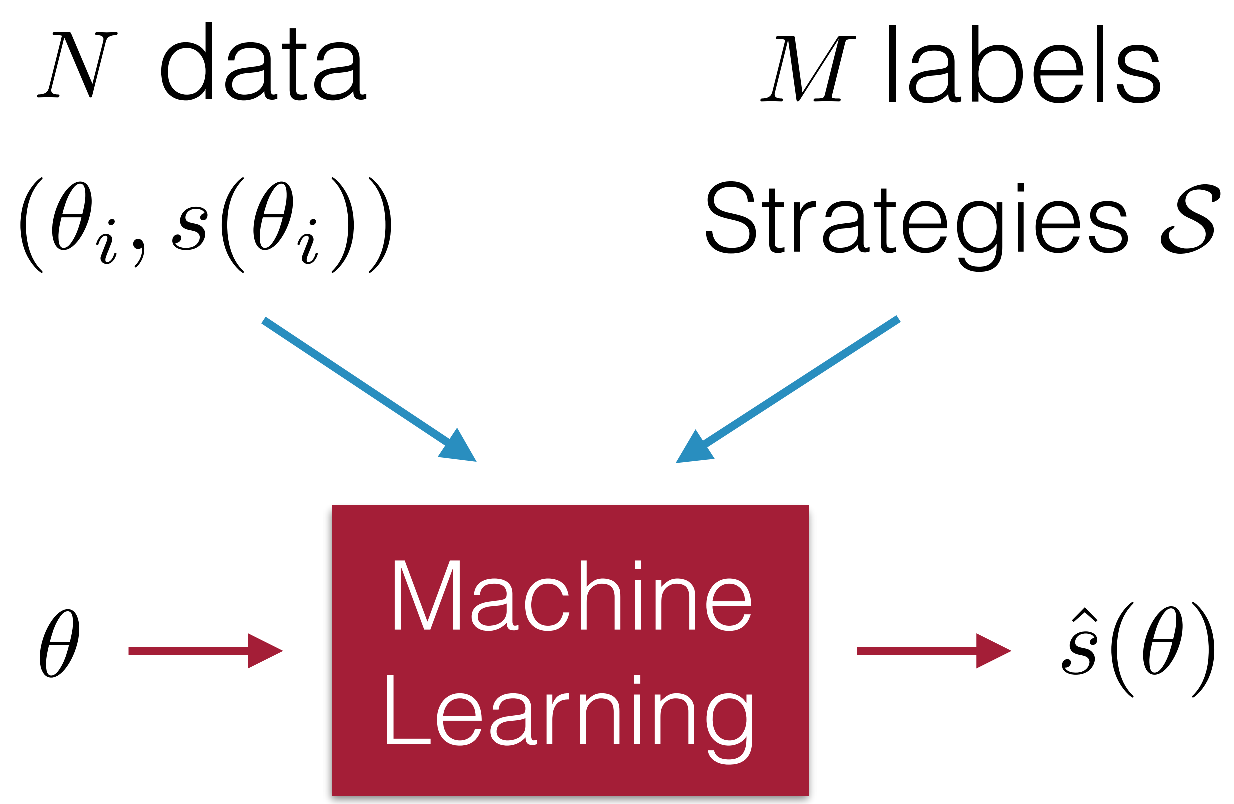


Selecting strategies is a classification problem

N data

$(\theta_i, s(\theta_i))$

M labels

Strategies $\mathcal{S}$

$\theta \longrightarrow$ **Machine Learning** $\longrightarrow \hat{s}(\theta)$

Optimal Trees

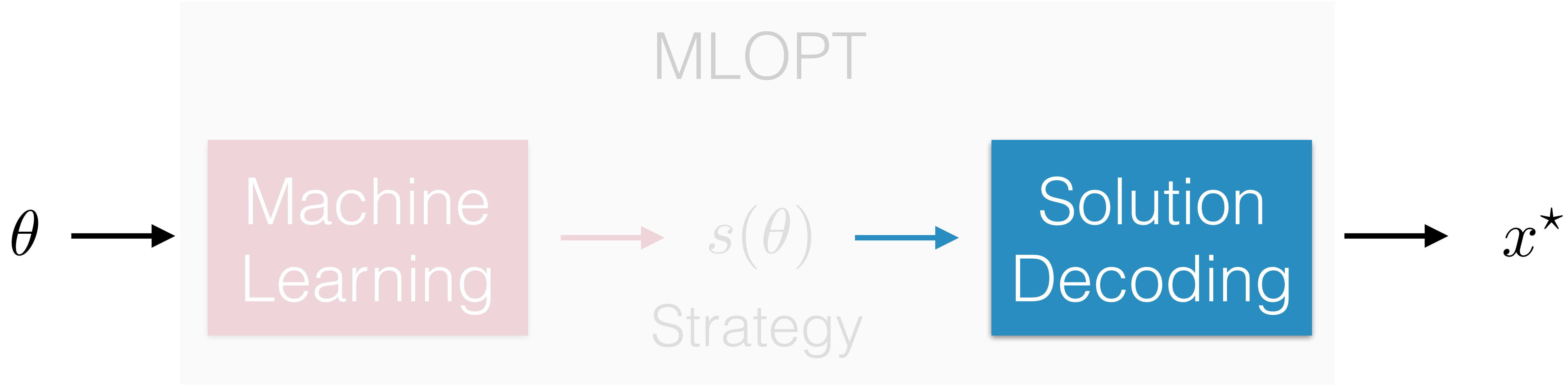
Interpretable AI

[Bertsimas and Dunn (2017)]

Neural Networks

PyTorch

Computing the solution from the optimal strategy



Solution decoding is just a linear system

$$\begin{array}{ll} \text{minimize} & c(\theta)^T x \\ \text{subject to} & A(\theta)x \leq b(\theta) \\ & x_{\mathcal{I}} \in \mathbf{Z}^d \end{array}$$

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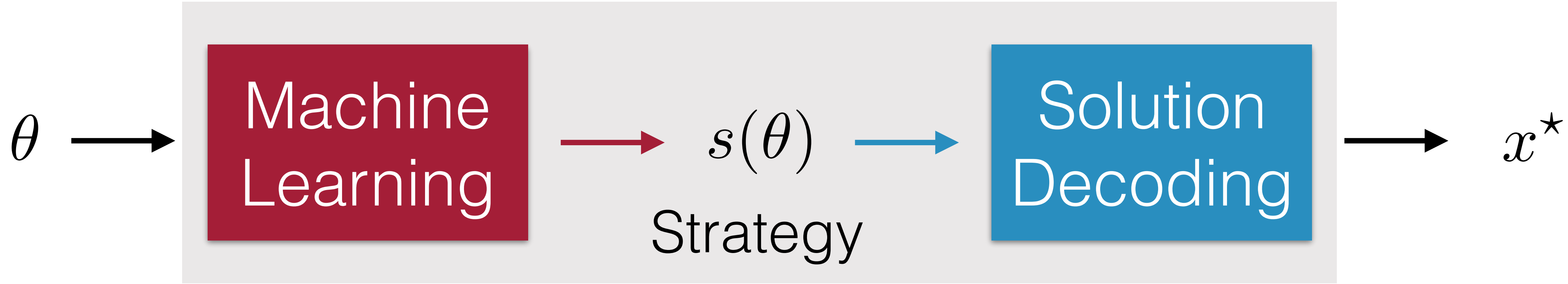
$$\Big\downarrow$$

KKT Linear System

MLOPT: Machine Learning Optimizer

Offline Learning

Online Optimization



Inventory management

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Inventory

Order

Inventory management

$$
\begin{aligned}
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Inventory

Order

Demand

Inventory management

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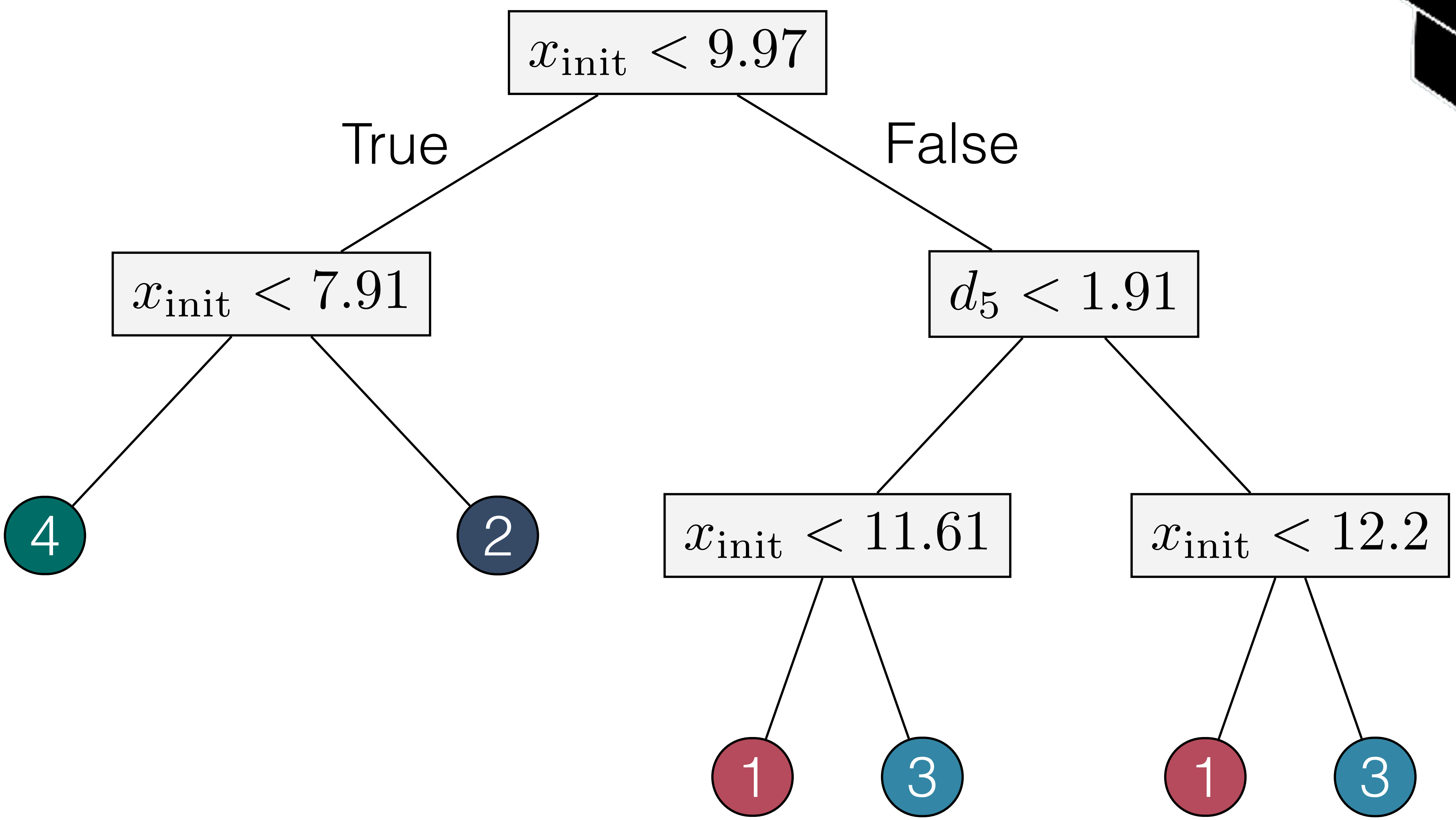
Parameters

Strategy selection using Optimal Trees



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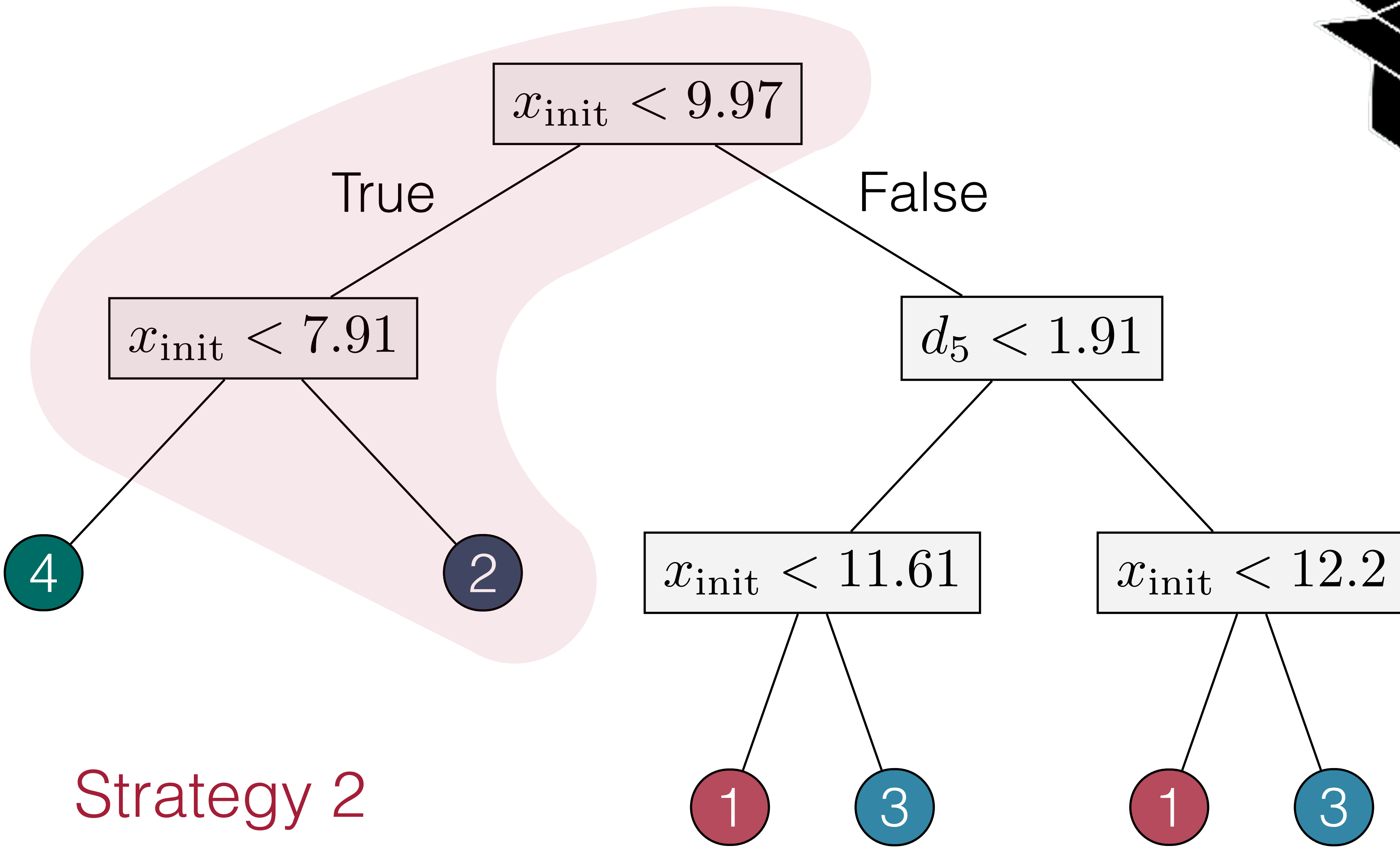


Strategy 2

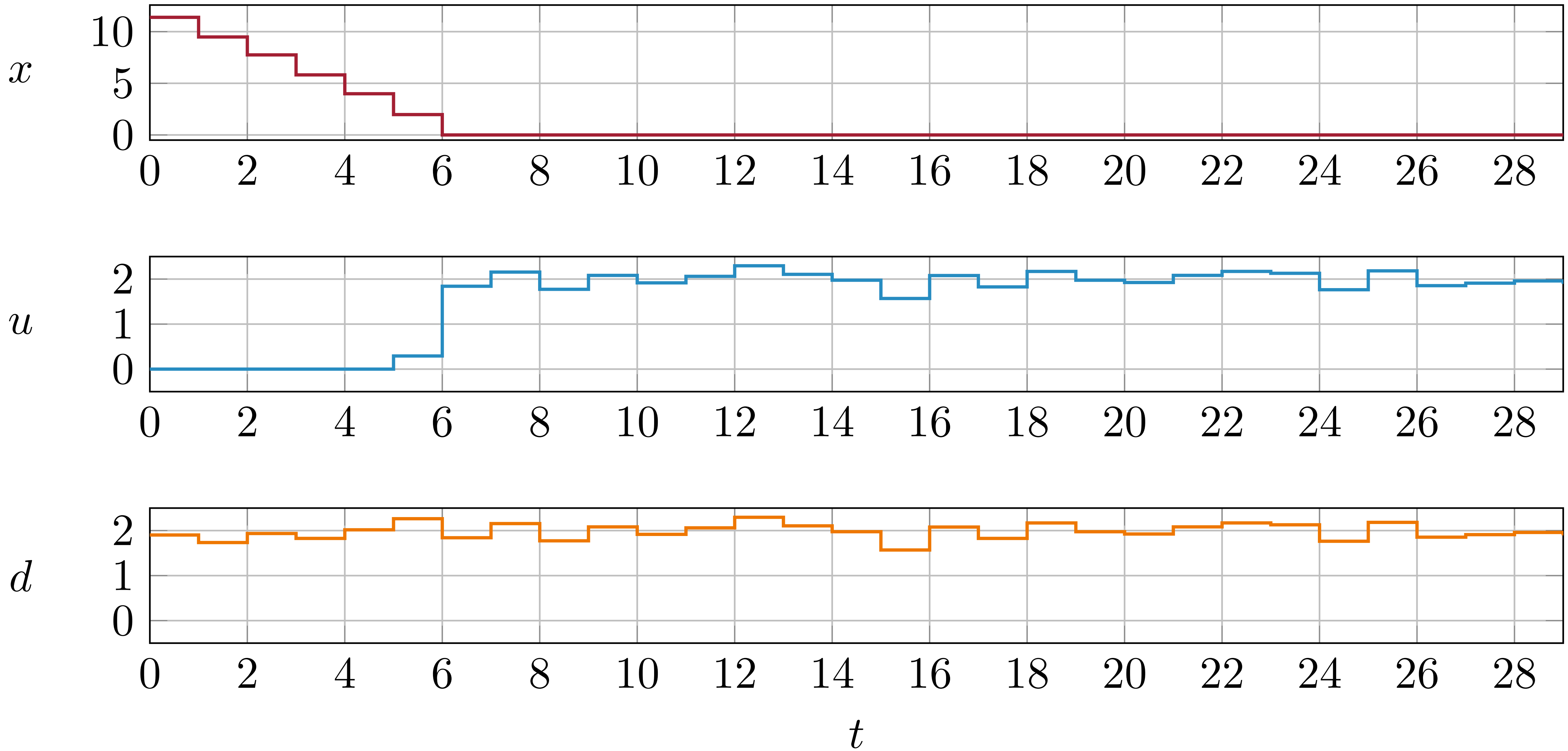
$$u_t = 0 \qquad t \le 4$$

$$0 \le u_t \le M \qquad t > 4$$

Strategies for inventory

Strategy 2

$$\begin{array}{ll} u_t = 0 & t \leq 4 \\ 0 \leq u_t \leq M & t > 4 \end{array}$$



Real-world Example

Sparse portfolio optimization

$$
\begin{array}{ll}
\text{maximize} & r^T w - \gamma w^T \Sigma w \\
\text{subject to} & \mathbf{1}^T w = 1 \\
& \mathbf{card}(w) \leq c
\end{array}
$$

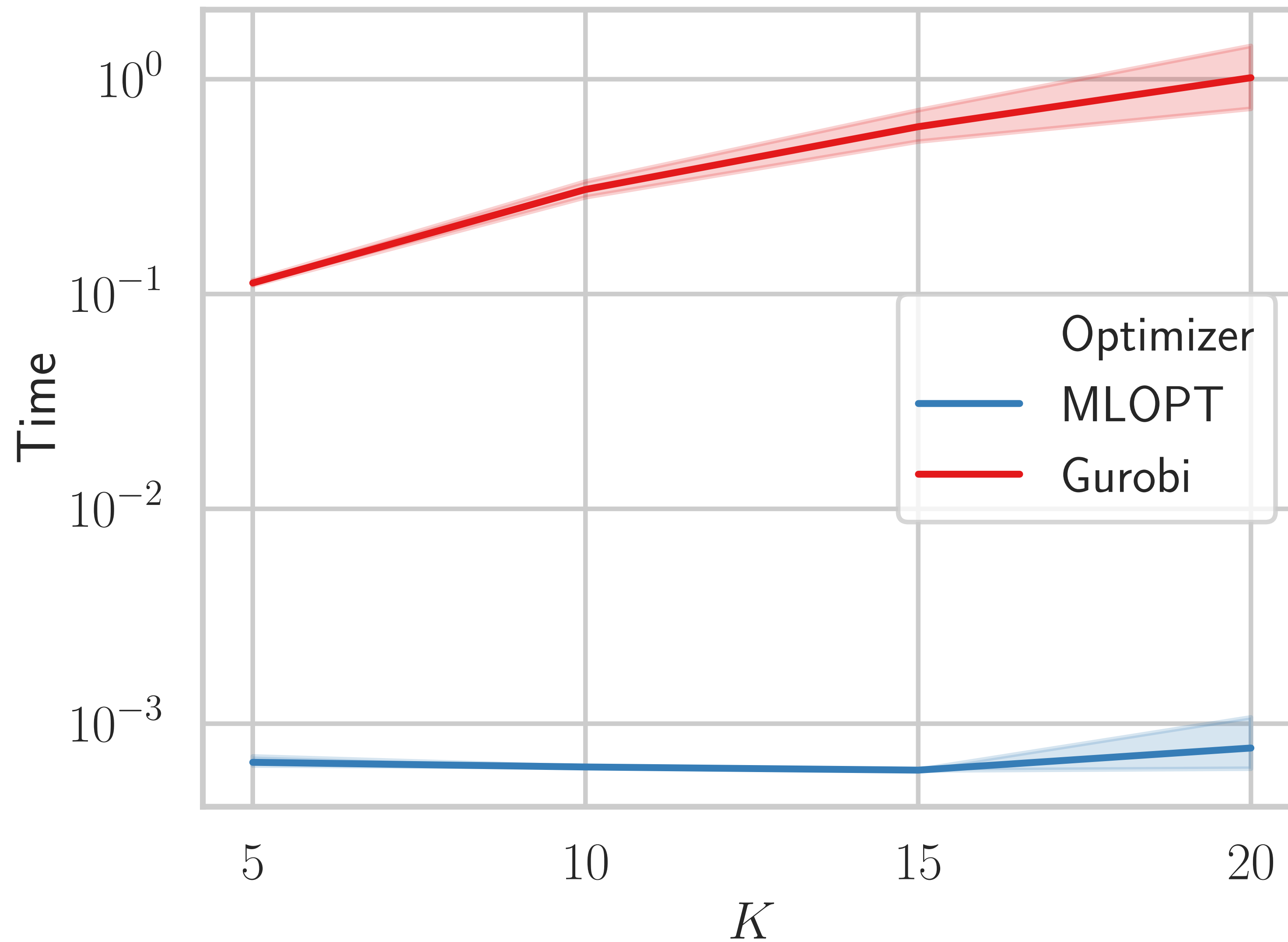
k-factor Risk Model

$$\Sigma = F \Sigma^k F^T + D$$

Parameters

$$\theta = (r, \Sigma^k, F, D)$$

S&P100 backtesting: high speed and accuracy



K	M	acc [%]	suboptimality	infeasibility
5	1027	99.11	9.88×10^{-3}	2.25×10^{-9}
10	1083	99.21	7.01×10^{-3}	8.19×10^{-7}
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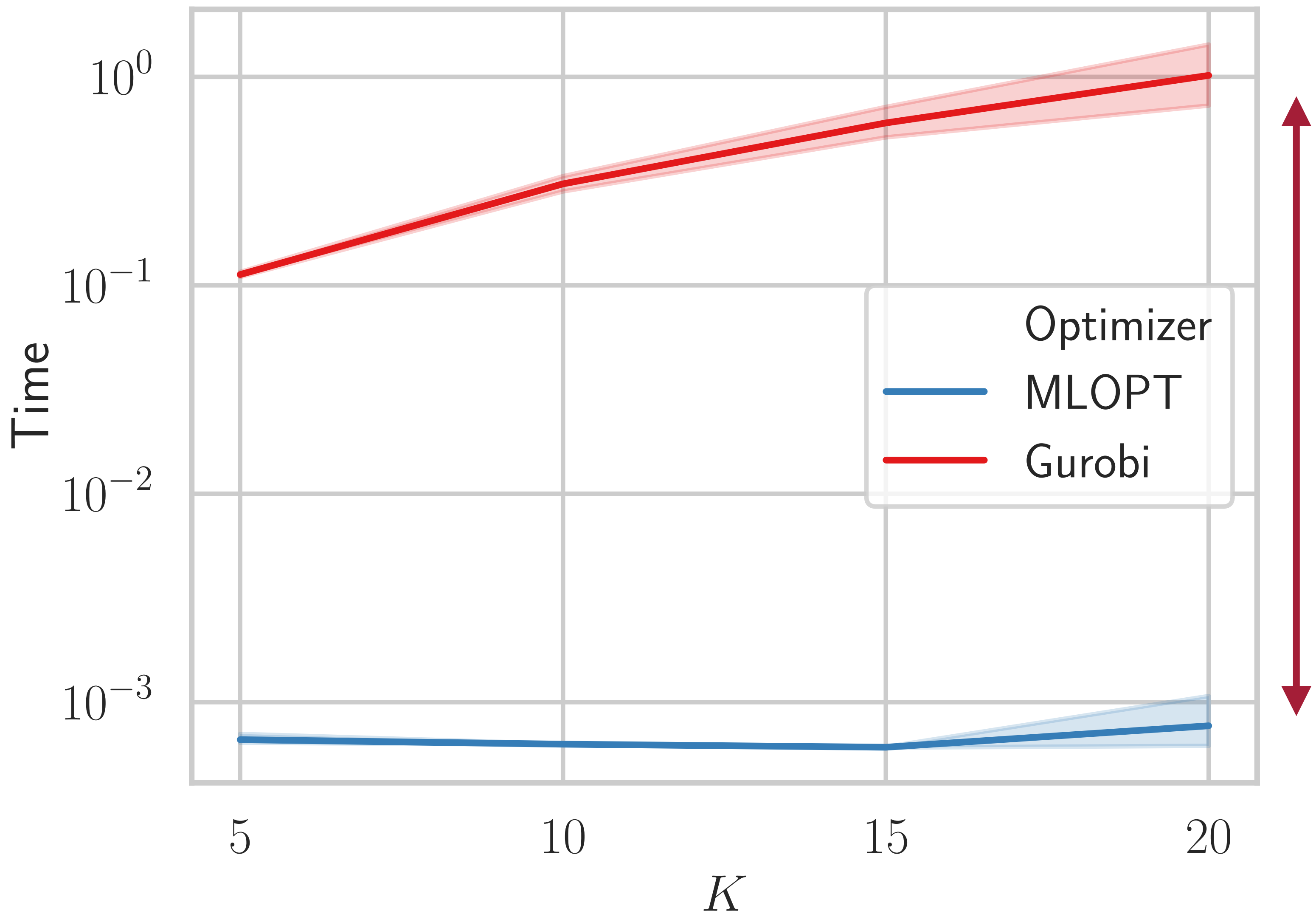

1000x faster

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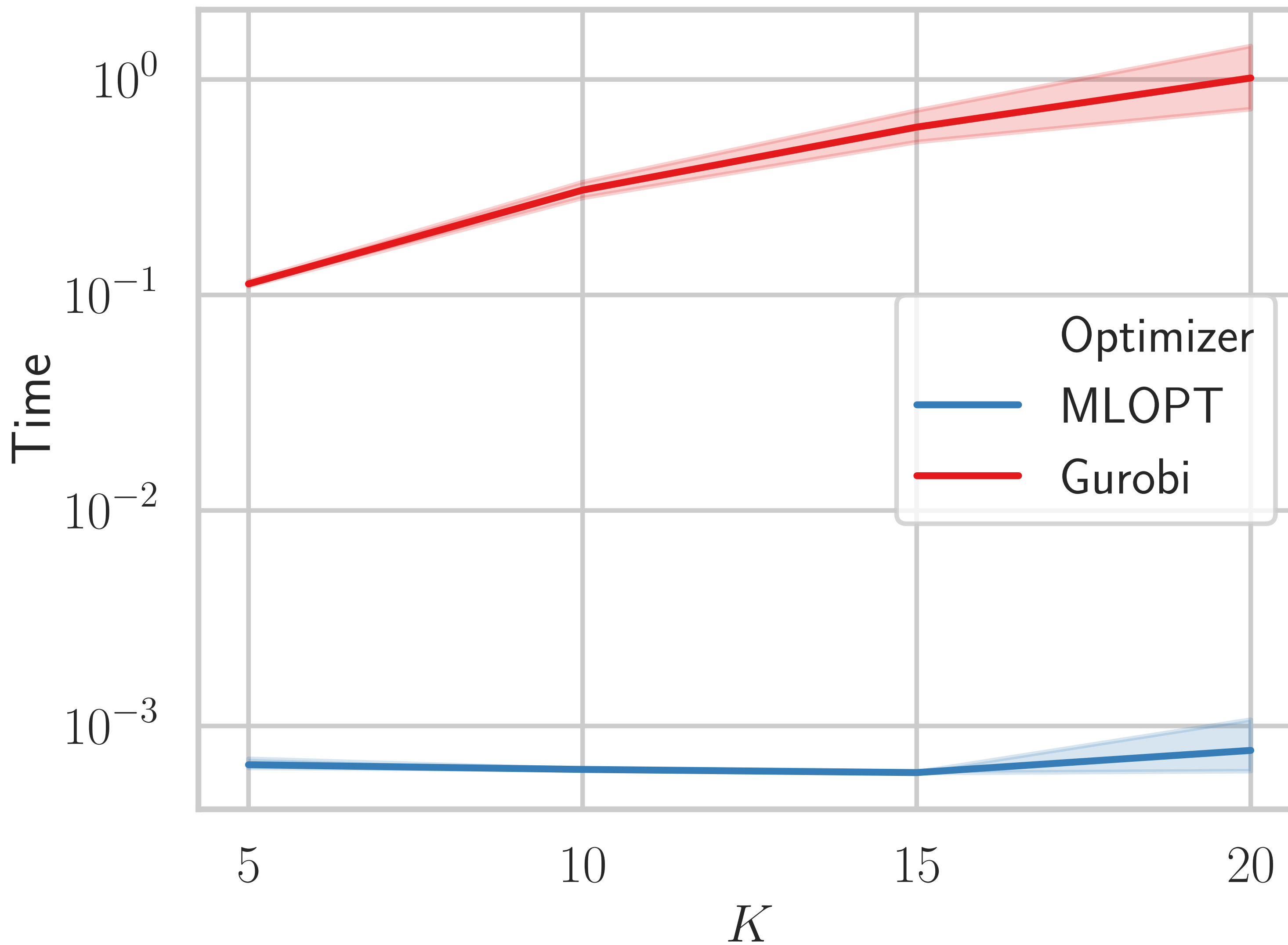

1000x faster

Low suboptimality and infeasibility

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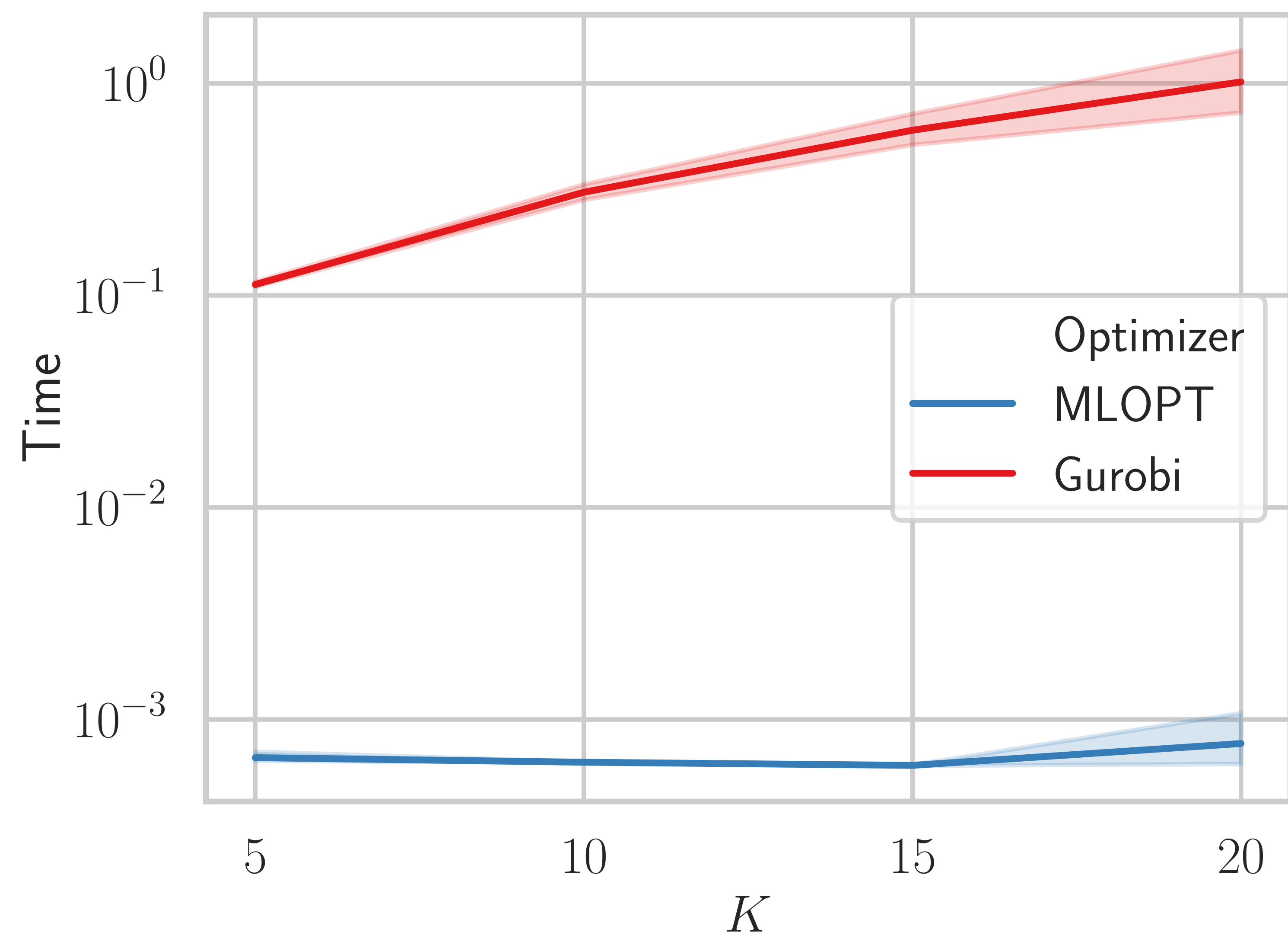

1000x faster

High accuracy

Low suboptimality and infeasibility

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Optimization
links data
to decisions

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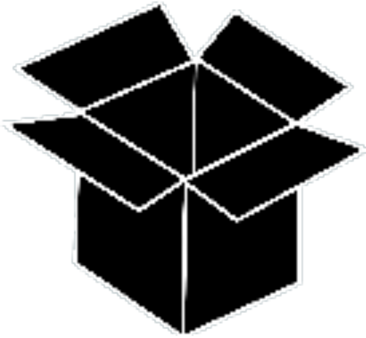
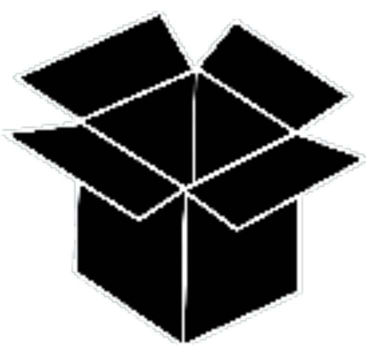


Understand
using ML

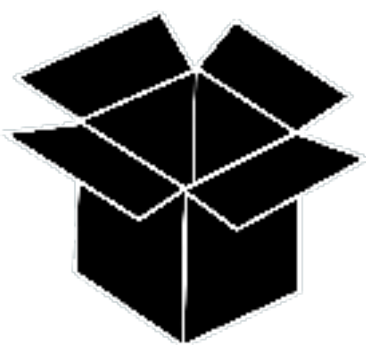
Optimization links data to decisions



Understand using ML

Solve really fast

Optimization links data to decisions



Understand using ML

Solve really fast

Bertsimas, D., Stellato, B., *The Voice of Optimization*, arXiv:1812.09991

Bertsimas, D., Stellato, B., *Online Mixed-Integer Optimization in Milliseconds*, arXiv:1907.02206

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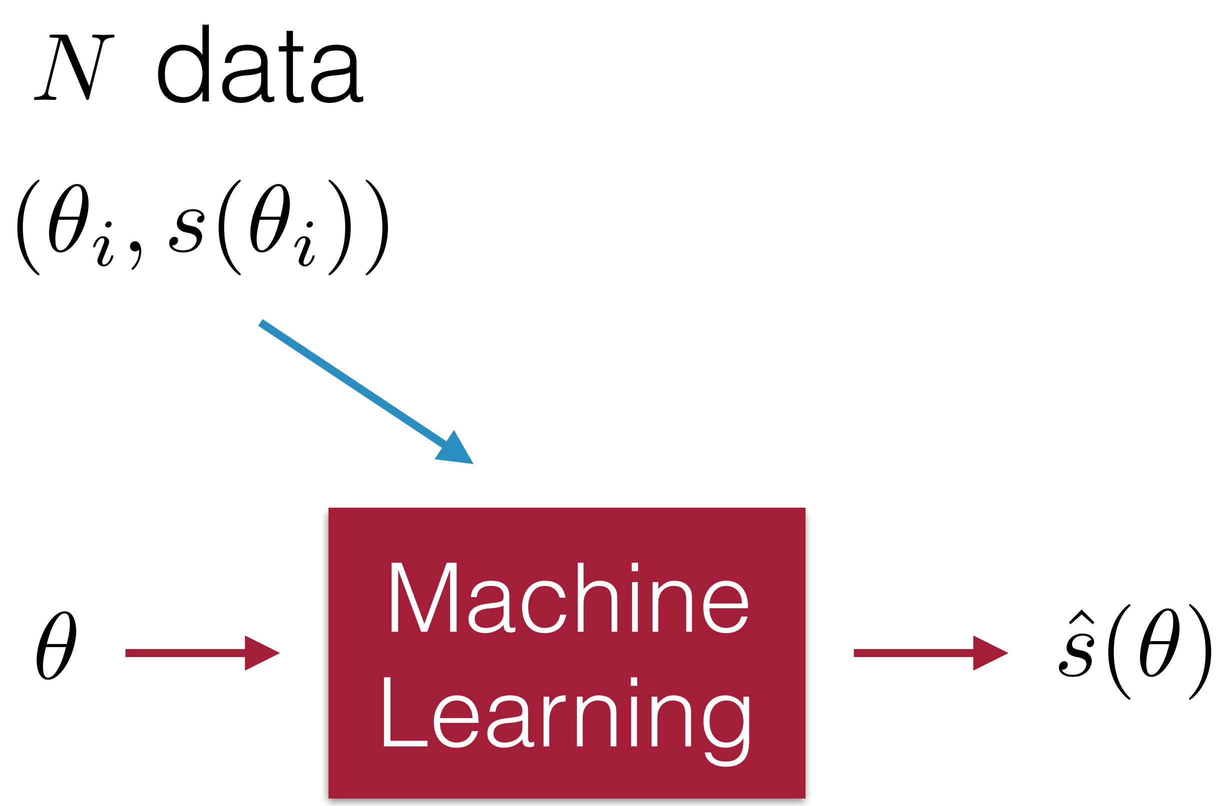
@b_stellato

Backup

Classification problem

$\theta \longrightarrow$ **Machine Learning** $\longrightarrow \hat{s}(\theta)$

Classification problem



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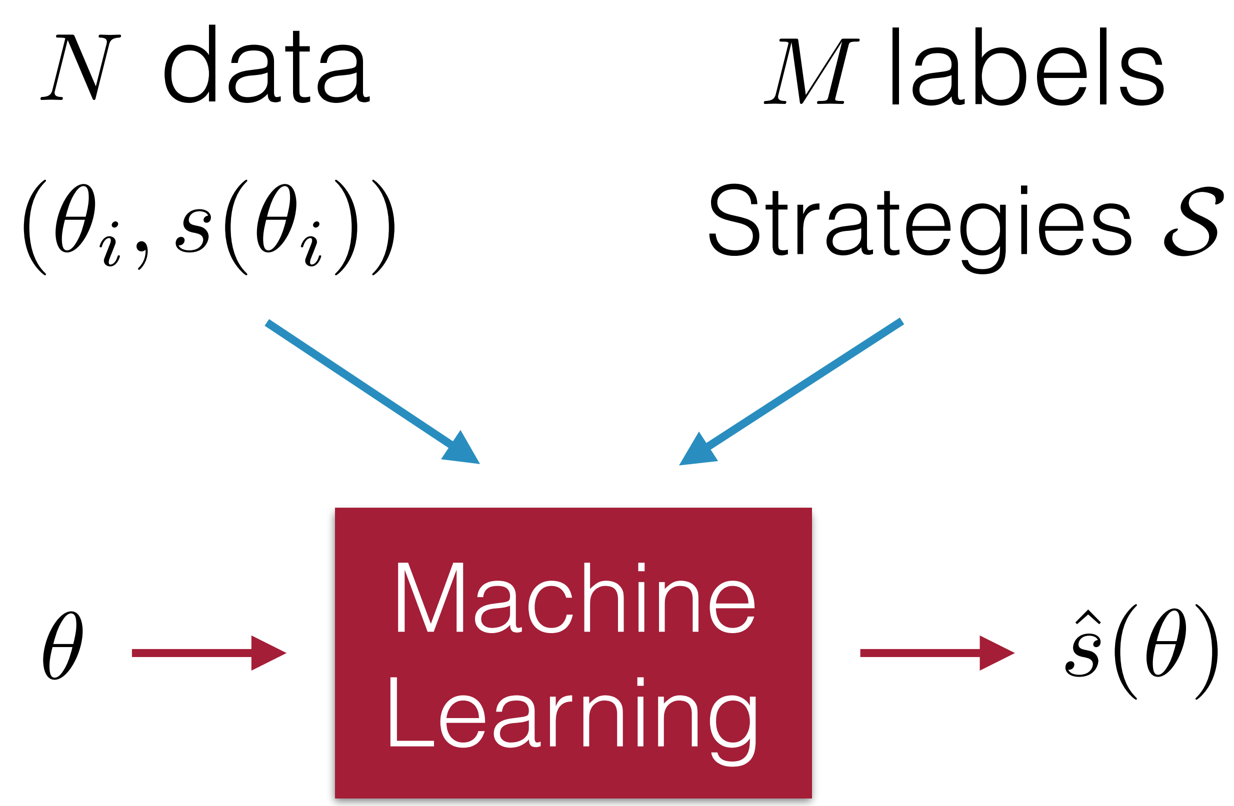

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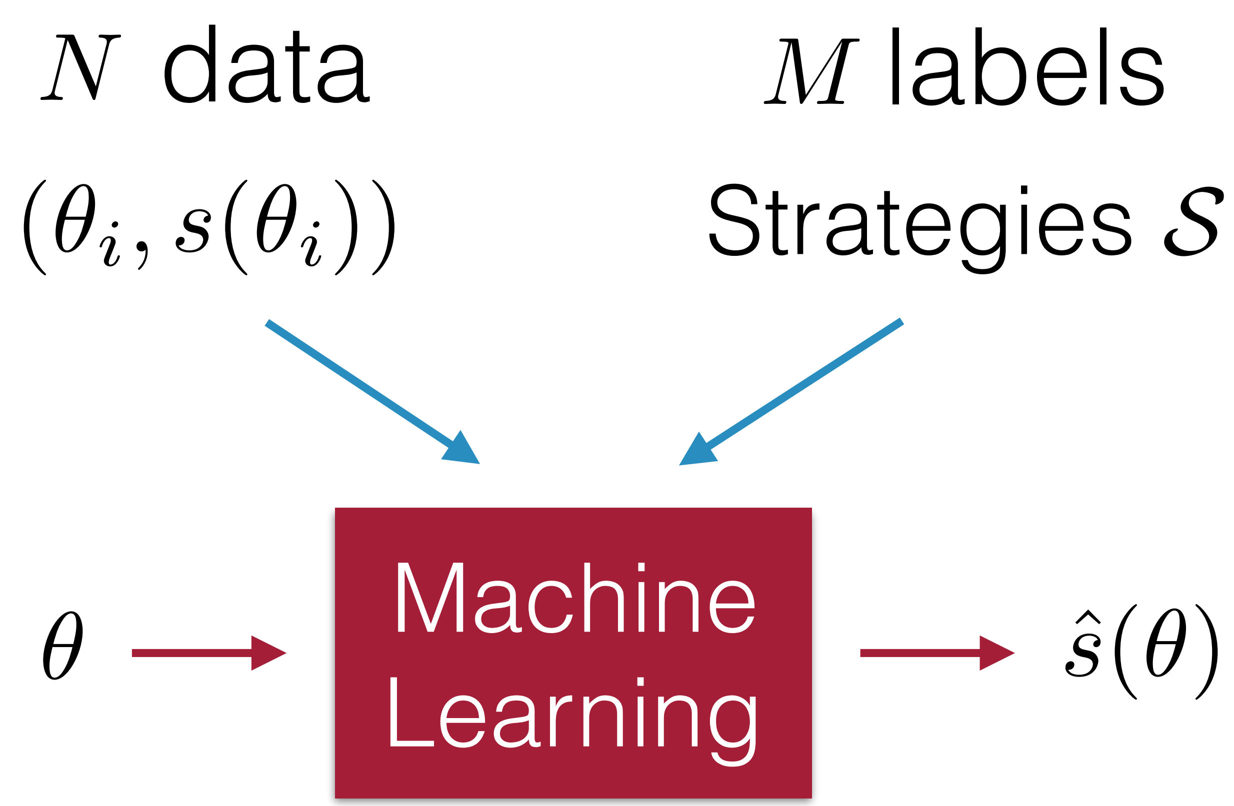
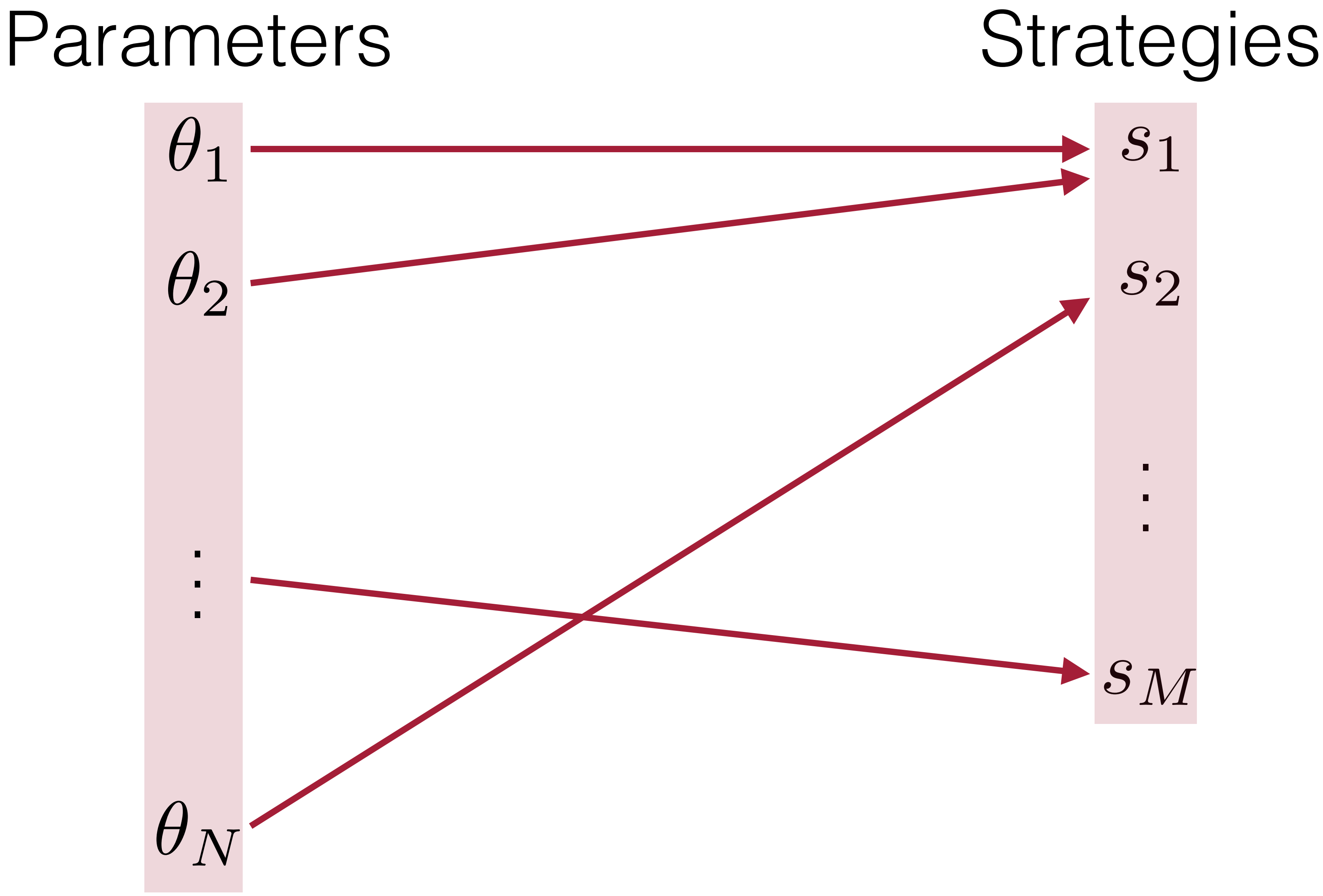


Have we seen enough data?

Will we find new strategies?



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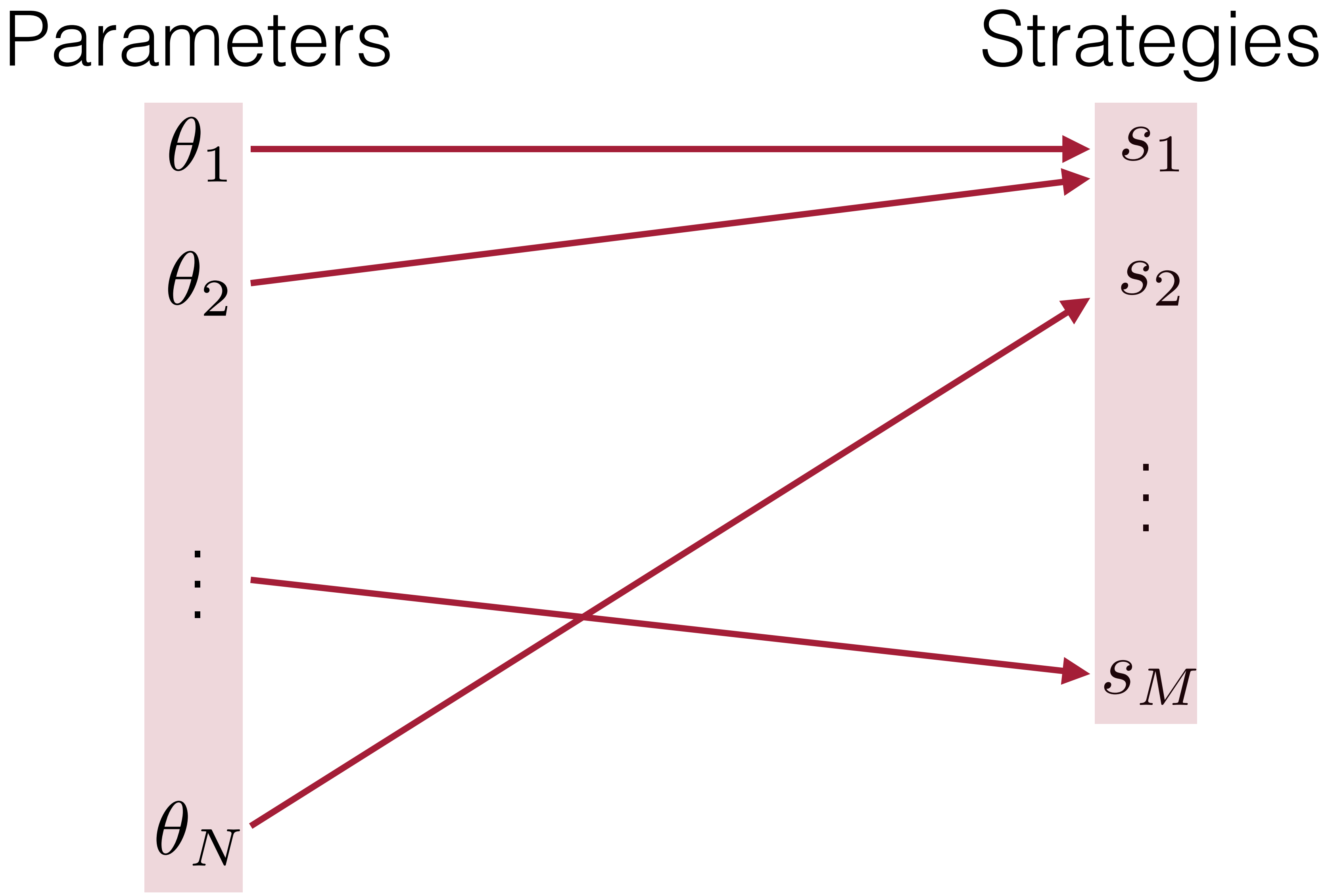

θ_{N+1}?

Alan Turing already knew that

Decoded the **Enigma Machine**
in
World War II

Only some words (labels) needed

Good-Turing estimator

s_1	12 times
s_2	45 times
$\vdots$	$\vdots$
s_M	2 times

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Probability of unseen strategies

$$GT = \frac{N_1}{N} \approx \mathbf{P}(\theta_{N+1} \in \mathbf{R}^p \mid s(\theta_{N+1}) \notin \mathcal{S}(\Theta_N))$$

Good-Turing estimator

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$\vdots$	$\vdots$
s_M	2 times

\# strategies appeared once

Probability of unseen strategies

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$\vdots$	$\vdots$
s_M	2 times

$$GT = \frac{\overset{\text{\# strategies appeared once}}{N_1}}{\underset{\text{\# samples}}{N}} \approx \overbrace{\mathbf{P}(\theta_{N+1} \in \mathbf{R}^p \mid s(\theta_{N+1}) \notin \mathcal{S}(\Theta_N))}^{\text{Probability of unseen strategies}}$$

Good-Turing estimator

s_1	12 times
s_2	45 times
$\vdots$	$\vdots$
s_M	2 times

\# strategies appeared once

Probability of unseen strategies

$$GT = \frac{N_1}{N} \approx \mathbf{P}(\theta_{N+1} \in \mathbf{R}^p \mid s(\theta_{N+1}) \notin \mathcal{S}(\Theta_N))$$

\# samples

Bound

$$\mathbf{P}(\theta_{N+1} \in \mathbf{R}^p \mid s(\theta_{N+1}) \notin \mathcal{S}(\Theta_N)) \leq GT + c\sqrt{(1/N)\ln(3/\beta)}$$

Simple sampling scheme

Bound

Repeat until $GT + c\sqrt{(1/N)\ln(3/\beta)} \leq \epsilon$

1. sample θ_i
2. compute $s(\theta_i)$
3. update estimator $GT = \frac{N_1}{N}$

Sampling scheme

Algorithm 1 Strategies exploration

given $\epsilon, \beta, \Theta = \emptyset, \mathcal{S} = \emptyset, u = \infty$
for $k = 1, \ldots,$ **do**
  Sample θ_k and compute $s(\theta_k)$ ▷ Sample parameter and strategy.
  $\Theta \leftarrow \Theta \cup \{\theta_k\}$ ▷ Update set of samples.
  if $s(\theta_k) \notin \mathcal{S}$ **then**
    $\mathcal{S} \leftarrow \mathcal{S} \cup \{s(\theta_k)\}$ ▷ Update strategy set if new strategy found
  end if
  if $G + c\sqrt{(1/k)\ln(3/\beta)} \leq \epsilon$ **then** ▷ Break if bound less than ϵ
    break
  end if
end for
return $k, \Theta, \mathcal{S}$