OSQP

An Operator Splitting Solver for Quadratic Programs

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joint work with Goran Banjac,

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Why quadratic programming?

AN ALGORITHM FOR QUADRATIC PROGRAMMING

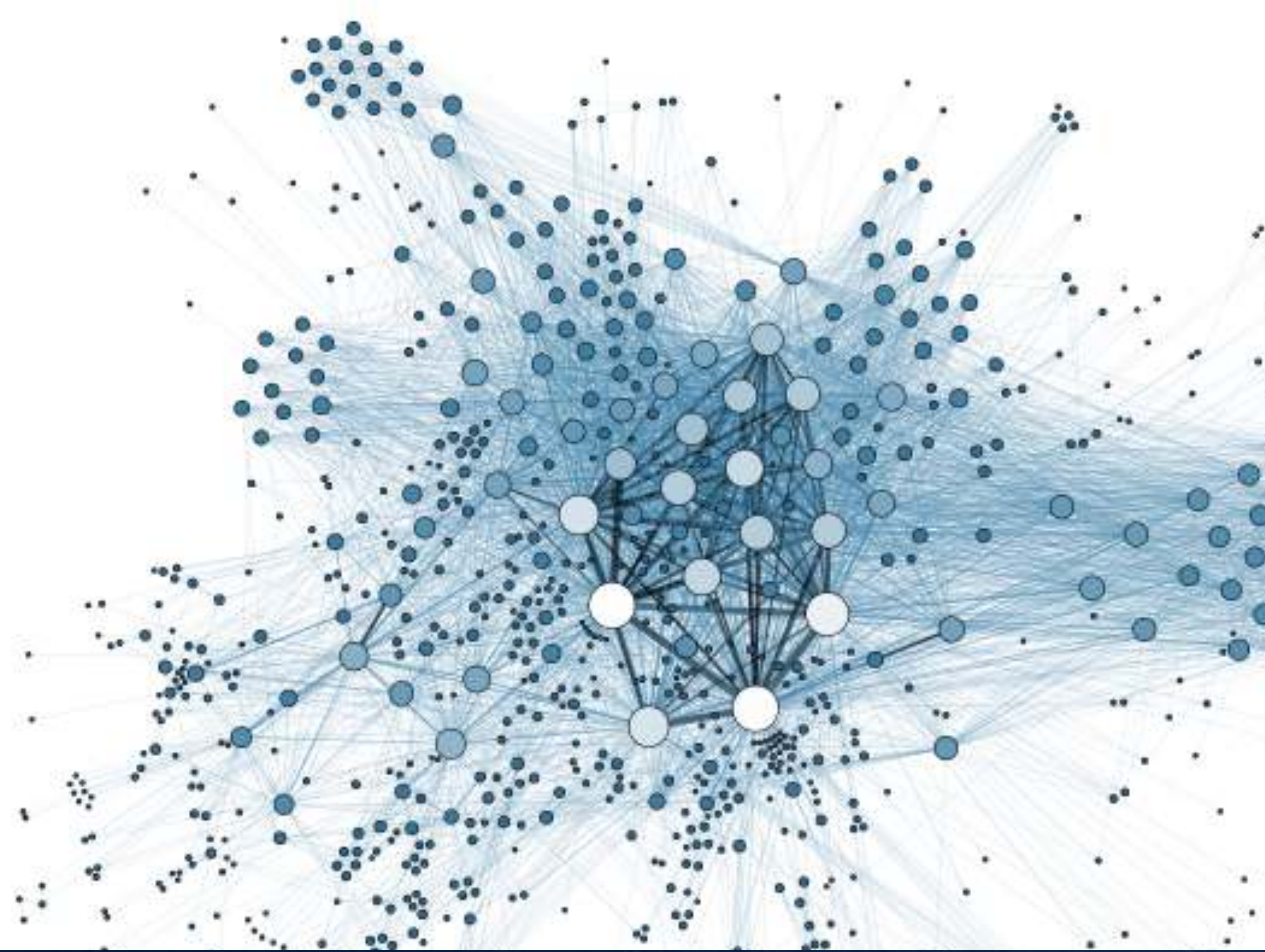
Marguerite Frank and Philip Wolfe[1]
Princeton University

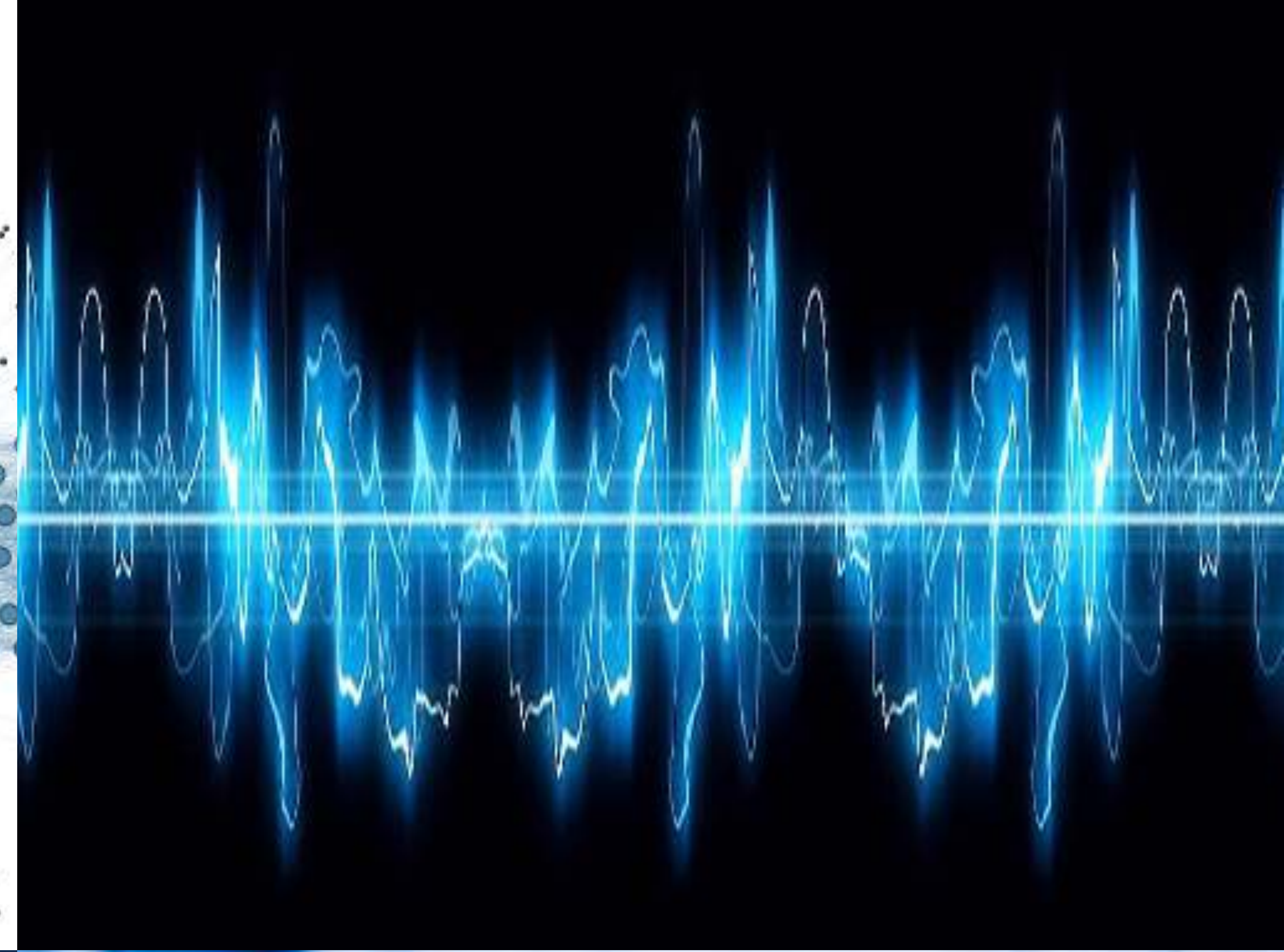
A finite iteration method for calculating the solution of quadratic programming problems is described. Extensions to more general non-linear problems are suggested.

1. INTRODUCTION

The problem of maximizing a concave quadratic function whose variables are subject to linear inequality constraints has been the subject of several recent studies, from both the computational side and the theoretical (see Bibliography). Our aim here has been to develop a method for solving this non-linear programming problem which should be particularly well adapted to high-speed machine computation.

March 1956!






123.74
145
140
135
130
125
Tues.
142.78
Chg

First-order methods

Pros	Cons
Warm starting	Low accuracy solutions
Handle large-scale problems	Can't detect infeasibility
Embeddable	Problem data dependent

General Purpose Solver

Based on first-order methods

Robust

Accurate

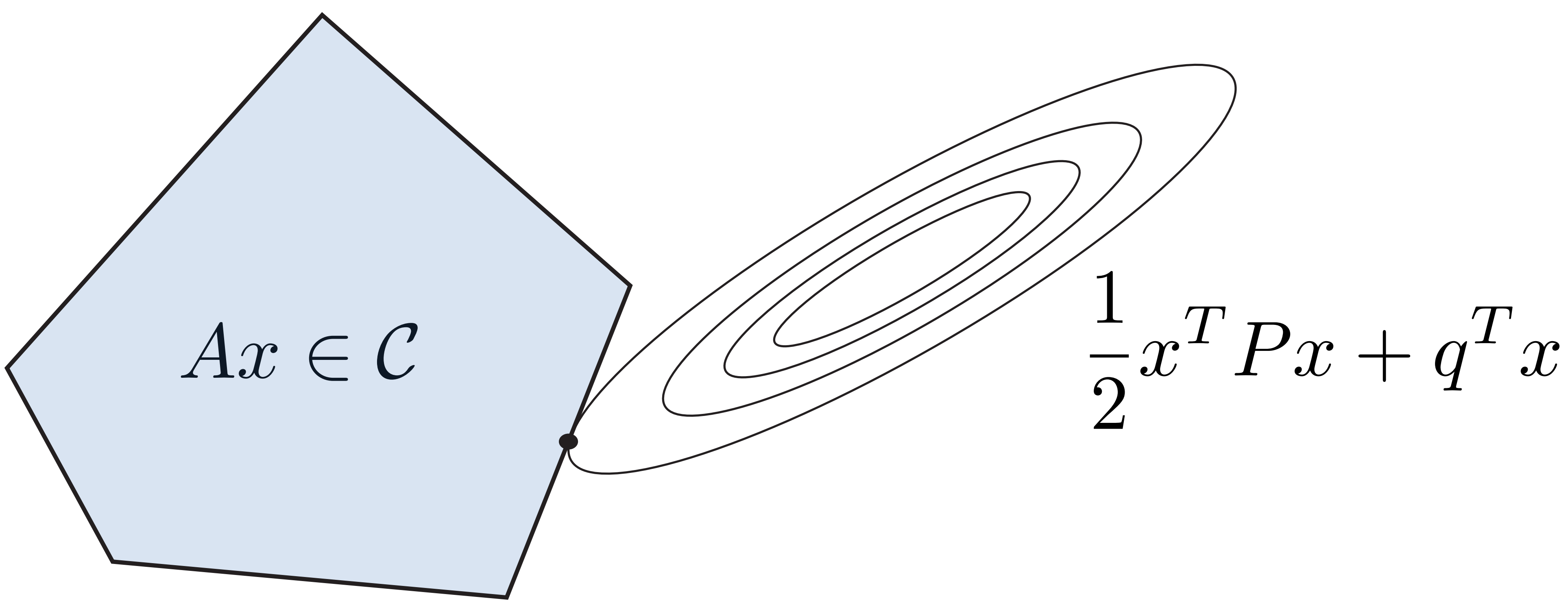
Detects Infeasibility

The OSQP Solver

The problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x^T P x + q^T x \\ \text{subject to} & Ax \in \mathcal{C} \end{array}$$

Quadratic Program $\quad \mathcal{C} = [l, u]$



ADMM

minimize $f(x) + g(x)$ $\longrightarrow$ minimize $f(\tilde{x}) + g(x)$
subject to $\tilde{x} = x$

$$\text{minimize} \quad f(x) + g(x) \quad \longrightarrow \quad \begin{array}{ll} \text{minimize} & f(\tilde{x}) + g(x) \\ \text{subject to} & \tilde{x} = x \end{array}$$

ADMM

minimize $\quad f(x) + g(x)$ $\longrightarrow$ minimize $\quad f(\tilde{x}) + g(x)$
subject to $\quad \tilde{x} = x$

1 $$\tilde{x}^{k+1} \leftarrow \underset{\tilde{x}}{\operatorname{argmin}} \left(f(\tilde{x}) + \frac{\rho}{2} \left\| \tilde{x} - x^k + \frac{y^k}{\rho} \right\|^2 \right)$$

2 $$x^{k+1} \leftarrow \underset{x}{\operatorname{argmin}} \left(g(x) + \frac{\rho}{2} \left\| x - \tilde{x}^{k+1} - \frac{y^k}{\rho} \right\|^2 \right)$$

3 $$y^{k+1} \leftarrow y^k + \rho \left(\tilde{x}^{k+1} - x^{k+1} \right)$$

How to split the QP?

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x^T P x + q^T x \\ \text{subject to} & Ax = z \\ & z \in \mathcal{C} \end{array}$$

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\tilde{x}^T P \tilde{x} + q^T \tilde{x} + \mathcal{I}_{Ax=z}(\tilde{x}, \tilde{z}) + \mathcal{I}_{\mathcal{C}}(z) \\ \text{subject to} & (\tilde{x}, \tilde{z}) = (x, z) \end{array}$$

How to split the QP?

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x^T P x + q^T x \\ \text{subject to} & Ax = z \\ & z \in \mathcal{C} \end{array}$$

f (braces the objective $\frac{1}{2}x^T P x + q^T x$ and the constraint $Ax = z$)

$$\begin{array}{ll} \text{minimize} & \underbrace{\frac{1}{2}\tilde{x}^T P \tilde{x} + q^T \tilde{x} + \mathcal{I}_{Ax=z}(\tilde{x}, \tilde{z})}_{f} + \mathcal{I}_{\mathcal{C}}(z) \\ \text{subject to} & (\tilde{x}, \tilde{z}) = (x, z) \end{array}$$

How to split the QP?

$$\begin{array}{ll} \text{minimize} & \underbrace{\tfrac{1}{2}x^T P x + q^T x}_{f} \\ \text{subject to} & Ax = z \quad \Big\} f \\ & z \in \mathcal{C} \quad \Big\} g \end{array}$$

$$\begin{array}{ll} \text{minimize} & \overbrace{\tfrac{1}{2}\tilde{x}^T P \tilde{x} + q^T \tilde{x} + \mathcal{I}_{Ax=z}(\tilde{x}, \tilde{z})}^{f} + \overbrace{\mathcal{I}_{\mathcal{C}}(z)}^{g} \\ \text{subject to} & (\tilde{x}, \tilde{z}) = (x, z) \end{array}$$

ADMM iterations

1. $$(x^{k+1}, \tilde{z}^{k+1}) \leftarrow \underset{(x,z):Ax=z}{\text{argmin}} \ \frac{1}{2}x^T P x + q^T x + \frac{\sigma}{2}\left\|x - x^k\right\|^2 + \frac{\rho}{2}\left\|z - z^k + \frac{y^k}{\rho}\right\|^2$$

2. $$z^{k+1} \leftarrow \Pi\left(\tilde{z}^{k+1} + \frac{y^k}{\rho}\right)$$

3. $$y^{k+1} \leftarrow y^k + \rho\left(\tilde{z}^{k+1} - z^{k+1}\right)$$

ADMM iterations

Inner QP

1 $$(x^{k+1}, \tilde{z}^{k+1}) \leftarrow \underset{(x,z):Ax=z}{\text{argmin}} \ \frac{1}{2}x^T P x + q^T x + \frac{\sigma}{2}\left\|x - x^k\right\|^2 + \frac{\rho}{2}\left\|z - z^k + \frac{y^k}{\rho}\right\|^2$$

2 $$z^{k+1} \leftarrow \Pi\left(\tilde{z}^{k+1} + \frac{y^k}{\rho}\right)$$

3 $$y^{k+1} \leftarrow y^k + \rho\left(\tilde{z}^{k+1} - z^{k+1}\right)$$

ADMM iterations

Inner QP

1 $$(x^{k+1}, \tilde{z}^{k+1}) \leftarrow \underset{(x,z):Ax=z}{\text{argmin}} \; \frac{1}{2}x^T P x + q^T x + \frac{\sigma}{2}\left\|x - x^k\right\|^2 + \frac{\rho}{2}\left\|z - z^k + \frac{y^k}{\rho}\right\|^2$$

2 $$z^{k+1} \leftarrow \Pi\left(\tilde{z}^{k+1} + \frac{y^k}{\rho}\right)$$

Projection onto $\mathcal{C}$

3 $$y^{k+1} \leftarrow y^k + \rho\left(\tilde{z}^{k+1} - z^{k+1}\right)$$

Solving the inner QP

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x^T P x + q^T x + \frac{\sigma}{2}\left\|x - x^k\right\|^2 + \frac{\rho}{2}\left\|z - z^k + \frac{y^k}{\rho}\right\|^2 \\ \text{subject to} & Ax = z \end{array}$$

Reduced KKT system

$$\begin{bmatrix} P + \sigma I & A^T \\ A & -\frac{1}{\rho} I \end{bmatrix} \begin{bmatrix} x \\ \nu \end{bmatrix} = \begin{bmatrix} \sigma x^k - q \\ z^k - \frac{1}{\rho} y^k \end{bmatrix}$$

$$\tilde{z}^{k+1} = z^k + \frac{1}{\rho}(\nu - y^k)$$

Solving the inner QP

minimize $\frac{1}{2}x^T P x + q^T x + \frac{\sigma}{2}\left\|x - x^k\right\|^2 + \frac{\rho}{2}\left\|z - z^k + \frac{y^k}{\rho}\right\|^2$

subject to $Ax = z$

Reduced KKT system

$$\begin{bmatrix} P + \sigma I & A^T \\ A & -\frac{1}{\rho}I \end{bmatrix} \begin{bmatrix} x \\ \nu \end{bmatrix} = \begin{bmatrix} \sigma x^k - q \\ z^k - \frac{1}{\rho}y^k \end{bmatrix}$$

Always solvable!

$$\tilde{z}^{k+1} = z^k + \frac{1}{\rho}(\nu - y^k)$$

Solving the linear system

Direct Method

$$\begin{bmatrix} P + \sigma I & A^T \\ A & -\frac{1}{\rho} I \end{bmatrix} \begin{bmatrix} x \\ \nu \end{bmatrix} = \begin{bmatrix} \sigma x^k - q \\ z^k - \frac{1}{\rho} y^k \end{bmatrix}$$

Solving the linear system

Direct Method

Quasi-definite matrix

$$\begin{bmatrix} P + \sigma I & A^T \\ A & -\frac{1}{\rho} I \end{bmatrix} \begin{bmatrix} x \\ \nu \end{bmatrix} = \begin{bmatrix} \sigma x^k - q \\ z^k - \frac{1}{\rho} y^k \end{bmatrix}$$

Well defined LDL^T factorization

Solving the linear system

Direct Method

Quasi-definite matrix

$$\begin{bmatrix} P + \sigma I & A^T \\ A & -\frac{1}{\rho} I \end{bmatrix} \begin{bmatrix} x \\ \nu \end{bmatrix} = \begin{bmatrix} \sigma x^k - q \\ z^k - \frac{1}{\rho} y^k \end{bmatrix}$$

Well defined LDL^T factorization

Factorization caching

Solving the linear system

Indirect Method

$$\left(P + \sigma I + \rho A^T A\right) x = \sigma x^k - q + A^T (\rho z^k - y^k)$$

Solving the linear system

Indirect Method

Positive definite matrix

$$\left(P + \sigma I + \rho A^T A\right) x = \sigma x^k - q + A^T(\rho z^k - y^k)$$

Conjugate gradient

Solving the linear system

Indirect Method

Positive definite matrix

$$\left(P + \sigma I + \rho A^T A\right) x = \sigma x^k - q + A^T (\rho z^k - y^k)$$

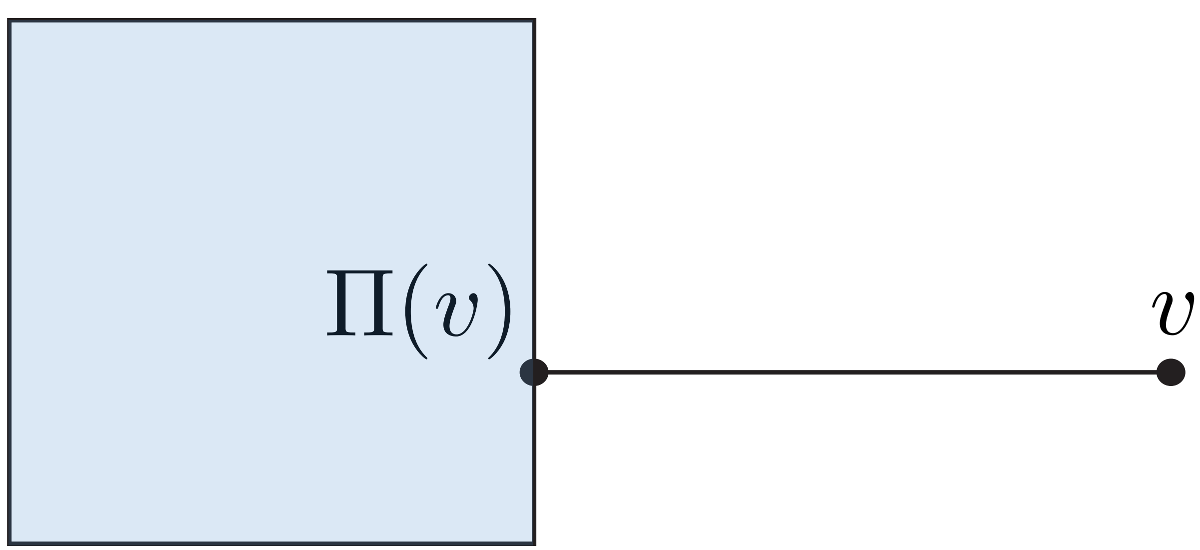
Conjugate gradient

Solve very large systems

Computing the projection

Box projection

$$\Pi(v) = \text{max}(\text{min}(v, u), l)$$



Final algorithm

Problem

$$\begin{array}{ll}\text{minimize} & \frac{1}{2}x^T P x + q^T x \\ \text{subject to} & l \leq Ax \leq u\end{array}$$

Algorithm

1 $$\begin{cases} (x^{k+1}, \nu^{k+1}) \leftarrow \text{solve} \begin{bmatrix} P + \sigma I & A^T \\ A & -\frac{1}{\rho} I \end{bmatrix} \begin{bmatrix} x^{k+1} \\ \nu^{k+1} \end{bmatrix} = \begin{bmatrix} \sigma x^k - q \\ z^k - \frac{1}{\rho} y^k \end{bmatrix} \\ \tilde{z}^{k+1} \leftarrow z^k + \frac{1}{\rho}(\nu^{k+1} - y^k) \end{cases}$$

2 $$\left\{ z^{k+1} \leftarrow \Pi\left(\tilde{z}^{k+1} + \frac{1}{\rho} y^k\right) \right.$$

3 $$\left\{ y^{k+1} \leftarrow y^k + \rho\left(\tilde{z}^{k+1} - z^{k+1}\right) \right.$$

Final algorithm

Problem

$$\begin{array}{ll}\text{minimize} & \frac{1}{2}x^T P x + q^T x \\ \text{subject to} & l \le Ax \le u\end{array}$$

Algorithm

Linear system solve

$$(x^{k+1}, \nu^{k+1}) \leftarrow \text{solve} \begin{bmatrix} P + \sigma I & A^T \\ A & -\frac{1}{\rho} I \end{bmatrix} \begin{bmatrix} x^{k+1} \\ \nu^{k+1} \end{bmatrix} = \begin{bmatrix} \sigma x^k - q \\ z^k - \frac{1}{\rho} y^k \end{bmatrix}$$

$$\tilde{z}^{k+1} \leftarrow z^k + \tfrac{1}{\rho}(\nu^{k+1} - y^k)$$

$$z^{k+1} \leftarrow \Pi\left(\tilde{z}^{k+1} + \tfrac{1}{\rho} y^k\right)$$

$$y^{k+1} \leftarrow y^k + \rho\left(\tilde{z}^{k+1} - z^{k+1}\right)$$

Final algorithm

Problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x^T P x + q^T x \\ \text{subject to} & l \le Ax \le u \end{array}$$

Algorithm

Linear system solve

$$(x^{k+1}, \nu^{k+1}) \leftarrow \text{solve} \begin{bmatrix} P + \sigma I & A^T \\ A & -\frac{1}{\rho} I \end{bmatrix} \begin{bmatrix} x^{k+1} \\ \nu^{k+1} \end{bmatrix} = \begin{bmatrix} \sigma x^k - q \\ z^k - \frac{1}{\rho} y^k \end{bmatrix}$$

Easy operations

$$\tilde{z}^{k+1} \leftarrow z^k + \frac{1}{\rho}(\nu^{k+1} - y^k)$$

$$z^{k+1} \leftarrow \Pi\left(\tilde{z}^{k+1} + \frac{1}{\rho} y^k\right)$$

$$y^{k+1} \leftarrow y^k + \rho\left(\tilde{z}^{k+1} - z^{k+1}\right)$$

OSQP

osqp.org

Library free

Multiple interfaces

Embeddable

Interfaces

Languages

julia

python™

Rust

C++

R

Fortran

MATLAB®

Parsers

JuMP

CVXPY

YALMIP

Users

MIT

Los Alamos
NATIONAL LABORATORY
EST. 1943

Berkeley
UNIVERSITY OF CALIFORNIA

TOYOTA
Let's Go Places

NYU

UC DAVIS
UNIVERSITY OF CALIFORNIA

ETH zürich

Google

BLACKROCK®

iit
ISTITUTO
ITALIANO DI
TECNOLOGIA

LUND
UNIVERSITY

KU LEUVEN

APTIV

Universität Stuttgart

Stanford
University

Numerical Examples

Benchmark Problems

Random

Equality Constrained

Portfolio

Lasso

Huber Fitting

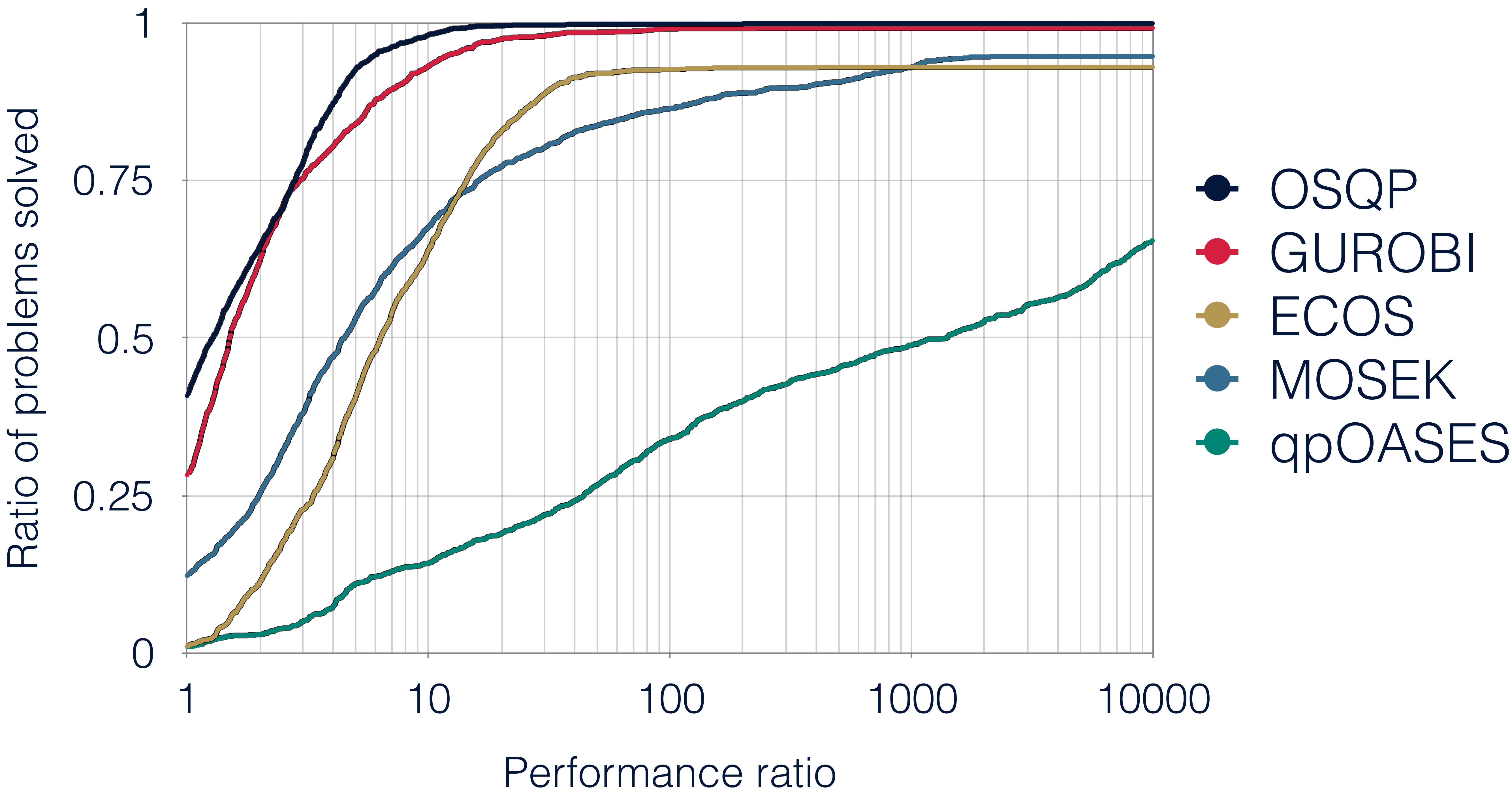
SVM

Control

1400 Problems

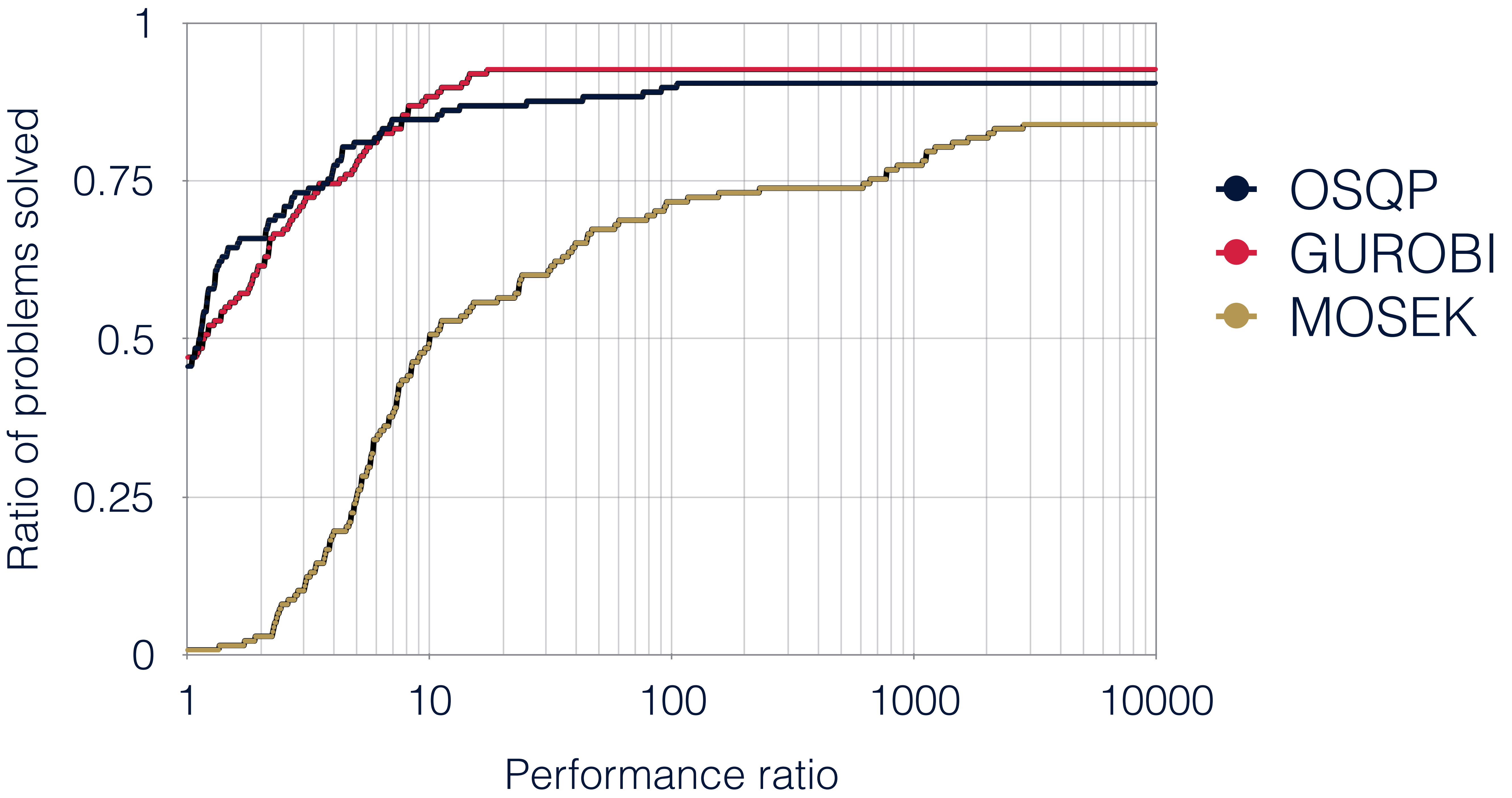
Performance Profiles

github.com/oxfordcontrol/osqp_benchmarks



Maros-Meszaros Benchmarks

`github.com/oxfordcontrol/osqp_benchmarks`



Infeasibility detection

What happens if the problem is infeasible?

$$Ax \notin \mathcal{C}$$

$$y^{k+1} \leftarrow y^k + \rho \left(\tilde{z}^{k+1} - z^{k+1} \right)$$

What happens if the problem is infeasible?

$$Ax \notin \mathcal{C}$$

$$y^{k+1} \leftarrow y^k + \rho \, \underbrace{\big(\underset{\underset{Ax}{\Cup}}{\tilde{z}^{k+1}} - \underset{\underset{\mathcal{C}}{\Cup}}{z^{k+1}}\big)}_{\neq 0}$$

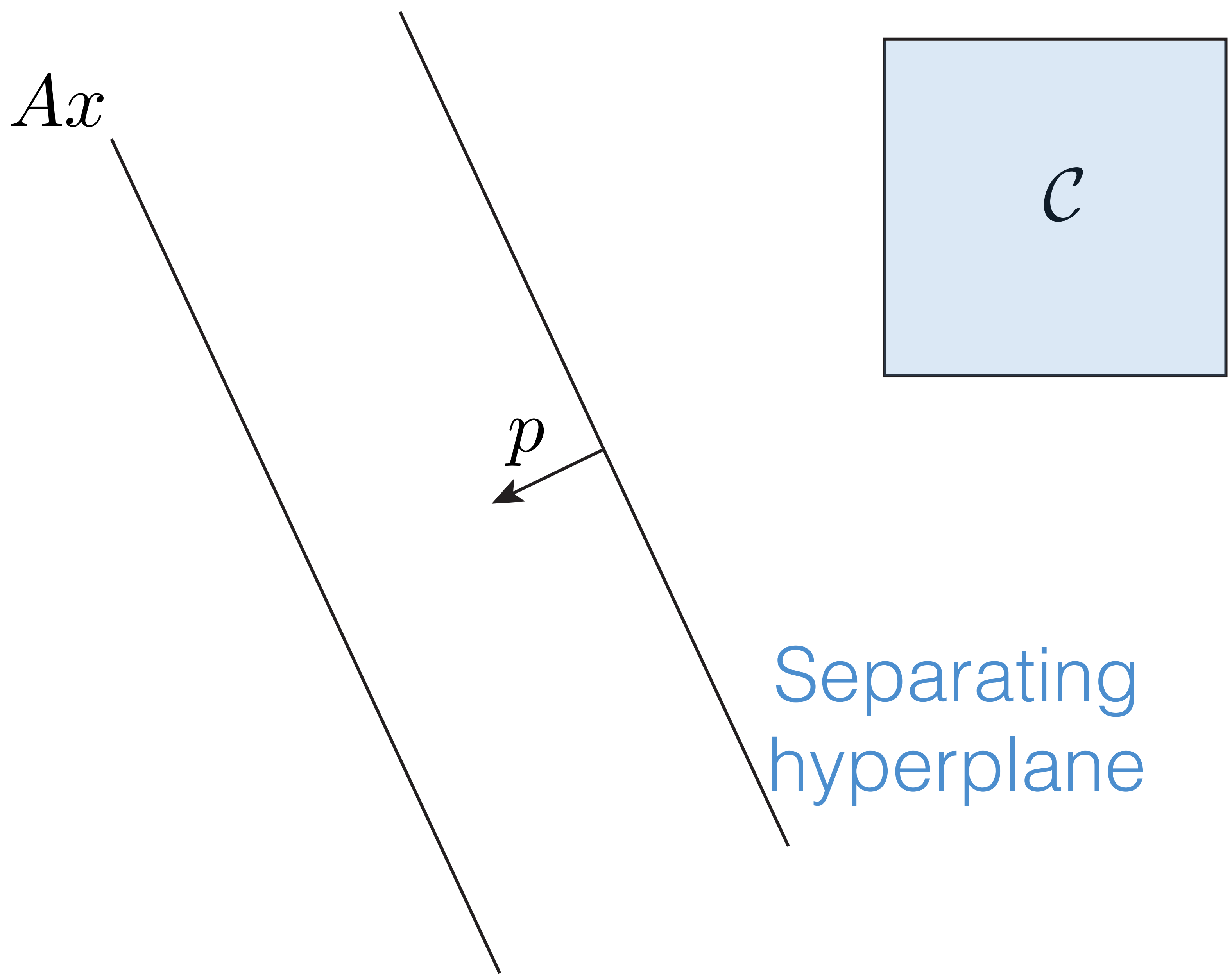
$\downarrow$

y^k does not converge!

Farkas' Lemma

Primal infeasibility

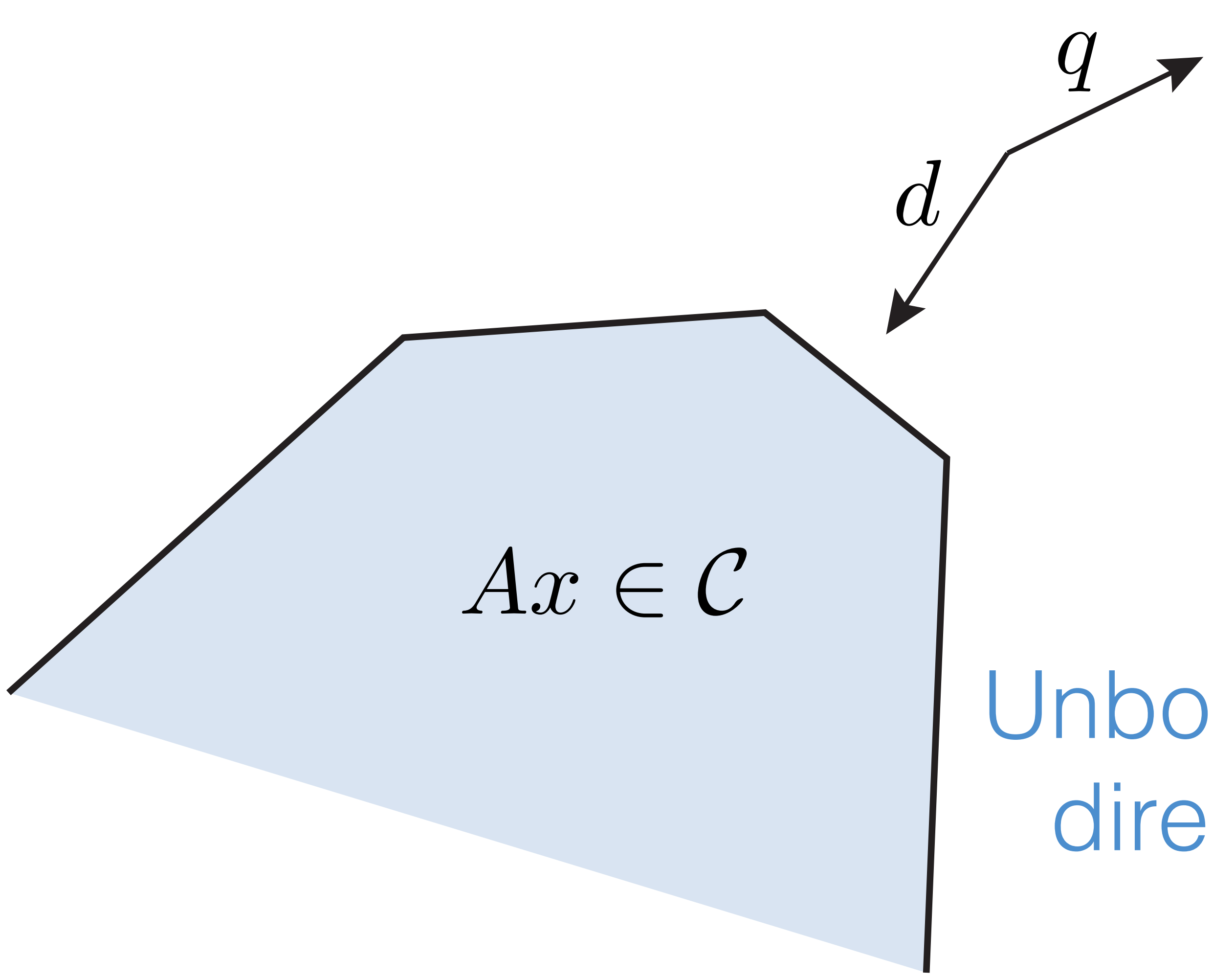
$$A^T p = 0 \qquad u^T p_+ + l^T p_- < 0$$



Farkas' Lemma

Dual infeasibility

$$Pd = 0 \qquad q^T d < 0 \qquad (Ad)_i \begin{cases} = 0 & l_i, u_i \neq \infty \\ \geq 0 & u_i = +\infty \\ \leq 0 & l_i = -\infty \end{cases}$$



Unbounded direction

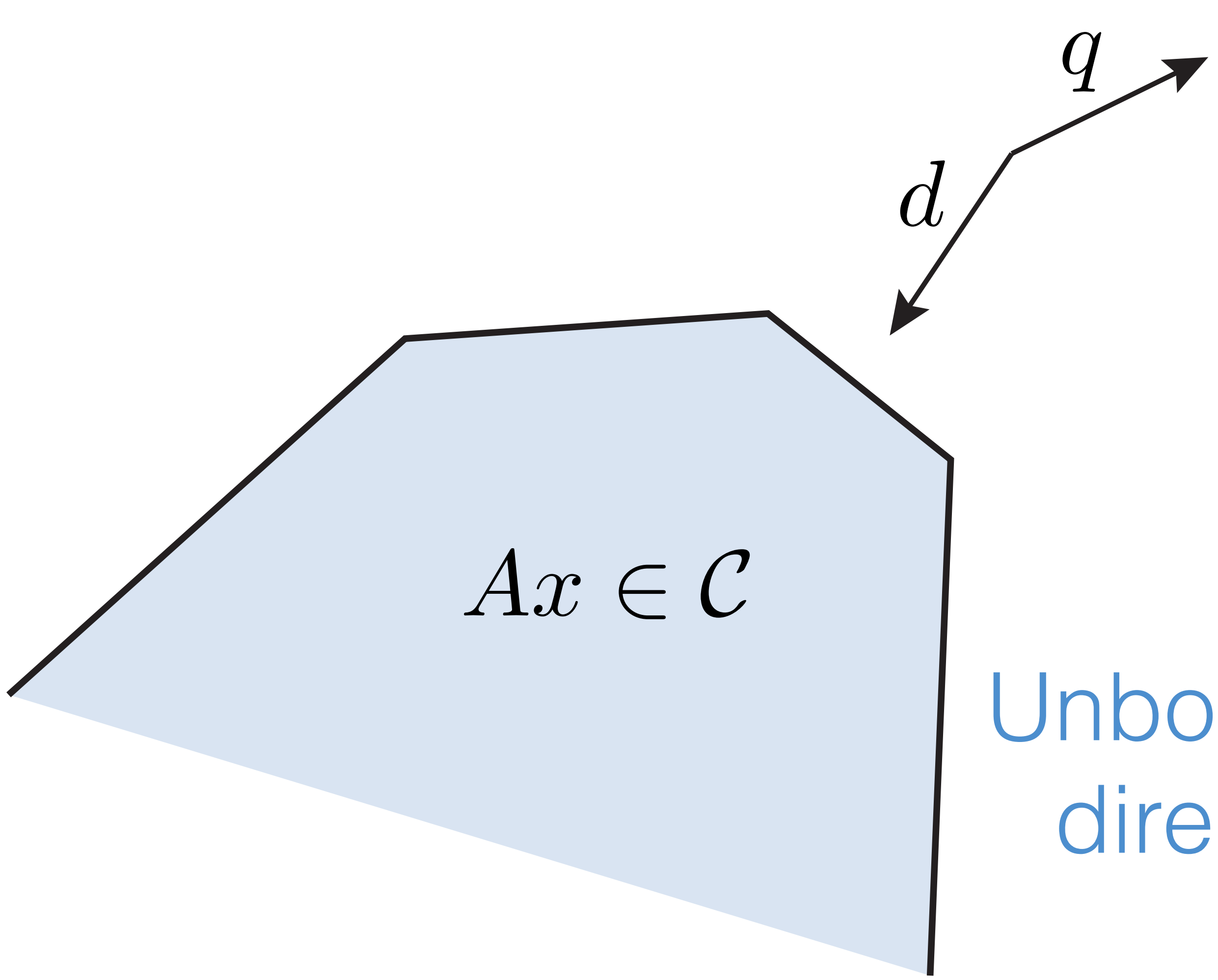
Farkas' Lemma

Dual infeasibility

$$Pd = 0 \qquad q^T d < 0 \qquad (Ad)_i \begin{cases} = 0 & l_i, u_i \neq \infty \\ \geq 0 & u_i = +\infty \\ \leq 0 & l_i = -\infty \end{cases}$$

$$Ad \in \mathcal{C}^\infty$$

Recession cone



Unbounded direction

Infeasibility detection

Primal infeasibility: $y^{k+1} - y^k \rightarrow \delta y \; (\neq 0)$

$$A^T \delta y = 0 \qquad u^T \delta y_+ + l^T \delta y_- < 0$$

Dual infeasibility: $x^{k+1} - x^k \rightarrow \delta x \; (\neq 0)$

$$P \delta x = 0 \qquad q^T \delta x < 0 \qquad (A \delta x)_i \begin{cases} = 0 & l_i, u_i \neq \infty \\ \geq 0 & u_i = +\infty \\ \leq 0 & l_i = -\infty \end{cases}$$

Conclusions

Acknowledgements

Goran Banjac

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Paul Goulart

Alberto Bemporad

Stephen Boyd

OSQP

Remarks

Simple

Robust

Embeddable

Warm-starting

Detects infeasibility

Future work

Algorithms

SDP

SQP

Mixed-Integer

Architecture

Acceleration

Linear System Solvers

References

`osqp.org`

B. Stellato, G. Banjac, P. Goulart, A. Bemporad and S. Boyd. ***OSQP: An Operator Splitting Solver for Quadratic Programs.*** *arXiv:1711.08013, 2018*

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G. Banjac, B.Stellato, N. Moehle, P. Goulart, A. Bemporad and S. Boyd. ***Embedded code generation using the OSQP solver.*** *IEEE Conference on Decision and Control, 2017*

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