OSQP

An Operator Splitting Solver for Quadratic Programs

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joint work with Goran Banjac,

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MIT Operations Research Center, 19 Jun 2017

Why quadratic programming?

AN ALGORITHM FOR QUADRATIC PROGRAMMING

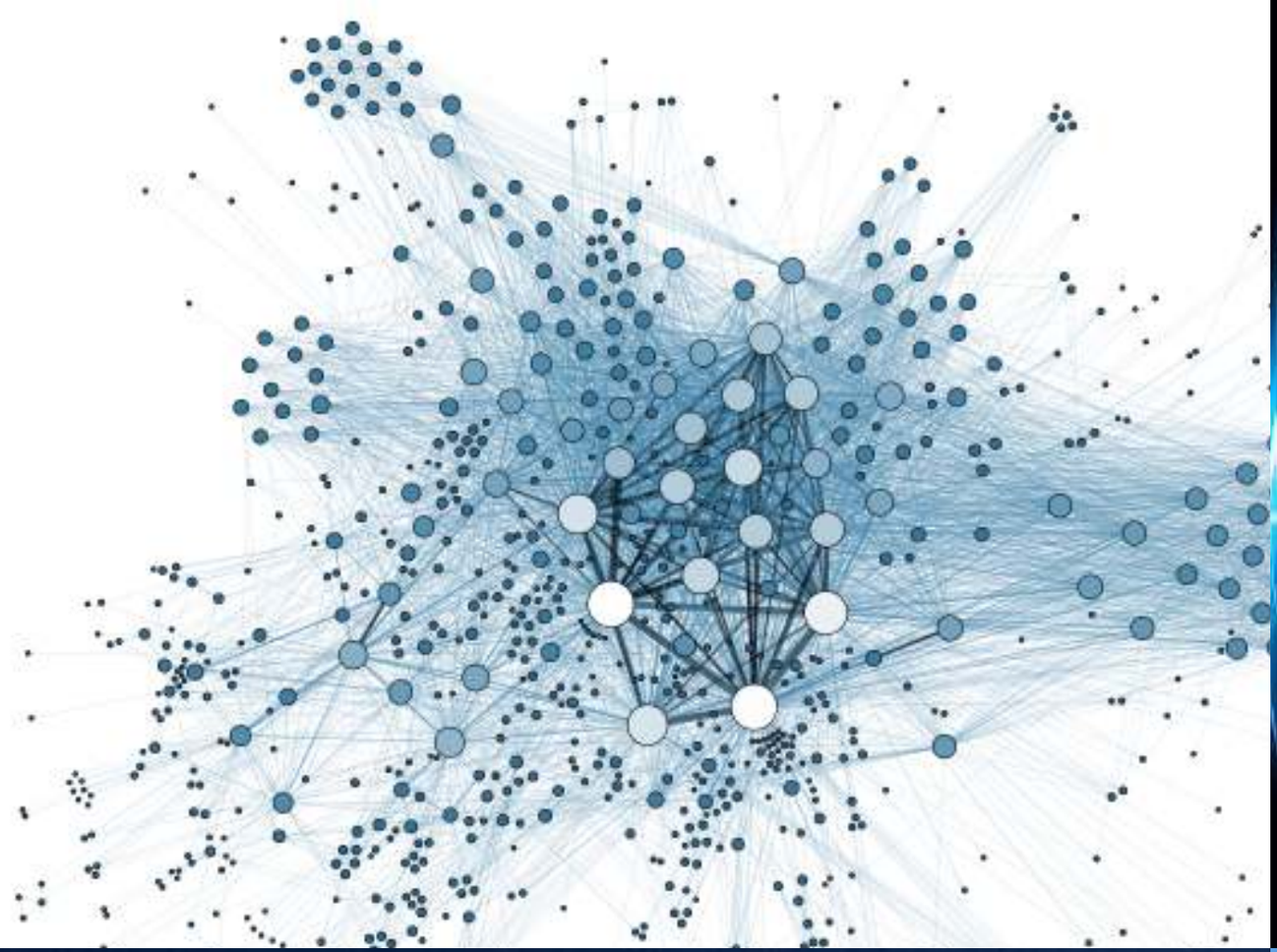
Marguerite Frank and Philip Wolfe[1]
Princeton University

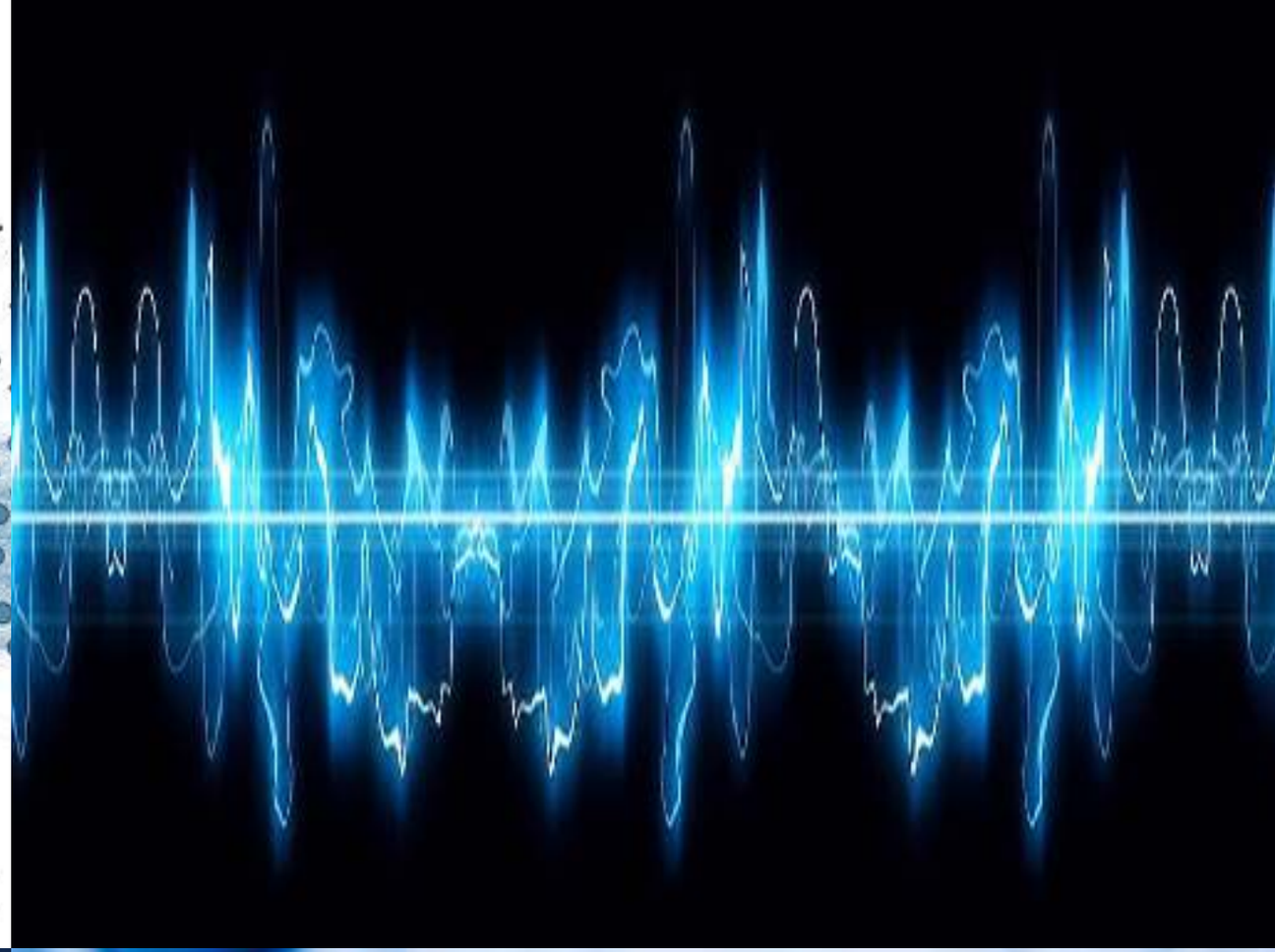
A finite iteration method for calculating the solution of quadratic programming problems is described. Extensions to more general non-linear problems are suggested.

1. INTRODUCTION

The problem of maximizing a concave quadratic function whose variables are subject to linear inequality constraints has been the subject of several recent studies, from both the computational side and the theoretical (see Bibliography). Our aim here has been to develop a method for solving this non-linear programming problem which should be particularly well adapted to high-speed machine computation.

March 1956!






145
140
135
130
125
123.74

First-order methods

Pros

Warm starting

Handle large-scale problems

Embeddable

Cons

Low accuracy solutions

Can’t detect infeasibility

Problem data dependent

General Purpose Solver

Based on first-order methods

Robust

Accurate

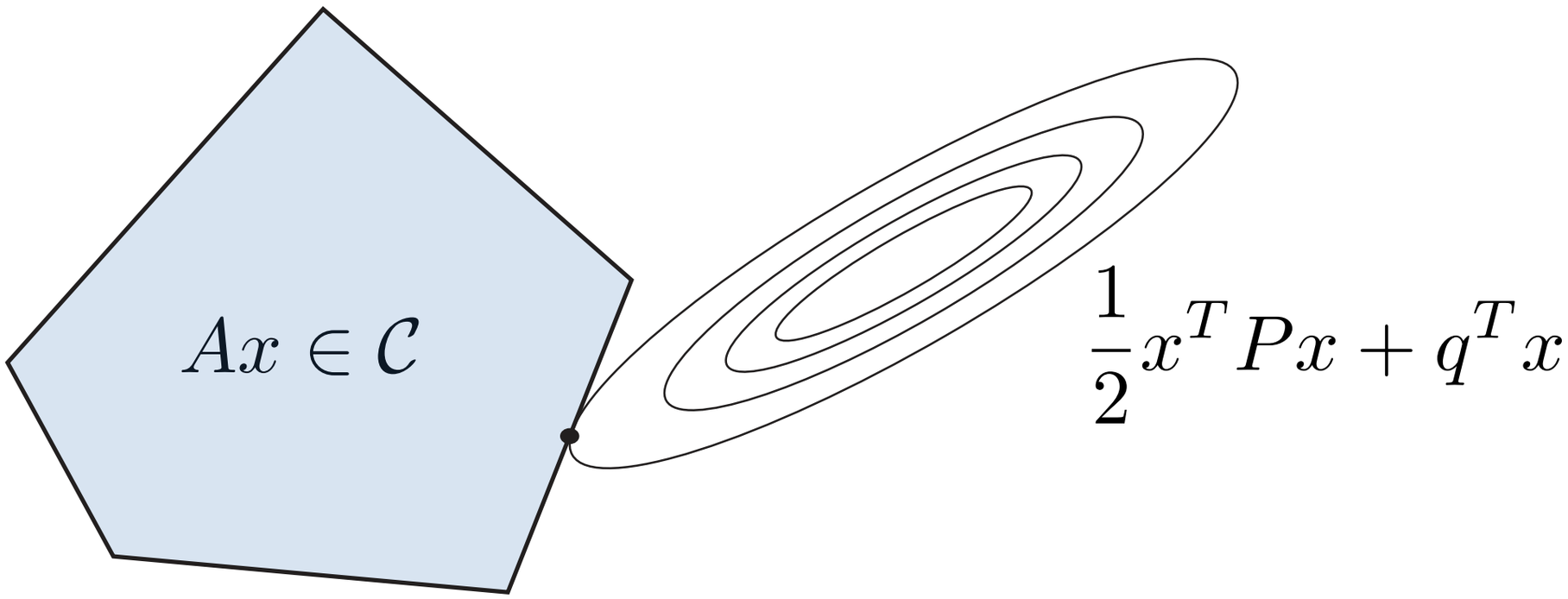
Detects Infeasibility

The OSQP Solver

The problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x^T P x + q^T x \\ \text{subject to} & Ax \in \mathcal{C} \end{array}$$

Quadratic Program $\quad \mathcal{C} = [l, u]$



ADMM

minimize $f(x) + g(x)$ ⟶ minimize $f(\tilde{x}) + g(x)$
subject to $\tilde{x} = x$

ADMM

$$\text{minimize} \quad f(x) + g(x) \longrightarrow \begin{array}{ll} \text{minimize} & f(\tilde{x}) + g(x) \\ \text{subject to} & \tilde{x} = x \end{array}$$

1 $$\tilde{x}^{k+1} \leftarrow \underset{\tilde{x}}{\text{argmin}} \left(f(\tilde{x}) + \frac{\rho}{2} \left\| \tilde{x} - x^k + \frac{y^k}{\rho} \right\|^2 \right)$$

2 $$x^{k+1} \leftarrow \underset{x}{\text{argmin}} \left(g(x) + \frac{\rho}{2} \left\| x - \tilde{x}^{k+1} - \frac{y^k}{\rho} \right\|^2 \right)$$

3 $$y^{k+1} \leftarrow y^k + \rho \left(\tilde{x}^{k+1} - x^{k+1} \right)$$

How to split the QP?

$$\begin{aligned} \text{minimize} \quad & \tfrac{1}{2} x^T P x + q^T x \\ \text{subject to} \quad & Ax = z \\ & z \in \mathcal{C} \end{aligned}$$

$$\begin{aligned} \text{minimize} \quad & \tfrac{1}{2} \tilde{x}^T P \tilde{x} + q^T \tilde{x} + \mathcal{I}_{Ax=z}(\tilde{x}, \tilde{z}) + \mathcal{I}_{\mathcal{C}}(z) \\ \text{subject to} \quad & (\tilde{x}, \tilde{z}) = (x, z) \end{aligned}$$

How to split the QP?

$$\begin{array}{ll} \text{minimize} & \left.\begin{array}{l} \frac{1}{2}x^T P x + q^T x \\ Ax = z \end{array}\right\} f \\ \text{subject to} & z \in \mathcal{C} \end{array}$$

$$\begin{array}{ll} & \overbrace{\phantom{\frac{1}{2}\tilde{x}^T P\tilde{x} + q^T\tilde{x} + \mathcal{I}_{Ax=z}(\tilde{x},\tilde{z})}}^{f} \\ \text{minimize} & \frac{1}{2}\tilde{x}^T P\tilde{x} + q^T\tilde{x} + \mathcal{I}_{Ax=z}(\tilde{x},\tilde{z}) + \mathcal{I}_{\mathcal{C}}(z) \\ \text{subject to} & (\tilde{x},\tilde{z}) = (x,z) \end{array}$$

How to split the QP?

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x^T P x + q^T x \\ \text{subject to} & Ax = z \\ & z \in \mathcal{C} \end{array} \quad \left.\begin{array}{l} \\ \\ \end{array}\right\} f \quad \left.\begin{array}{l} \\ \\ \end{array}\right\} g$$

$$\begin{array}{ll} \text{minimize} & \overbrace{\frac{1}{2}\tilde{x}^T P \tilde{x} + q^T \tilde{x} + \mathcal{I}_{Ax=z}(\tilde{x}, \tilde{z})}^{f} + \overbrace{\mathcal{I}_{\mathcal{C}}(z)}^{g} \\ \text{subject to} & (\tilde{x}, \tilde{z}) = (x, z) \end{array}$$

ADMM iterations

1 $$(x^{k+1}, \tilde{z}^{k+1}) \leftarrow \underset{(x,z):Ax=z}{\text{argmin}} \frac{1}{2}x^T Px + q^T x + \frac{\sigma}{2}\left\|x - x^k\right\|^2 + \frac{\rho}{2}\left\|z - z^k + \frac{y^k}{\rho}\right\|^2$$

2 $$z^{k+1} \leftarrow \Pi\left(\tilde{z}^{k+1} + \frac{y^k}{\rho}\right)$$

3 $$y^{k+1} \leftarrow y^k + \rho\left(\tilde{z}^{k+1} - z^{k+1}\right)$$

ADMM iterations

Inner QP

1 $$(x^{k+1}, \tilde{z}^{k+1}) \leftarrow \underset{(x,z):Ax=z}{\text{argmin}} \frac{1}{2}x^T Px + q^T x + \frac{\sigma}{2}\left\|x - x^k\right\|^2 + \frac{\rho}{2}\left\|z - z^k + \frac{y^k}{\rho}\right\|^2$$

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ADMM iterations

Inner QP

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2 $$z^{k+1} \leftarrow \Pi\left(\tilde{z}^{k+1} + \frac{y^k}{\rho}\right)$$

Projection onto $\mathcal{C}$

3 $$y^{k+1} \leftarrow y^k + \rho\left(\tilde{z}^{k+1} - z^{k+1}\right)$$

Solving the inner QP

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x^T P x + q^T x + \frac{\sigma}{2}\left\|x - x^k\right\|^2 + \frac{\rho}{2}\left\|z - z^k + \frac{y^k}{\rho}\right\|^2 \\ \text{subject to} & Ax = z \end{array}$$

Reduced KKT system

$$\begin{bmatrix} P + \sigma I & A^T \\ A & -\frac{1}{\rho}I \end{bmatrix} \begin{bmatrix} x \\ \nu \end{bmatrix} = \begin{bmatrix} \sigma x^k - q \\ z^k - \frac{1}{\rho}y^k \end{bmatrix}$$

$$\tilde{z}^{k+1} = z^k + \frac{1}{\rho}(\nu - y^k)$$

Solving the inner QP

$$\begin{aligned} \text{minimize} \quad & \tfrac{1}{2}x^T P x + q^T x + \frac{\sigma}{2}\left\|x - x^k\right\|^2 + \frac{\rho}{2}\left\|z - z^k + \frac{y^k}{\rho}\right\|^2 \\ \text{subject to} \quad & Ax = z \end{aligned}$$

Reduced KKT system

$$\begin{bmatrix} P + \sigma I & A^T \\ A & -\frac{1}{\rho}I \end{bmatrix} \begin{bmatrix} x \\ \nu \end{bmatrix} = \begin{bmatrix} \sigma x^k - q \\ z^k - \frac{1}{\rho}y^k \end{bmatrix}$$

Always solvable!

$$\tilde{z}^{k+1} = z^k + \tfrac{1}{\rho}(\nu - y^k)$$

Solving the linear system

Direct Method

$$\begin{bmatrix} P + \sigma I & A^T \\ A & -\frac{1}{\rho} I \end{bmatrix} \begin{bmatrix} x \\ \nu \end{bmatrix} = \begin{bmatrix} \sigma x^k - q \\ z^k - \frac{1}{\rho} y^k \end{bmatrix}$$

Solving the linear system

Direct Method

Quasi-definite matrix

$$\begin{bmatrix} P+\sigma I & A^T \\ A & -\frac{1}{\rho}I \end{bmatrix} \begin{bmatrix} x \\ \nu \end{bmatrix} = \begin{bmatrix} \sigma x^k - q \\ z^k - \frac{1}{\rho} y^k \end{bmatrix}$$

Well defined LDL^T factorization

Solving the linear system

Direct Method

Quasi-definite matrix

$$\begin{bmatrix} P+\sigma I & A^T \\ A & -\frac{1}{\rho}I \end{bmatrix} \begin{bmatrix} x \\ \nu \end{bmatrix} = \begin{bmatrix} \sigma x^k - q \\ z^k - \frac{1}{\rho} y^k \end{bmatrix}$$

Well defined LDL^T factorization

Factorization caching

Solving the linear system

Indirect Method

$$\left(P + \sigma I + \rho A^T A\right) x = \sigma x^k - q + A^T(\rho z^k - y^k)$$

Solving the linear system

Indirect Method

Positive definite matrix

$$\left(P + \sigma I + \rho A^T A\right) x = \sigma x^k - q + A^T(\rho z^k - y^k)$$

Conjugate gradient

Solving the linear system

Indirect Method

Positive definite matrix

$$\left(P + \sigma I + \rho A^T A\right) x = \sigma x^k - q + A^T(\rho z^k - y^k)$$

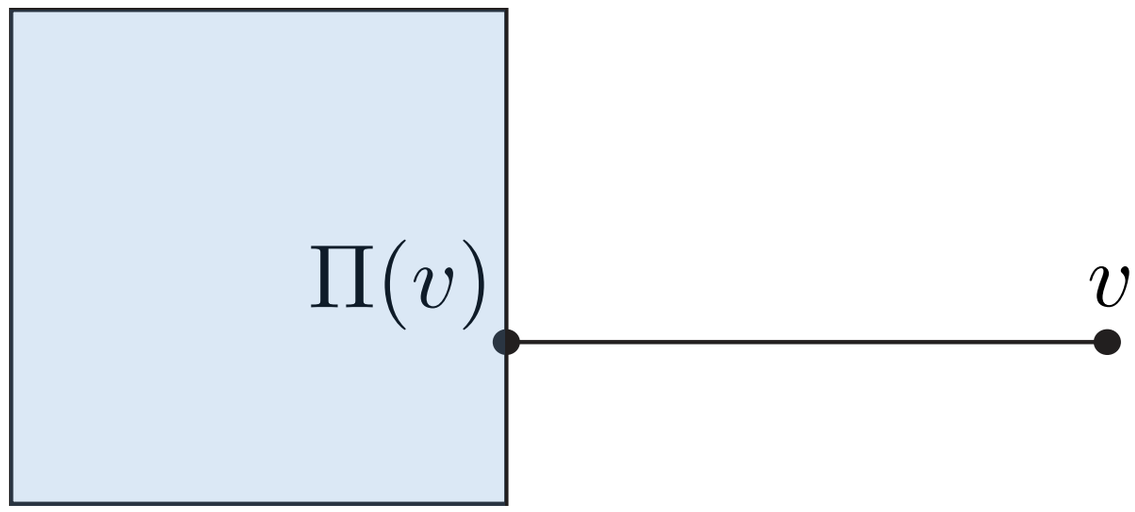
Conjugate gradient

Solve very large systems

Computing the projection

Box projection

$$\Pi(v) = \mathsf{max}(\mathsf{min}(v, u), l)$$



Final algorithm

Problem

$$\begin{aligned} \text{minimize} \quad & \tfrac{1}{2} x^T P x + q^T x \\ \text{subject to} \quad & l \leq Ax \leq u \end{aligned}$$

Algorithm

1

$$\begin{cases} (x^{k+1}, \nu^{k+1}) \leftarrow \text{solve} \begin{bmatrix} P + \sigma I & A^T \\ A & -\frac{1}{\rho} I \end{bmatrix} \begin{bmatrix} x^{k+1} \\ \nu^{k+1} \end{bmatrix} = \begin{bmatrix} \sigma x^k - q \\ z^k - \frac{1}{\rho} y^k \end{bmatrix} \\ \tilde{z}^{k+1} \leftarrow z^k + \frac{1}{\rho}(\nu^{k+1} - y^k) \end{cases}$$

2

$$\begin{cases} z^{k+1} \leftarrow \Pi\left(\tilde{z}^{k+1} + \frac{1}{\rho} y^k\right) \end{cases}$$

3

$$\begin{cases} y^{k+1} \leftarrow y^k + \rho\left(\tilde{z}^{k+1} - z^{k+1}\right) \end{cases}$$

Final algorithm

Problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x^T P x + q^T x \\ \text{subject to} & l \leq Ax \leq u \end{array}$$

Algorithm

Linear system solve

$$(x^{k+1}, \nu^{k+1}) \leftarrow \text{solve} \begin{bmatrix} P + \sigma I & A^T \\ A & -\frac{1}{\rho} I \end{bmatrix} \begin{bmatrix} x^{k+1} \\ \nu^{k+1} \end{bmatrix} = \begin{bmatrix} \sigma x^k - q \\ z^k - \frac{1}{\rho} y^k \end{bmatrix}$$

$$\tilde{z}^{k+1} \leftarrow z^k + \frac{1}{\rho}(\nu^{k+1} - y^k)$$

$$z^{k+1} \leftarrow \Pi\left(\tilde{z}^{k+1} + \frac{1}{\rho} y^k\right)$$

$$y^{k+1} \leftarrow y^k + \rho\left(\tilde{z}^{k+1} - z^{k+1}\right)$$

Final algorithm

Problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x^T P x + q^T x \\ \text{subject to} & l \leq Ax \leq u \end{array}$$

Algorithm

Linear system solve

$$(x^{k+1}, \nu^{k+1}) \leftarrow \text{solve} \begin{bmatrix} P + \sigma I & A^T \\ A & -\frac{1}{\rho} I \end{bmatrix} \begin{bmatrix} x^{k+1} \\ \nu^{k+1} \end{bmatrix} = \begin{bmatrix} \sigma x^k - q \\ z^k - \frac{1}{\rho} y^k \end{bmatrix}$$

Easy operations

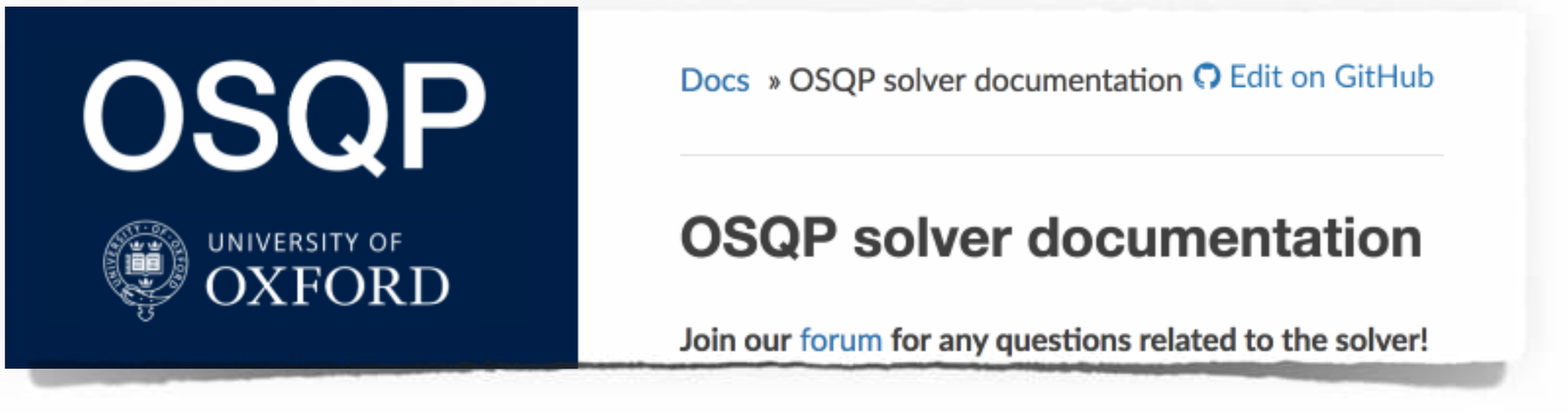
$$\tilde{z}^{k+1} \leftarrow z^k + \frac{1}{\rho}(\nu^{k+1} - y^k)$$

$$z^{k+1} \leftarrow \Pi\left(\tilde{z}^{k+1} + \frac{1}{\rho} y^k\right)$$

$$y^{k+1} \leftarrow y^k + \rho\left(\tilde{z}^{k+1} - z^{k+1}\right)$$

OSQP

`osqp.readthedocs.io`



Library free

Multiple interfaces

Embeddable

Interfaces

Languages

Parsers

JuMP

CVXPY

YALMIP

OSQP interface


```python
# Create OSQP object
m = osqp.OSQP()

# Initialize solver
m.setup(P, q, A, l, u,
        settings)

# Solve
results = m.solve()

# Update cost with q_new
m.update(q=q_new)

# Solve again
results_new = m.solve()
```


```matlab
% Create OSQP object
m = osqp();

% Initialize solver
m.setup(P, q, A, l, u,
        settings);

% Solve
results = m.solve();

% Update cost with q_new
m.update('q', q_new);

% Solve again
results_new = m.solve();
```

Numerical Example

Lasso

$$\text{minimize} \quad \|Ax - b\|_2^2 + \lambda \|x\|_1$$

Features
n

Data points
$m = 100n$

Lasso

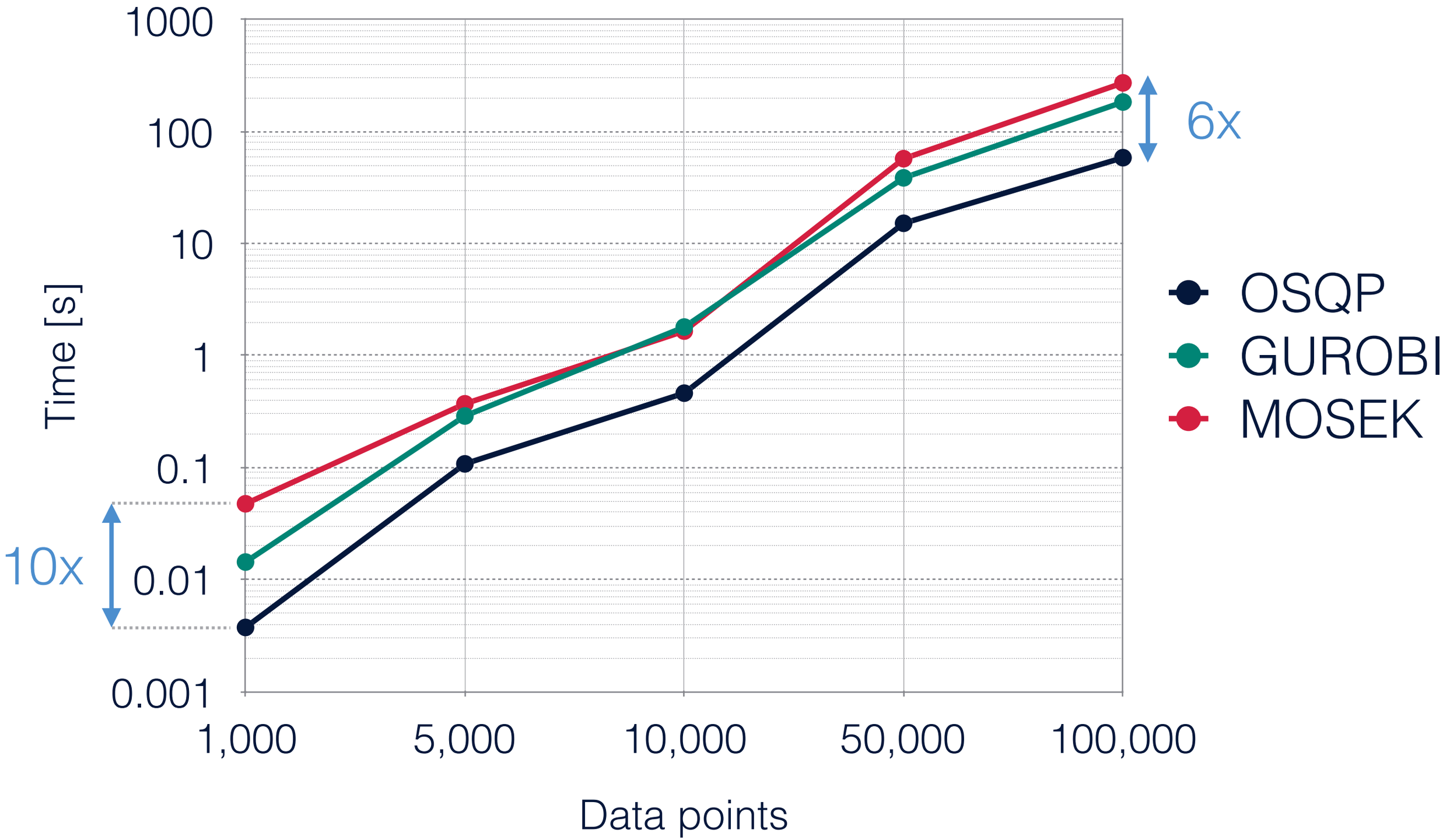
Weighting parameter

$$\text{minimize} \quad \|Ax - b\|_2^2 + \lambda \|x\|_1$$

Features
n

Data points
$m = 100n$

Lasso timings



Solution Polishing

Obtaining the active set

Lower active

$$\mathcal{L} = \{i \mid (Ax)_i = l_i \wedge y_i < 0\}$$

Upper active

$$\mathcal{U} = \{i \mid (Ax)_i = u_i \wedge y_i > 0\}$$

Solving single linear system

$$\begin{bmatrix} P & A_{\mathcal{L}}^T & A_{\mathcal{U}}^T \\ A_{\mathcal{L}} & & \\ A_{\mathcal{U}} & & \end{bmatrix} \begin{bmatrix} x \\ y_{\mathcal{L}} \\ y_{\mathcal{U}} \end{bmatrix} = \begin{bmatrix} -q \\ l_{\mathcal{L}} \\ u_{\mathcal{U}} \end{bmatrix}$$

Inactive constraints

$$y_i = 0 \quad i \notin (\mathcal{L} \cup \mathcal{U})$$

Solving single linear system

$$\underbrace{\begin{bmatrix} P & A_{\mathcal{L}}^T & A_{\mathcal{U}}^T \\ A_{\mathcal{L}} & & \\ A_{\mathcal{U}} & & \end{bmatrix}}_{M} \begin{bmatrix} x \\ y_{\mathcal{L}} \\ y_{\mathcal{U}} \end{bmatrix} = \underbrace{\begin{bmatrix} -q \\ l_{\mathcal{L}} \\ u_{\mathcal{U}} \end{bmatrix}}_{g}$$

Inactive constraints

$$y_i = 0 \quad i \notin (\mathcal{L} \cup \mathcal{U})$$

Iterative refinement

Perturbed system

$$\begin{bmatrix} P+\delta I & A_{\mathcal{L}}^T & A_{\mathcal{U}}^T \\ A_{\mathcal{L}} & -\delta I & \\ A_{\mathcal{U}} & & -\delta I \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y}_{\mathcal{L}} \\ \hat{y}_{\mathcal{U}} \end{bmatrix} = \begin{bmatrix} -q \\ l_{\mathcal{L}} \\ u_{\mathcal{U}} \end{bmatrix}$$

Iterative refinement

Perturbed system

Quasi-definite

$$\begin{bmatrix} P+\delta I & A_{\mathcal{L}}^T & A_{\mathcal{U}}^T \\ A_{\mathcal{L}} & -\delta I & \\ A_{\mathcal{U}} & & -\delta I \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y}_{\mathcal{L}} \\ \hat{y}_{\mathcal{U}} \end{bmatrix} = \begin{bmatrix} -q \\ l_{\mathcal{L}} \\ u_{\mathcal{U}} \end{bmatrix}$$

Iterative refinement

Perturbed system

Quasi-definite

$$\underbrace{\begin{bmatrix} P+\delta I & A_{\mathcal{L}}^T & A_{\mathcal{U}}^T \\ A_{\mathcal{L}} & -\delta I & \\ A_{\mathcal{U}} & & -\delta I \end{bmatrix}}_{M+\Delta} \begin{bmatrix} \hat{x} \\ \hat{y}_{\mathcal{L}} \\ \hat{y}_{\mathcal{U}} \end{bmatrix} = \underbrace{\begin{bmatrix} -q \\ l_{\mathcal{L}} \\ u_{\mathcal{U}} \end{bmatrix}}_{g}$$

Iterative refinement

Perturbed system

Quasi-definite

$$\begin{bmatrix} P+\delta I & A_{\mathcal{L}}^T & A_{\mathcal{U}}^T \\ A_{\mathcal{L}} & -\delta I & \\ A_{\mathcal{U}} & & -\delta I \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y}_{\mathcal{L}} \\ \hat{y}_{\mathcal{U}} \end{bmatrix} = \begin{bmatrix} -q \\ l_{\mathcal{L}} \\ u_{\mathcal{U}} \end{bmatrix}$$

$M + \Delta$ $\qquad$ g

Iterations

$$\delta t^k \leftarrow \text{solve } (M + \Delta)\delta t^k = g - Kt^k$$

$$t^{k+1} \leftarrow t^k + \delta t^k$$

Infeasibility detection

What happens if the problem is infeasible?

$$y^{k+1} \leftarrow y^k + \rho \underbrace{\left(\tilde{z}^{k+1} - z^{k+1}\right)}_{\neq 0}$$

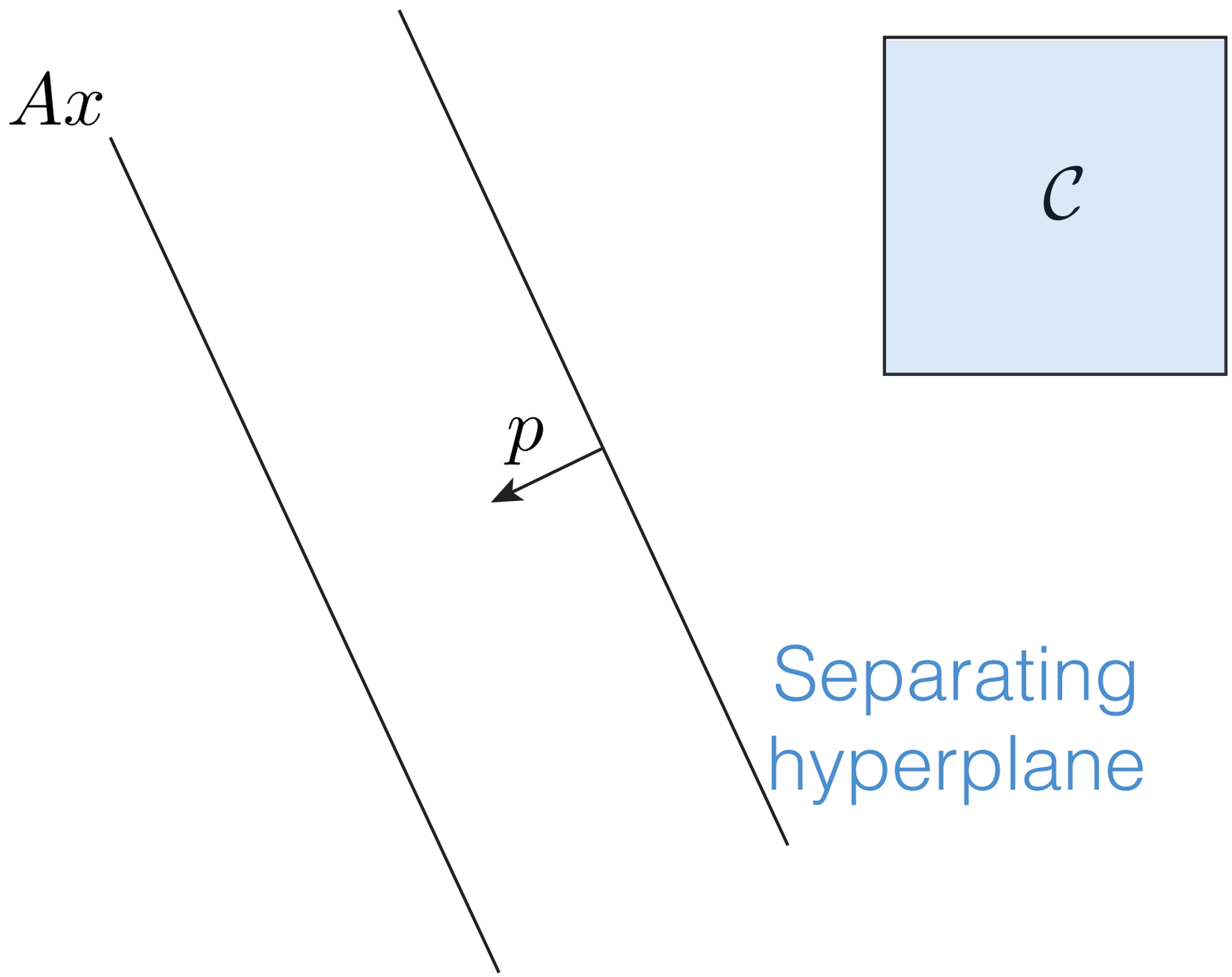
$\downarrow$

y^k diverge!

Farkas' Lemma

Primal infeasibility

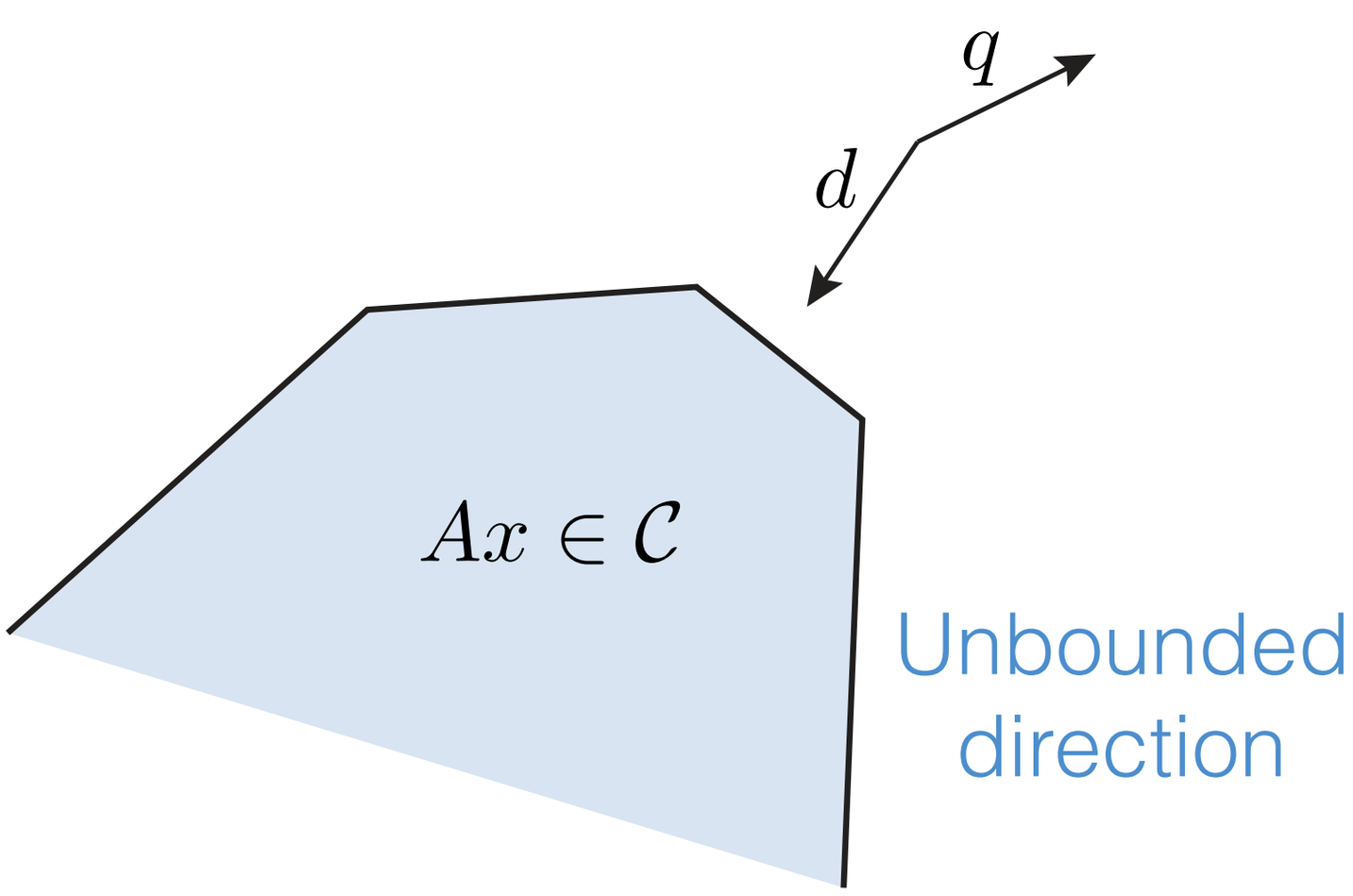
$$A^T p = 0 \qquad u^T p_+ + l^T p_- < 0$$



Farkas' Lemma

Dual infeasibility

$$Pd = 0 \qquad q^T d < 0 \qquad (Ad)_i \begin{cases} = 0 & l_i, u_i \neq \infty \\ \geq 0 & u_i = +\infty \\ \leq 0 & l_i = -\infty \end{cases}$$

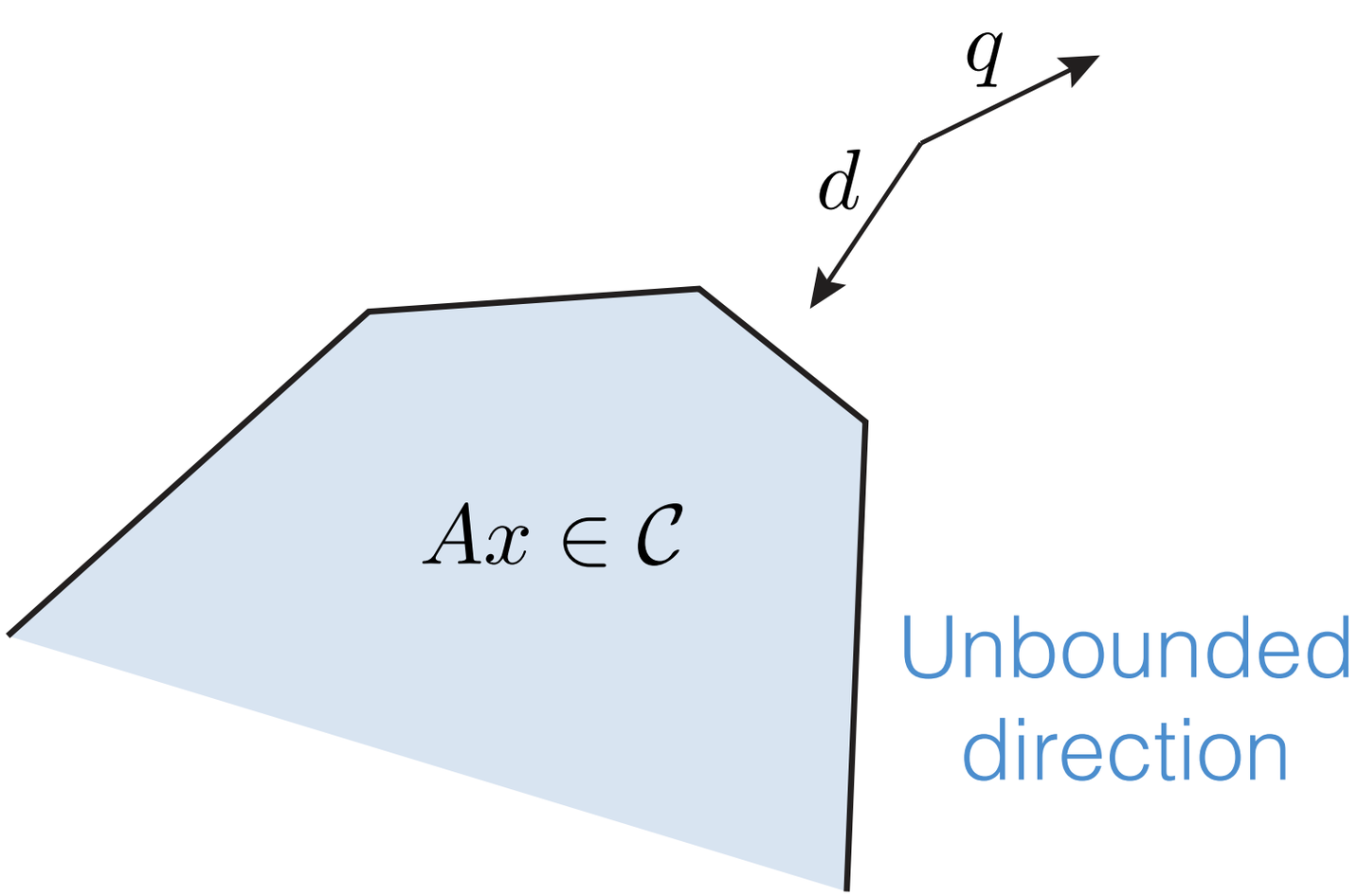


Unbounded direction

Farkas' Lemma

Dual infeasibility

$$Pd = 0 \qquad q^T d < 0 \qquad (Ad)_i \begin{cases} = 0 & l_i, u_i \neq \infty \\ \geq 0 & u_i = +\infty \\ \leq 0 & l_i = -\infty \end{cases}$$



Unbounded direction

$Ad \in \mathcal{C}^\infty$ Recession cone

OSQP Algorithm

Reformulation

$$(x^{k+1}, \tilde{z}^k) \leftarrow \underset{(x,z):Ax=z}{\text{argmin}} \frac{1}{2}x^T Px + q^T x + \frac{\sigma}{2}\left\|x - x^k\right\|^2 + \frac{\rho}{2}\left\|z - (2\Pi - I)(v^k)\right\|^2$$

$$v^{k+1} = \tilde{z}^k + (I - \Pi)(v^k)$$

Averaged non-expansive operator

$$(x^{k+1}, v^{k+1}) = T(x^k, v^k)$$

Original variables

$$z^k = \Pi(v^k) \qquad\qquad y^k = \rho(I - \Pi)(v^k)$$

Asymptotic behavior

Averaged non-expansive operator

$$(x^{k+1}, v^{k+1}) = T(x^k, v^k)$$

Differences

$$\delta x^k = x^k - x^{k-1} \qquad \delta v^k = v^k - v^{k-1}$$

Convergence to smallest vector

$$\lim_{k\to\infty} (\delta x^k, \delta v^k) = (\delta x, \delta v) \in \underset{\overline{\text{ran}}(T-I)}{\text{argmin}} \|(\delta x, \delta v)\|$$

[A. Pazy, 1971]

Auxiliary results

Difference of dual iterates: δy

$$A^T \delta y = 0 \qquad u^T \delta y_- + l^T \delta y_+ = -\frac{1}{\rho}\|\delta y\|^2$$

Difference of primal iterates: δx

$$P\delta x = 0 \qquad q^T \delta x = -\sigma\|\delta x\|^2 - \rho\|A\delta x\|^2 \qquad A\delta x \in C^\infty$$

Infeasibility detection

Main result

Primal infeasibility: $\delta y \neq 0$

$$A^T \delta y = 0 \qquad u^T \delta y_+ + l^T \delta y_- < 0$$

Dual infeasibility: $\delta x \neq 0$

$$P\delta x = 0 \qquad q^T \delta x < 0 \qquad (A\delta x)_i \begin{cases} = 0 & l_i, u_i \neq \infty \\ \geq 0 & u_i = +\infty \\ \leq 0 & l_i = -\infty \end{cases}$$

Example

Simple QP

$$\begin{aligned} \text{minimize} \quad & \tfrac{1}{2} x^T \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x \\ \text{subject to} \quad & 1 \leq \begin{bmatrix} 1 & 0 \end{bmatrix} \leq 2 \end{aligned}$$

Optimal solution

$$x = (1, 0)$$

Example

Primal infeasible

$$\text{minimize} \quad \frac{1}{2} x^T \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x$$

$$\text{subject to} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} \leq \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \leq \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

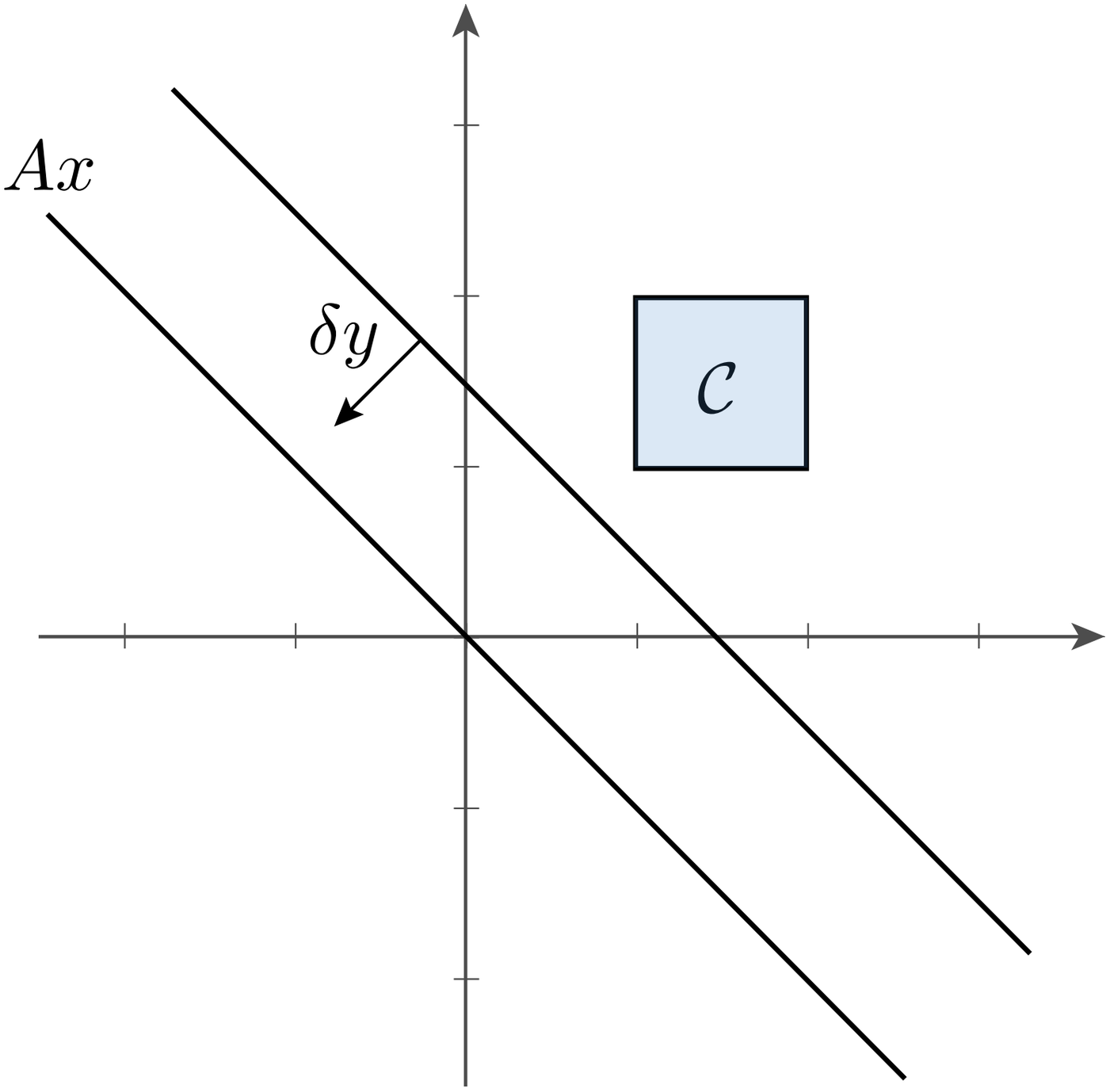
Conflicting constraint

Certificate

$$\delta y = (-1, -1)$$

Example

Primal infeasible



Certificate

$$\delta y = (-1, -1)$$

Example

Dual infeasible

$$\begin{aligned} \text{minimize} \quad & \tfrac{1}{2} x^T \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \end{bmatrix} x \\ \text{subject to} \quad & 1 \leq \begin{bmatrix} 1 & 0 \end{bmatrix} \leq 2 \end{aligned}$$

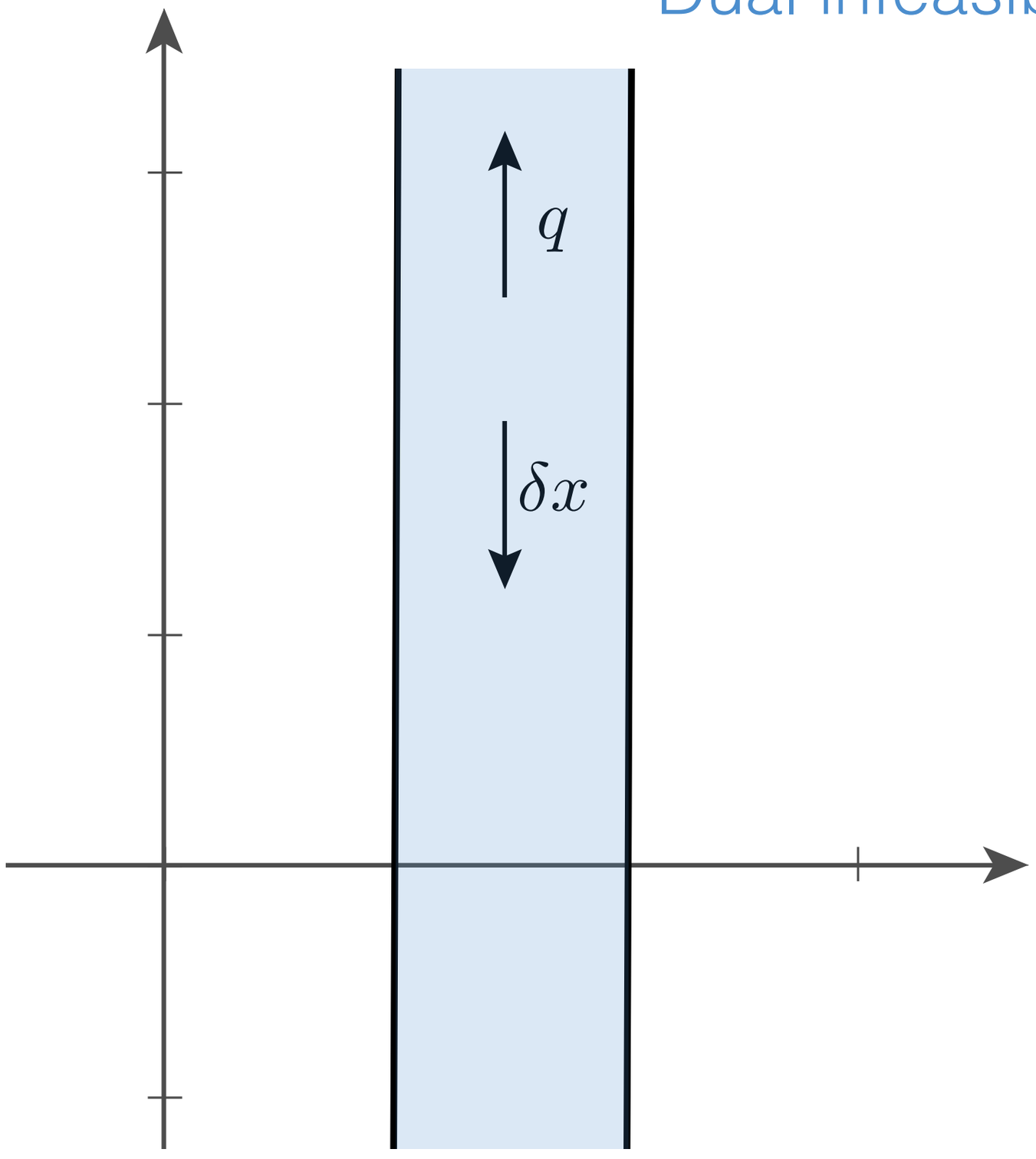
Unbounded direction

Certificate

$$\delta x = (0, -1)$$

Example

Dual infeasible



Certificate

$$\delta x = (0, -1)$$

Example

Primal and dual infeasible

$$\text{minimize} \quad \frac{1}{2}x^T \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

Unbounded direction

$$\text{subject to} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} \leq \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \leq \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Conflicting constraint

Certificates

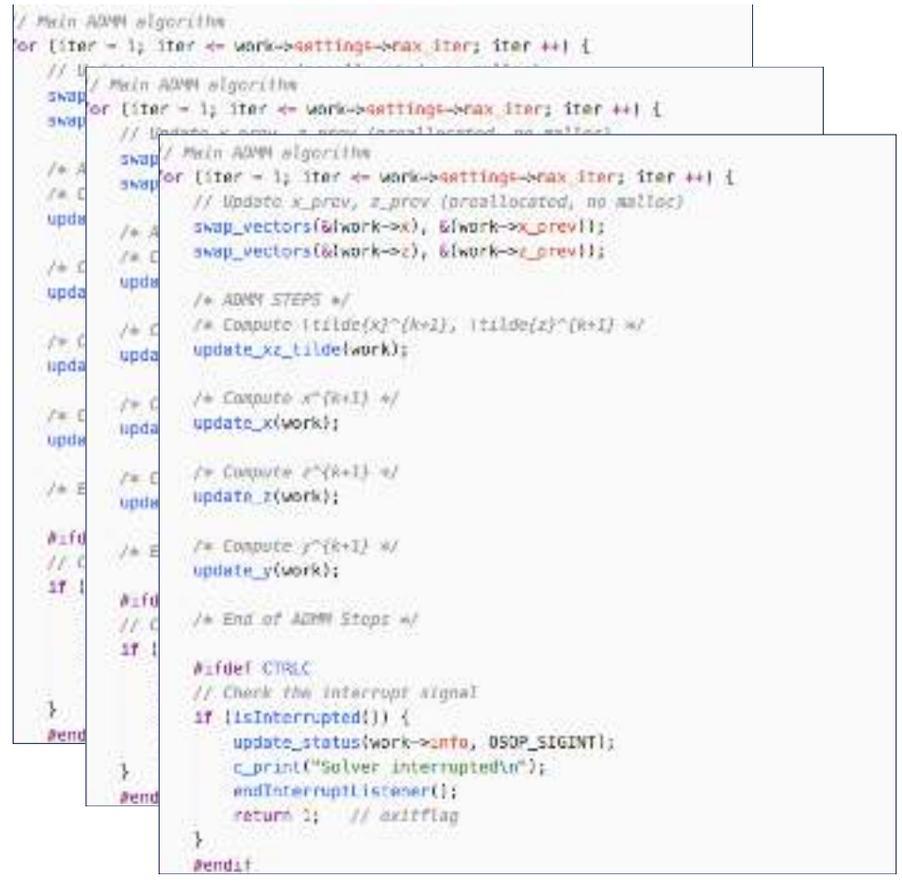
$$\delta x = (0, -1)$$
$$\delta y = (-1, -1)$$

Extensions

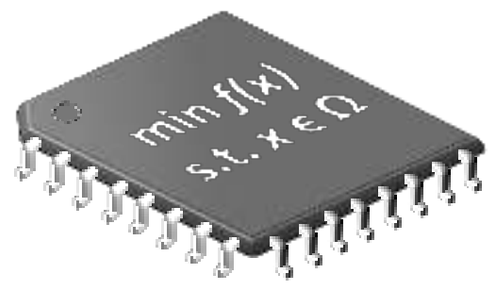
Code generation

```
# Create OSQP object
m = osqp.OSQP()

# Initialize solver
m.setup(P, q, A, l, u,
        settings)


# Generate C code
m.codegen('folder_name')
```

Optimized
C code

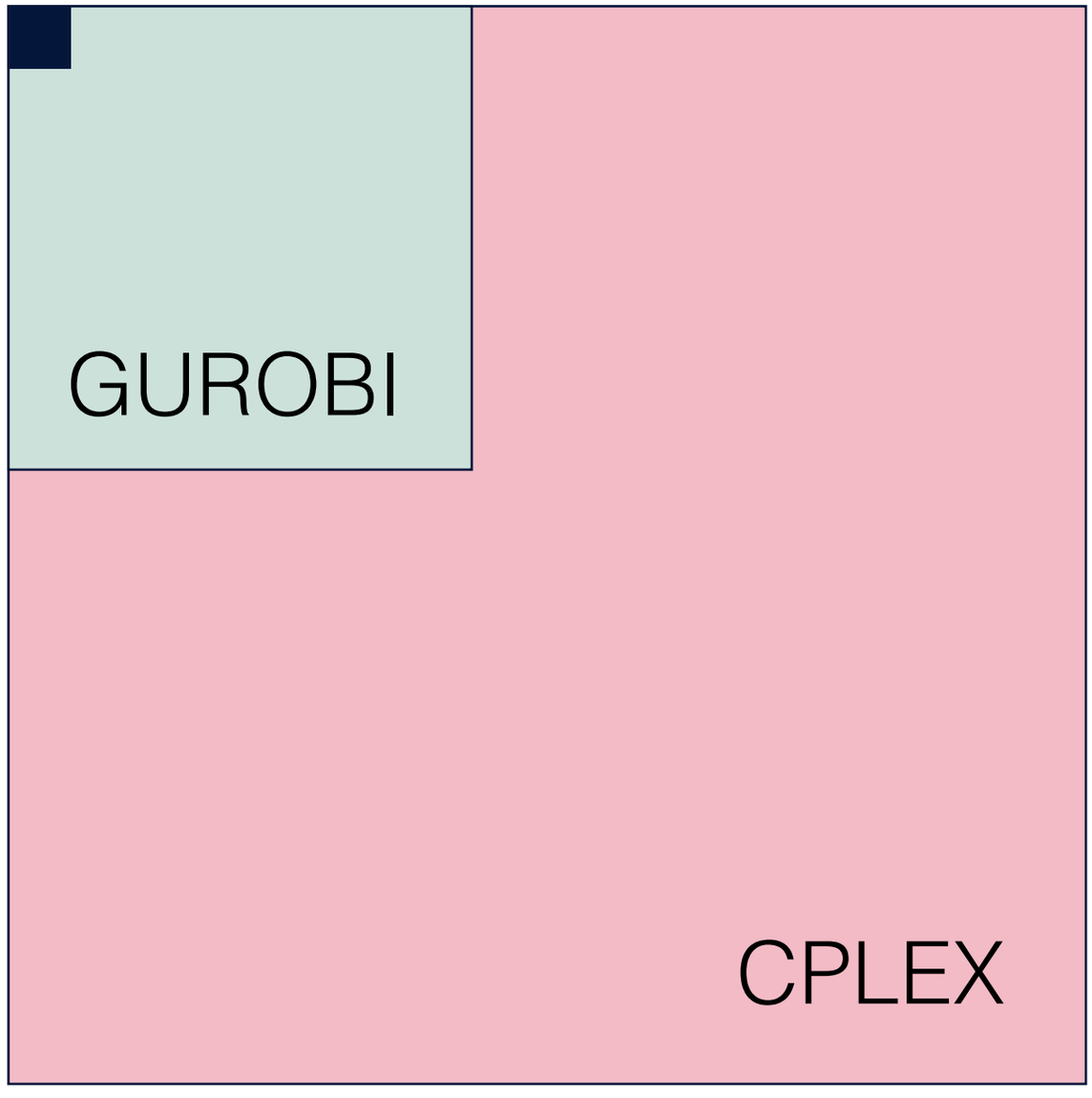


Embedded
hardware



Compiled code size ~80kb

OSQP →



300x
Reduction!

Mixed-Integer Quadratic Programs

$$
\begin{array}{ll}
\text{minimize} & \frac{1}{2}x^T P x + q^T x \\
\text{subject to} & l \leq Ax \leq u \\
& x_i \in \mathbf{Z} \quad \forall i \in \mathbf{I}
\end{array}
$$

Mixed-Integer Quadratic Programs

$$\begin{aligned} \text{minimize} \quad & \tfrac{1}{2} x^T P x + q^T x \\ \text{subject to} \quad & l \leq Ax \leq u \\ & x_i \in \mathbf{Z} \quad \forall i \in \mathbf{I} \end{aligned}$$

Integer constraints

Mixed-Integer Quadratic Programs

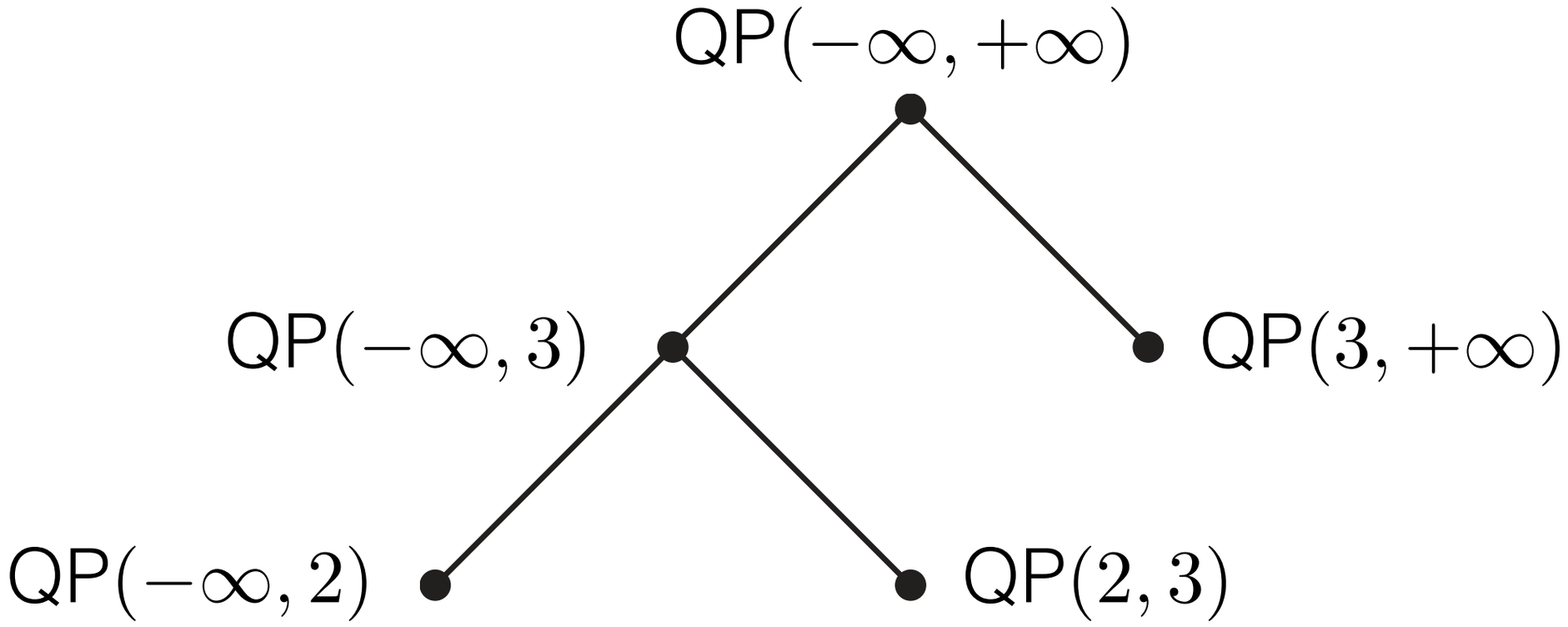
$$\begin{aligned} \text{minimize} \quad & \tfrac{1}{2} x^T P x + q^T x \qquad f(x) \\ \text{subject to} \quad & l \leq Ax \leq u \\ & x_i \in \mathbf{Z} \quad \forall i \in \mathbf{I} \end{aligned}$$

Integer constraints

Mixed-Integer Quadratic Programs

$$\begin{aligned} \text{minimize} \quad & \tfrac{1}{2} x^T P x + q^T x \qquad f(x) \\ \text{subject to} \quad & l \leq Ax \leq u \\ & x_i \in \mathbf{Z} \quad \forall i \in \mathbf{I} \end{aligned}$$

Integer constraints



Inner QPs

$\text{QP}(\underline{x}, \overline{x})$

$$
\begin{aligned}
\text{minimize} \quad & \tfrac{1}{2} x^T P x + q^T x \\
\text{subject to} \quad & l \leq Ax \leq u \\
& \underline{x}_i \leq x_i \leq \overline{x}_i \quad \forall i \in \mathbf{I}
\end{aligned}
$$

Inner QPs

$\text{QP}(\underline{x}, \overline{x})$

minimize $\frac{1}{2}x^T P x + q^T x$

subject to $l \leq Ax \leq u$

$\underline{x}_i \leq x_i \leq \overline{x}_i \quad \forall i \in \mathbf{I}$

Changing bounds

↓

Reusing factorization

Branch-and-bound

initialize $U \leftarrow \infty$, $\mathcal{H} \leftarrow \text{QP}(-\infty, \infty)$
while $\mathcal{H} \neq \emptyset$ **do**
    $\tilde{x}$, $f(\tilde{x}) \leftarrow$ pick $\text{QP}(\underline{x}, \overline{x})$ from $\mathcal{H}$ and solve
    if $\text{QP}(\underline{x}, \overline{x})$ is infeasible **or** $f(\tilde{x}) > U$ **then**
        prune
    else if $\tilde{x}$ is integer feasible **then**
        $U \leftarrow f(\tilde{x})$, $x^\star \leftarrow \tilde{x}$, prune
    else
        branch
    end if
end while

Branch-and-bound

initialize $U \leftarrow \infty$, $\mathcal{H} \leftarrow \text{QP}(-\infty, \infty)$
while $\mathcal{H} \neq \emptyset$ **do**
    $\tilde{x}$, $f(\tilde{x}) \leftarrow$ pick $\text{QP}(\underline{x}, \overline{x})$ from $\mathcal{H}$ and solve
    if $\text{QP}(\underline{x}, \overline{x})$ is infeasible **or** $f(\tilde{x}) > U$ **then**
        prune
    else if $\tilde{x}$ is integer feasible **then**
        $U \leftarrow f(\tilde{x})$, $x^\star \leftarrow \tilde{x}$, prune
    else
        branch
    end if
end while

Bad node

Branch-and-bound

initialize $U \leftarrow \infty$, $\mathcal{H} \leftarrow \text{QP}(-\infty, \infty)$
while $\mathcal{H} \neq \emptyset$ **do**
  $\tilde{x}$, $f(\tilde{x}) \leftarrow$ pick $\text{QP}(\underline{x}, \overline{x})$ from $\mathcal{H}$ and solve
  if $\text{QP}(\underline{x}, \overline{x})$ is infeasible **or** $f(\tilde{x}) > U$ **then**
    prune
  else if $\tilde{x}$ is integer feasible **then**
    $U \leftarrow f(\tilde{x})$, $x^\star \leftarrow \tilde{x}$, prune
  else
    branch
  end if
end while

Bad node

New solution

Branch-and-bound

initialize $U \leftarrow \infty$, $\mathcal{H} \leftarrow \text{QP}(-\infty, \infty)$
while $\mathcal{H} \neq \emptyset$ **do**
  $\tilde{x}$, $f(\tilde{x}) \leftarrow$ pick $\text{QP}(\underline{x}, \overline{x})$ from $\mathcal{H}$ and solve
  if $\text{QP}(\underline{x}, \overline{x})$ is infeasible **or** $f(\tilde{x}) > U$ **then**
    prune
  else if $\tilde{x}$ is integer feasible **then**
    $U \leftarrow f(\tilde{x})$, $x^\star \leftarrow \tilde{x}$, prune
  else
    branch
  end if
end while

Bad node

New solution

Continue

Saving computations

Factorization caching + Warm starting

ADMM

QP($\underline{x}, \overline{x}$)

Repeated MIQPs

Numerical example

Hybrid Model Predictive Control

$$\begin{aligned}
\text{minimize} \quad & \sum_{k=0}^{N-1} \gamma^k l(x(k)) + \gamma^N V(x(N)) \\
\text{subject to} \quad & x(k+1) = Ax(k) + Bu(k) \\
& x(0) = x_0 \\
& x(k) \in \mathcal{X}, u(k) \in \{-1, 0, 1\}^3
\end{aligned}$$

Numerical example

Hybrid Model Predictive Control

$$\text{minimize} \quad \sum_{k=0}^{N-1} \gamma^k l(x(k)) + \gamma^N V(x(N))$$

$$\text{subject to} \quad x(k+1) = Ax(k) + Bu(k) \qquad \text{Dynamics}$$

$$x(0) = x_0$$

$$x(k) \in \mathcal{X}, u(k) \in \{-1, 0, 1\}^3$$

Numerical example

Hybrid Model Predictive Control

minimize $$\sum_{k=0}^{N-1} \gamma^k l(x(k)) + \gamma^N V(x(N))$$

subject to $x(k+1) = Ax(k) + Bu(k)$ Dynamics

$x(0) = x_0$ Initial state

$x(k) \in \mathcal{X}, u(k) \in \{-1, 0, 1\}^3$

Numerical example

Hybrid Model Predictive Control

$$\text{minimize} \quad \sum_{k=0}^{N-1} \gamma^k l(x(k)) + \gamma^N V(x(N))$$

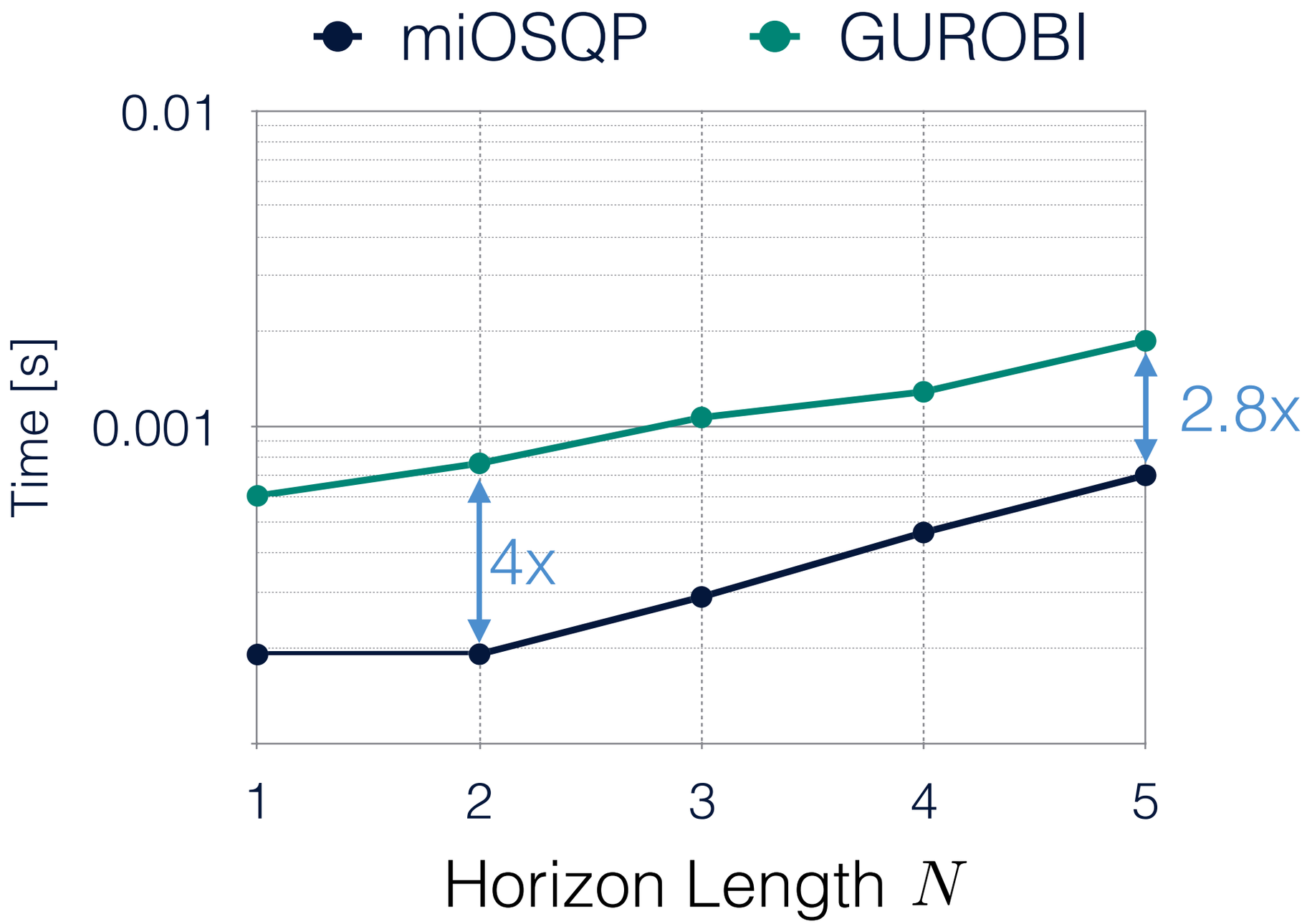
$$\text{subject to} \quad x(k+1) = Ax(k) + Bu(k) \qquad \text{Dynamics}$$

$$x(0) = x_0 \qquad \text{Initial state}$$

$$x(k) \in \mathcal{X}, u(k) \in \{-1, 0, 1\}^3 \qquad \text{State/input constraints}$$

MIQP Timings

Hybrid Model Predictive Control



Simple Python Implementation!

Conclusions

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Oxford

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IMT Lucca

Stephen Boyd

Stanford

Final remarks

OSQP

Simple

Robust

Embeddable

Warm-starting

Precise

Detects infeasibility

Future work

Semidefinite programs

"Meta-algorithms"

Mixed-Integer

SQP

References

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G. Banjac, P. Goulart, B. Stellato, and S. Boyd. ***Infeasibility detection in the alternating direction method of multipliers for convex optimization.*** optimization-online.org, 2017

G. Banjac, B.Stellato, N. Moehle, P. Goulart, A. Bemporad and S. Boyd. ***Embedded code generation using the OSQP solver.*** *IEEE Conference on Decision and Control (CDC) (submitted), 2017*

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